Solutions of the Homeworks

HW1: (a) $p_\mu p^\mu - m^2c^2 = \underbrace{(\beta^\kappa p_\kappa - mc)(\gamma^\lambda p_\lambda + mc)}_{} = 0$

$$= \beta^\kappa \gamma^\lambda p_\kappa p_\lambda + (\beta^\kappa - \gamma^\kappa) p_\kappa mc - m^2c^2$$

$$p_\mu p_\nu g^{\mu\nu} - \underbrace{\beta^\kappa \gamma^\lambda p_\kappa p_\lambda}_{\beta^\mu \gamma^\nu p_\mu p_\nu} - mc(\beta^\kappa - \gamma^\kappa) p_\kappa = 0$$

$$(g^{\mu\nu} - \beta^\mu \gamma^\nu) p_\mu p_\nu - mc \underbrace{(\beta^\kappa - \gamma^\kappa)}_{=0} p_\kappa = 0 \rightarrow \boxed{\beta^\kappa = \gamma^\kappa, \ \forall \kappa}$$

(b)

$$(g^{\mu\nu} - \underbrace{\gamma^\mu \gamma^\nu}_{}) \underbrace{p_\mu p_\nu}_{\text{symmetric in } \mu,\nu} = 0$$

$$\gamma^\mu \gamma^\nu \rightarrow \underbrace{\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\}}_{S^{\mu\nu}} + \underbrace{\tfrac{1}{2}[\gamma^\mu, \gamma^\nu]}_{A^{\mu\nu}} \qquad S^{\mu\nu} = S^{\nu\mu} \ \& \ A^{\mu\nu} = -A^{\nu\mu}$$

$$(g^{\mu\nu} - S^{\mu\nu} - A^{\mu\nu}) p_\mu p_\nu = 0$$

$A^{\mu\nu} p_\mu p_\nu$ $=0$ since symmetric × Antisymmetric

$$\underbrace{(g^{\mu\nu} - S^{\mu\nu})}_{=0} p_\mu p_\nu = 0 \rightarrow S^{\mu\nu} = g^{\mu\nu} \rightarrow \boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$$

(c) $\mu=\nu=0$, $\quad 2(\gamma^0)^2 = 2g^{00} \rightarrow (\gamma^0)^2 = 1$
$\mu=\nu=i$, $\quad (\gamma^i)^2 = -1$
$\mu=0, \nu=i$ $\quad 2\gamma^0\gamma^i = 0$
(If γ^0 & γ^i's are some commuting numbers

choose $\gamma^0 = +1$, $\gamma^i = i$
$(\gamma^0)^2 = 1$ ✓, $(\gamma^i)^2 = -1$ ✓, $\gamma^0\gamma^i = 0$ ✗

→ No numbers could satisfy the algebra

(d) (α) $\mu=0, \nu=i \rightarrow \gamma^0\gamma^i+\gamma^i\gamma^0=2g^{0i}=0$

$$\gamma^0 / \gamma^0\gamma^i=-\gamma^i\gamma^0$$

$$\underbrace{(\gamma^0)^2}_{=1}\gamma^i=-\gamma^0\gamma^i\gamma^0$$

$$\gamma^i=-\gamma^0\gamma^i\gamma^0$$

$$\mathrm{Tr}[\gamma^i]=-\mathrm{Tr}[\gamma^0\gamma^i\gamma^0] \quad (\mathrm{Tr}[ABC]=\mathrm{Tr}[BCA]=\dots)$$

$$=-\mathrm{Tr}[\gamma^i\underbrace{\gamma^0\gamma^0}_{=1}]=-\mathrm{Tr}[\gamma^i]$$

$$\rightarrow 2\,\mathrm{Tr}[\gamma^i]=0 \rightarrow \boxed{\mathrm{Tr}[\gamma^i]=0}$$

same holds for γ^0; $\mathrm{Tr}[\gamma^0]=0$.

(β) For γ^0: since $(\gamma^0)^2=1$, $(\gamma^0)^2-1=0 \rightarrow (\gamma^0-1)(\gamma^0+1)=0$

$\downarrow$

$\underline{\gamma^0=\pm 1}$

$\exists$ a basis where γ^0 is diagonal with ± 1 eign.

For γ^i: since $(\gamma^i)^2=-1 \rightarrow (\gamma^i)^2+1=0$

$\downarrow$

$\gamma^i=\pm i$

(e) $R\gamma^0R^\dagger=\gamma^0_{Diag}=\begin{pmatrix} +1 & & & & \\ & -1 & & & \\ & & +1 & & \\ & & & -1 & \\ & & & & \ddots \end{pmatrix}$

Since γ^0_{Diag} is traceless, $\exists$ $(+1,-1)$ pairs in the diagonal

$\downarrow$

$N\equiv$ Dim of γ^0 is <u>even.</u>

Same for γ^i.

$N=2,4,6,\dots$

$N=2$ case, Pauli matrices σ^i's & $I_{2\times 2}$

Take $\gamma^0=I_{2\times 2}$, $\gamma^i=i\sigma^i$

$\downarrow$

Not satisfying properties

one must go $N=4$.

(c) If $\gamma^0_{Dirac} = \begin{pmatrix} 1_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & -1_{2\times2} \end{pmatrix}$, $\gamma^i_{Dirac} = \begin{pmatrix} 0_{2\times2} & \sigma^i \\ -\sigma^i & 0_{2\times2} \end{pmatrix}$

One can show that $(\gamma^0_{Dirac})^2 = 1$, $(\gamma^i_D)^2 = -1$ & $\gamma^0_D\gamma^i_D + \gamma^i_D\gamma^0_D = 0$

$\rightarrow \gamma^5_D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Alternative representation

$$\gamma^0_{Chiral} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i_{Chiral} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \rightarrow \gamma^5_{Chiral} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

HW2: Show that $H_D = \frac{\hbar c}{i}\vec{\alpha}\cdot\vec{\nabla} + mc^2\beta$ for a free Dirac particle

$$(\not{p} - mc)\psi = 0, \quad \not{p} = \gamma^\mu p_\mu = i\hbar\gamma^\mu\partial_\mu$$

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$$

$$\gamma^0 /\ i\hbar\gamma^0\partial_0\psi + i\hbar\gamma^i\partial_i\psi - mc\psi = 0$$

$$c /\ i\hbar\underbrace{(\gamma^0)^2}_{1}\underbrace{\partial_0}_{\frac{1}{c}\frac{\partial}{\partial t}}\psi + i\hbar\underbrace{\gamma^0\gamma^i}_{\alpha_i}\partial_i\psi - mc\gamma^0\psi = 0$$

$$i\hbar\frac{\partial\psi}{\partial t} = -i\hbar c\underbrace{\gamma^0\gamma^i}_{\alpha^i}\partial_i\psi + mc^2\overbrace{\gamma^0}^{\beta}\psi$$

$$= \underbrace{\left(-i\hbar c\,\vec{\alpha}\cdot\vec{\nabla} + mc^2\beta\right)}_{\equiv H_D}\psi$$

HW3: (a) $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$

From H_D which is hermitian, both γ^0 & $\gamma^0\gamma^i$ must be hermitian as well.

- $\gamma^{0\dagger} = \gamma^0$
- $(\gamma^0\gamma^i)^\dagger = \gamma^0\gamma^i \to \gamma^{i\dagger} \underset{\gamma^0}{\underset{\parallel}{\gamma^{0\dagger}}} = \gamma^0\gamma^i \,/\, \gamma^0$

$\gamma^{i\dagger} \underbrace{\gamma^0\gamma^0}_{=1} = \gamma^0\gamma^i\gamma^0 \to \underline{\gamma^{i\dagger} = \gamma^0\gamma^i\gamma^0}$

Also, $\gamma^{0\dagger} = \underbrace{\gamma^0\gamma^0}_{=\mathbb{1}}\gamma^0 = \gamma^0\gamma^0\gamma^0$ ✓

(b) $(S^{\alpha\beta})^\dagger = -\gamma^0 S^{\alpha\beta}\gamma^0$

$S^{\alpha\beta} = \frac{1}{4}[\gamma^\alpha, \gamma^\beta] \to (S^{\alpha\beta})^\dagger = \frac{1}{4}[\gamma^\alpha, \gamma^\beta]^\dagger$

$= \frac{1}{4}[\gamma^{\beta\dagger}, \gamma^{\alpha\dagger}]$

$= \frac{1}{4}[\gamma^0\gamma^\beta\gamma^0, \gamma^0\gamma^\alpha\gamma^0]$

$= \frac{1}{4}\left(\gamma^0\gamma^\beta\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}\gamma^\alpha\gamma^0 - \gamma^0\gamma^\alpha\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}\gamma^\beta\gamma^0\right)$

$= \cancel{\frac{1}{4}}\left(\gamma^0 \underbrace{(\gamma^\beta\gamma^\alpha - \gamma^\alpha\gamma^\beta)}_{[\gamma^\beta, \gamma^\alpha] = -[\gamma^\alpha, \gamma^\beta] = -\cancel{4} S^{\alpha\beta}}\gamma^0\right)$

$= -\gamma^0 S^{\alpha\beta}\gamma^0$

(c) $S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma^0$

$S[\Lambda] = e^{\frac{1}{2}\Omega_{\alpha\beta}S^{\alpha\beta}} \to S[\Lambda]^\dagger = e^{1/2\,\Omega_{\alpha\beta}\overbrace{(S^{\alpha\beta})^\dagger}^{-\gamma^0 S^{\alpha\beta}\gamma^0}}$

$= e^{-\frac{1}{2}\Omega_{\alpha\beta}\gamma^0 S^{\alpha\beta}\gamma^0}$

$$S[\Lambda]^\dagger = e^{-\frac{1}{2}\gamma^0 \Omega\cdot S\gamma^0} \qquad \left(e^x = 1 + x + \frac{x^2}{2!} + \dots\right)$$

$$= \underset{\underset{\gamma^0\gamma^0}{\shortparallel}}{1} - \frac{1}{2}\gamma^0\Omega\cdot S\gamma^0 + \frac{1}{2!}\frac{1}{4}\underbrace{(\gamma^0\Omega\cdot S\gamma^0)^2}_{} + \dots$$

$$= \gamma^0\Omega\cdot S\underbrace{\gamma^0\gamma^0}_{=\mathbb{1}}\Omega\cdot S\gamma^0$$

$$= \gamma^0(\Omega\cdot S)^2\gamma^0$$

$$= \gamma^0\underbrace{\left(1 - \frac{1}{2}\Omega\cdot S + \frac{1}{2!}\left(\frac{1}{2}\Omega\cdot S\right)^2 + \dots\right)}_{e^{-\frac{1}{2}\Omega\cdot S} = S[\Lambda]^{-1}}\gamma^0$$

$$\underline{S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma^0}$$

(d) $\bar{\psi}\psi$ is a Lorentz scalar.

$$\psi(x) \xrightarrow{LT} S[\Lambda]\psi(\Lambda^{-1}x)$$

$$\bar{\psi}(x) \xrightarrow{LT} \overline{S[\Lambda]\psi(\Lambda^{-1}x)} = (S[\Lambda]\psi(\Lambda^{-1}x))^\dagger\gamma^0$$

$$= \psi^\dagger(\Lambda^{-1}x)\underset{\gamma^0\gamma^0}{\wedge} S[\Lambda]^\dagger\gamma^0$$

$$= \bar{\psi}(\Lambda^{-1}x)\underbrace{\gamma^0 S[\Lambda]^\dagger\gamma^0}_{S[\Lambda]^{-1}} \text{ (from (c))}$$

$$= \bar{\psi}(\Lambda^{-1}x)S[\Lambda]^{-1}$$

$$\bar{\psi}(x)\psi(x) \longrightarrow \bar{\psi}(\Lambda^{-1}x)\underbrace{S[\Lambda]^{-1}S[\Lambda]}_{=\mathbb{1}}\psi(\Lambda^{-1}x)$$

$$= \bar{\psi}(\Lambda^{-1}x)\cdot\psi(\Lambda^{-1}x) \checkmark$$

HW6

(e) $J = \int d^4x \, \mathcal{L}(\psi, \bar\psi, \partial_\mu\psi, \partial_\mu\bar\psi)$

$$\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$$

$$\underbrace{\partial_\alpha \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\bar\psi)}}_{=0} - \underbrace{\frac{\partial\mathcal{L}}{\partial\bar\psi}}_{-m\psi + i\gamma^\mu\partial_\mu\psi} = 0 \quad \to (i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\underbrace{\partial_\alpha \frac{\mathcal{L}}{\partial(\partial_\alpha\psi)}}_{\bar\psi i\gamma^\mu \delta^\alpha_\mu} - \underbrace{\frac{\partial\mathcal{L}}{\partial\psi}}_{-m\bar\psi} = 0 \to \bar\psi(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0$$