Ödev 1: Denklem 4 ile verilen Lagrangiyen'deki alanlar için hareket denklemlerini elde ederek $\mathcal{F}$ alanının çözümünün sıfır olması gerektiğini gösteriniz.

Çözüm:

$$\mathcal{L} = -\partial^{\mu}\phi\partial_{\mu}\phi^{\dagger} + i\psi^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\psi + F^{\dagger}F$$

$\mathcal{L}$ ile verilmiş olan Lagrangiyen'deki alanlara ait hareket denklemleri her bir alan için Euler-Lagrange denklemleri yazılarak elde edilebilir:

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A)}\right) - \frac{\partial\mathcal{L}}{\partial A} = 0$$

Burada $A = \phi, \phi^{\dagger}, \psi, \psi^{\dagger}, F, F^{\dagger}$

$\phi^{\dagger}$:

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^{\dagger})}\right) - \frac{\partial\mathcal{L}}{\partial\phi^{\dagger}} = 0 \quad : \quad \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^{\dagger})} = -\partial^{\mu}\phi$$

$$\frac{\partial\mathcal{L}}{\partial\phi^{\dagger}} = 0$$

$$\boxed{-\partial_{\mu}\partial^{\mu}\phi = 0}$$

Kütlesiz bir skaler alanın hareket denklemi:
Klein-Gordon Denklemi

$\psi^{\dagger}$:

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi^{\dagger})}\right) - \frac{\partial\mathcal{L}}{\partial\psi^{\dagger}} = 0 \quad : \quad \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi^{\dagger})} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\psi^{\dagger}} = i\bar{\sigma}^{\mu}\partial_{\mu}\psi$$

$$0 - i\bar{\sigma}^{\mu}\partial_{\mu}\psi = 0$$

$$\boxed{i\bar{\sigma}^{\mu}\partial_{\mu}\psi}$$

Kütlesiz Weyl (2-) spinorlar için hareket denklemi:
Dirac Denklemi

$\mathcal{F}^{\dagger}$:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathcal{F}^{\dagger})} \right) - \frac{\partial \mathcal{L}}{\partial \mathcal{F}^{\dagger}} = 0 \quad : \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathcal{F}^{\dagger})} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}^{\dagger}} = \mathcal{F}$$

$$0 - \mathcal{F} = 0 \Rightarrow \boxed{\mathcal{F} = 0}$$

Yukarıda elde edilen hareket denklemleri, ancak gerçek (on-shell) parçacıkları ifade edebilirler. Başka bir deyişle sanal (virtual, off-shell) parçacıklar, yukarıda elde edilen hareket denklemlerine uymak zorunda değildir. Özellikle spinor alanların on-shell ve off-shell serbestlik derecelerinin farklı olmasının nedeni budur.

Ödev 2: Denklem 4 ile verilen Lagrangiyen'in Denklem 5 ile tanımlanan dönüşümler altında invaryant olduğunu gösteriniz.

Çözüm

$$\mathcal{L} = -\partial^\mu \phi \partial_\mu \phi^\dagger + i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + F^\dagger F$$

Lagrangiyen'inin, eğer alanlar aşağıdaki gibi dönüşürlerse;

$$\begin{array}{llll|ll}
\phi' & = Q\phi & = \phi + \delta\phi & & \delta\phi & = \epsilon\psi \\
\psi' & = Q\psi & = \psi + \delta\psi & & \delta\psi & = -i(\sigma^\mu \epsilon^\dagger)\partial_\mu \phi + \epsilon F \\
F' & = QF & = F + \delta F & & \delta F & = -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi
\end{array}$$

bu dönüşümler altında invaryant (değişmez) kalacağı söylenmişti. Bu değişmezliği görebilmek için, yukarıda verilen Lagrangien'i süpersimetrik dönüşümlere uğramış alanlar için tekrar yazalım.

$$\mathcal{L}' = -\partial_\mu \phi'^\dagger \partial^\mu \phi' + i\psi^{\dagger\prime} \bar{\sigma}^\mu \partial_\mu \psi' + F'^\dagger F'$$

Uygulanan dönüşümlerin sonsuz küçük bir dönüşüm olduğu kabul edilirse;

$$\left.\begin{aligned}
\phi' &= \phi + \delta\phi \\
\phi'^\dagger &= \phi^\dagger + \delta\phi^\dagger \\
\psi' &= \psi + \delta\psi \\
\psi^{\dagger\prime} &= \psi^\dagger + \delta\psi'
\end{aligned}\right\}$$

$$\mathcal{L}' = \mathcal{L} + \delta\mathcal{L}$$

Eğer $\mathcal{L}$, yukarıda verilen süpersimetrik dönüşümler altında değişmeden kalıyorsa $\mathcal{L}' = \mathcal{L} \Rightarrow \delta\mathcal{L} = 0$ olmalıdır.

$$\mathcal{L}'_{scalar} = -\partial_\mu \phi'^\dagger \partial^\mu \phi' = -\partial_\mu (\phi + \delta\phi)^\dagger \partial^\mu (\phi + \delta\phi)$$

$$= \underbrace{-\partial_\mu \phi^\dagger \partial^\mu \phi^\dagger}_{\mathcal{L}_{scalar}} \underbrace{- \partial_\mu (\delta\phi)^\dagger \partial^\mu \phi - \partial_\mu \phi^\dagger \partial^\mu (\delta\phi)}_{\delta\mathcal{L}_{scalar}}$$

$$\delta\mathcal{L}_{scalar} = -\partial_\mu (\epsilon\psi)^\dagger \partial^\mu \phi - \partial_\mu \phi^\dagger \partial^\mu (\epsilon\psi)$$

$$= -(\partial_\mu \psi^\dagger \partial^\mu \phi)\epsilon^\dagger - \epsilon(\partial_\mu \phi^\dagger \partial^\mu \psi)$$

$$\mathcal{L}'_{fermion} = i\psi'^\dagger \bar{\sigma}^\mu \partial_\mu \psi' = i(\psi + \delta\psi)^\dagger \bar{\sigma}^\mu \partial_\mu (\psi + \delta\psi)$$

$$= \underbrace{i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi}_{\mathcal{L}_{fermion}} + \underbrace{i\,\delta\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \delta\psi}_{\delta\mathcal{L}_{fermion}}$$

$$\delta\mathcal{L}_{fermion} := i\left[i(\sigma^\nu \epsilon^\dagger)^\dagger \partial_\nu \phi^\dagger + \epsilon^\dagger F^\dagger\right]\bar\sigma^\mu \partial_\mu \psi \; +$$

$$i\,\psi^\dagger \bar\sigma^\mu \underline{\partial_\mu}\left[-i(\sigma^\nu \epsilon^\dagger)\underline{\partial_\nu} \phi + \epsilon F\right]$$

$$= -\epsilon\sigma^\nu\bar\sigma^\mu \partial_\nu\phi^\dagger \partial_\mu\psi + i\epsilon^\dagger F^\dagger \bar\sigma^\mu \partial_\mu \psi \; +$$

$$\underbrace{\partial_\mu\left[\psi^\dagger \bar\sigma^\mu\sigma^\nu \partial_\nu\phi\, \epsilon^\dagger + \psi^\dagger \epsilon F\right]}_{} - \underbrace{\partial_\mu\psi^\dagger \bar\sigma^\mu}_{}\sigma^\nu \partial_\nu \phi\, \epsilon^\dagger - \partial_\mu\psi^\dagger \epsilon F$$

$$\mathcal{L}'_{aux} = F^{\dagger\prime} F' = (F+\delta F)^\dagger (F+\delta F) = \underbrace{F^\dagger F}_{\mathcal{L}_{aux}} + \underbrace{\delta F^\dagger F + F^\dagger \delta F}_{\delta\mathcal{L}_{aux}}$$

$$\delta\mathcal{L}_{aux} = (-i\epsilon^\dagger \bar\sigma^\mu \partial_\mu \psi)^\dagger F + F^\dagger(-i\epsilon^\dagger\bar\sigma^\mu \partial_\mu\psi)$$

$$= i\bar\sigma^\mu \epsilon\, \partial_\mu \psi\, F - i\epsilon^\dagger \bar\sigma^\mu \partial_\mu \psi\, F^\dagger \quad \ast$$

$$\delta\mathcal{L} = \delta\mathcal{L}_{scalar} + \delta\mathcal{L}_{fermion} + \delta\mathcal{L}_{aux}$$

$$= -(\partial_\mu \psi^\dagger \partial^\mu \phi)\epsilon^\dagger - \epsilon(\partial_\mu\phi^\dagger \partial^\mu \psi)$$

$$i\epsilon\sigma^\nu\bar\sigma^\mu \partial_\nu\phi^\dagger \partial_\mu\psi + i\epsilon^\dagger F^\dagger \cancel{\bar\sigma^\mu}^{(1)} \partial_\mu \psi \; +$$

$$+ \partial_\mu\left[\psi^\dagger \bar\sigma^\mu\sigma^\nu \partial_\nu\phi\, \epsilon^\dagger + \psi^\dagger \epsilon F\right] - \partial_\mu\psi^\dagger \bar\sigma^\mu\sigma^\nu \partial_\nu \phi\, \epsilon^\dagger - i\cancel{\partial_\mu \psi^\dagger}^{(2)} \epsilon F$$

$$+ i\bar\sigma^\mu \epsilon \cancel{\partial_\mu}^{(2)} \psi\, F - i\epsilon^\dagger \bar\sigma^\mu \cancel{\partial_\mu}^{(1)} \psi\, F^\dagger$$

$$\ast\ast$$

$$= -\frac{1}{2}\partial_\mu\psi^\dagger \partial^\mu \phi\, \epsilon^\dagger - \frac{1}{2}\epsilon\,\partial_\mu \phi^\dagger \partial^\mu\psi - i\epsilon\sigma^\nu\bar\sigma^\mu \partial_\nu\phi^\dagger \partial_\mu \psi - i\partial_\mu\psi^\dagger \bar\sigma^\mu \sigma^\nu \partial_\nu \phi\, \epsilon^\dagger$$

$\bar\sigma^\mu\sigma^\nu - \sigma^\nu\bar\sigma^\mu = -2\eta^{\mu\nu}$	$\sigma^\nu\bar\sigma^\mu - \bar\sigma^\mu\sigma^\nu = -2\eta^{\mu\nu}$
$\bar\sigma^\mu\sigma^\nu + \sigma^\nu\bar\sigma^\mu = \eta^{\mu\nu}$	$\sigma^\nu\bar\sigma^\mu + \bar\sigma^\mu\sigma^\nu = \eta^{\mu\nu}$
$\bar\sigma^\mu\sigma^\nu = -\frac{1}{2}\eta^{\mu\nu}$	$\sigma^\nu\bar\sigma^\mu = -\frac{1}{2}\eta^{\mu\nu}$

$$\delta\mathcal{L} = -\frac{1}{2}\partial_\mu\psi^\dagger \partial^\mu\phi\,\epsilon^\dagger + \frac{1}{2}\partial_\mu\psi^\dagger \eta^{\mu\nu}\partial_\nu\phi\,\epsilon'$$

$$-\frac{1}{2}\epsilon\,\partial_\mu\phi^\dagger \partial^\mu\psi + \frac{1}{2}\epsilon\,\eta^{\mu\nu}\partial_\nu\phi^\dagger \partial_\mu\psi$$

$$= \left[-\frac{1}{2}\partial_\mu\psi^\dagger \partial^\mu\phi + \frac{1}{2}\partial_\mu\psi^\dagger\partial^\mu\phi\right]\epsilon^\dagger$$

$$+\epsilon\left[-\frac{1}{2}\epsilon\,\partial_\mu\phi^\dagger\partial^\mu\psi + \frac{1}{2}\epsilon\,\partial_\mu\phi^\dagger\partial^\mu\psi\right] = 0$$