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Nonleptonic *B* meson decays to NNLO

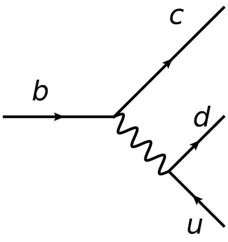
Implications of LHCb measurements and future prospects

Manuel Egner | CERN, October 23, 2024

INSTITUTE FOR THEORETICAL PARTICLE PHYSICS

based on 2406.19456 [Egner, Fael, Schönwald, Steinhauser (2024)]
and 2411.xxxxx [Egner, Fael, Lenz, Piscopo, Rusov, Schönwald, Steinhauser (in preparation)]

Motivation



- Heavy Quark Expansion (HQE): Decay width of $B \to X_c q_2 q_3$ and $B \to X_u q_2 q_3$ as sum of decay width of $b \to q_1 q_2 q_3$ and corrections suppressed by the mass m_b:

$$\Gamma(B) = \Gamma_3 + \Gamma_5 \frac{\langle \mathcal{O}_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_b^3} + \cdots + 16\pi^2 \left(\Gamma_6 \frac{\langle \tilde{\mathcal{O}}_6 \rangle}{m_b^3} + \Gamma_7 \frac{\langle \tilde{\mathcal{O}}_7 \rangle}{m_b^4} + \ldots \right).$$

- $\Gamma_3 \leftrightarrow$ decay of free b-quark.

- Our goal: Hadronic decay channels $b \to q_1 q_2 q_3$ at $\mathcal{O}\left(\alpha_s^2\right)$ with all charm mass effects.
- Input for the calculation of B-meson lifetimes in HQE.
- Knowing the $\mathcal{O}\left(\alpha_s^2\right)$ contributions will reduce the uncertainty induced by the renormalization scale variation.

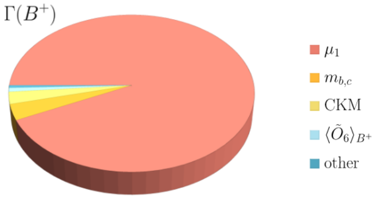


Figure: Uncertainty contributions on B-meson lifetimes [Albrecht, Bernlochner, Lenz, Rusov (2024)]

What is known?

$$\Gamma(B) = \Gamma_3 + \Gamma_5 \frac{\langle \mathcal{O}_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_b^3} + \cdots + 16\pi^2 \left(\Gamma_6 \frac{\langle \tilde{\mathcal{O}}_6 \rangle}{m_b^3} + \Gamma_7 \frac{\langle \tilde{\mathcal{O}}_7 \rangle}{m_b^4} + \dots \right).$$

$\Gamma_3^{SL,(3)}$	[Fael, Schönwald, Steinhauser (2020)] [Czakon, Czarnecki, Dowling (2021)]	$\Gamma_3^{NL,(2)}$ (incomplete)	[Czarnecki, Slusarczyk, Tkachov (2005)]
$\Gamma_5^{SL,(1)}$	[Alberti, Gambino, Nandi (2014)] [Mannel, Pivovarov, Rosenthal (2015)]	$\Gamma_5^{NL,(1)}$	[Mannel, Moreno, Pivovarov (2023)] [Mannel, Moreno, Pivovarov (2024)]
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$\tilde{\Gamma}_6^{SL,(1)}$	[Lenz, Rauh (2013)]	$\tilde{\Gamma}_6^{NL,(1)}$	[Beneke, Buchalla, Greub, Lenz, Nierste (2002)] [Franco, Lubicz, Mescia, Tarantino (2002)]
$\Gamma_7^{SL,(0)}$	[Dassinger, Mannel, Turczyk (2007)]		
$\Gamma_8^{SL,(0)}$	[Mannel, Turczyk, Uraltsev (2010)]		

What is known?

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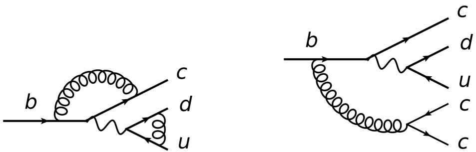
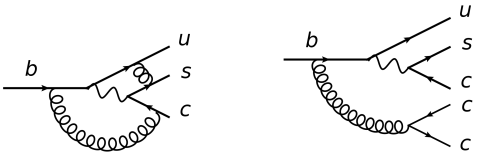
Nonleptonic decay channels at NNLO

Four decay channels with charm quarks in the final state at $\mathcal{O}(\alpha_s^2)$:

- $b \to c\bar{u}d$ ($\propto V_{cb}$)



- $b \to u\bar{c}s$ ($\propto V_{ub}$)



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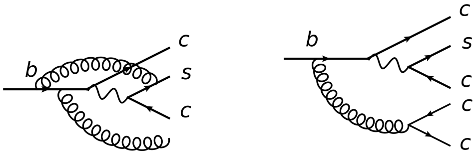

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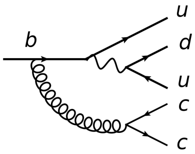


Calculation setup

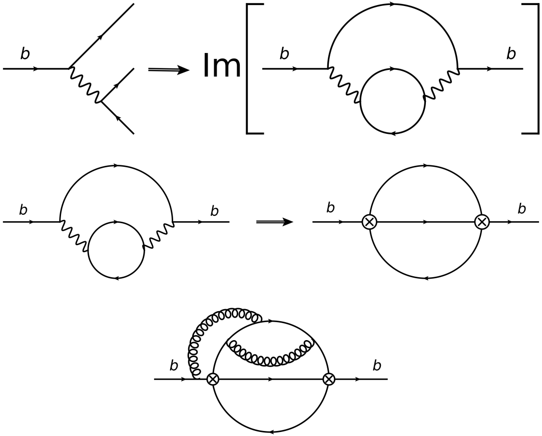
- Optical theorem:

$$\Gamma = \frac{1}{m_b} \mathrm{Im}\left[\mathcal{M}(b \to b)\right]$$

- Integrate out W-boson

$$\frac{1}{(m_W^2 - p^2)} \to \frac{1}{m_W^2}$$

- At $\mathcal{O}\left(\alpha_s^2\right)$ calculate imaginary part of 4-loop diagrams



Effective operators

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \sum_{q_{1,3}=u,c} \sum_{q_2=d,s} V_{q_1 b} V^*_{q_2 q_3} \left(C_1(\mu_b) O_1^{q_1 q_2 q_3} + C_2(\mu_b) O_2^{q_1 q_2 q_3} \right) + \text{h.c.}$$

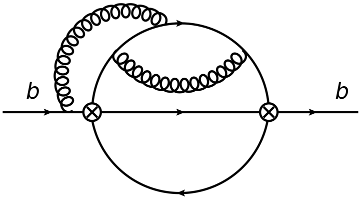
with physical operators

$$O_1^{q_1 q_2 q_3} = (\bar{q}_1^{\alpha} \gamma^{\mu} P_L b^{\beta})(\bar{q}_2^{\beta} \gamma_{\mu} P_L q_3^{\alpha}),$$
$$O_2^{q_1 q_2 q_3} = (\bar{q}_1^{\alpha} \gamma^{\mu} P_L b^{\alpha})(\bar{q}_2^{\beta} \gamma_{\mu} P_L q_3^{\beta}).$$

- Consider all possible combinations of operators:

$$\{O_1 \times O_1, O_1 \times O_2, O_2 \times O_1, \dots\}$$

- What to do with γ_5 in P_L?

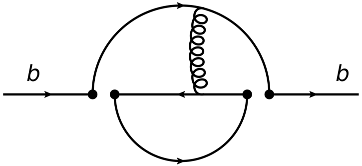


Treatment of γ_5

- At NLO and NNLO traces with one γ_5 matrix appears

$$\mathrm{Tr}\left[\gamma_5\gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma\right]$$

  in $d = 4 - 2\epsilon$ dimensions



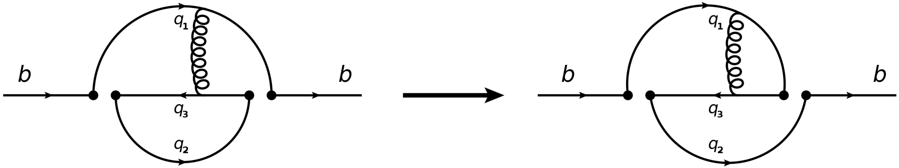
- Use anticommuting γ_5 to be consistent with calculation of matching coefficients and anomalous dimensions [Buras, Weisz (1990)], [Gorbahn, Haisch (2005)]
- Solution: Fierz identities [Bagan, Ball, Braun, Gosdzinsky (1994)]:

$$O_1^{q_1 q_2 q_3} = (\bar{q}_1^\alpha \gamma^\mu P_L b^\beta)(\bar{q}_2^\beta \gamma_\mu P_L q_3^\alpha) \xrightarrow{\text{Fierz}} (\bar{q}_2^\beta \gamma^\mu P_L b^\beta)(\bar{q}_1^\alpha \gamma_\mu P_L q_3^\alpha)$$

Treatment of γ_5

- Fierzed operators correspond to unfierzed operators with switched color indices

$$O_1^{q_1 q_2 q_3} = (\bar{q}_1^{\alpha}\gamma^{\mu}P_L b^{\beta})(\bar{q}_2^{\beta}\gamma_{\mu}P_L q_3^{\alpha}) \xrightarrow{\text{Fierz}} (\bar{q}_2^{\beta}\gamma^{\mu}P_L b^{\beta})(\bar{q}_1^{\alpha}\gamma_{\mu}P_L q_3^{\alpha}) = O_2^{q_2 q_1 q_3}$$



- Applying Fierz identities to one vertex leads to one instead of two spin lines:

$$\mathrm{Tr}\left[\ldots\gamma_5\ldots\right]\mathrm{Tr}\left[\ldots\gamma_5\ldots\right] \to \mathrm{Tr}\left[\ldots\gamma_5\ldots\gamma_5\ldots\right]$$

- We can use anticommuting γ_5!
- But: Fierz identities are valid in $d = 4$ dimensions!

Evanescent operators

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- Fierz symmetry can be restored order by order in perturbation theory by choosing the correct evanescent operators [Buras, Weisz (1990)], [Herrlich, Nierste (1994)]

$$E_1^{(1),q_1q_2q_3} = (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3} P_L b^\beta)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3} P_L q_3^\alpha) - (16 - 4\epsilon + A_1\epsilon^2) O_1^{q_1q_2q_3},$$
$$E_2^{(1),q_1q_2q_3} = (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3} P_L b^\alpha)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3} P_L q_3^\beta) - (16 - 4\epsilon + A_1\epsilon^2) O_2^{q_1q_2q_3},$$
$$E_1^{(2),q_1q_2q_3} = (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L b^\beta)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L q_3^\alpha) - (256 - 224\epsilon + B_1\epsilon^2) O_1^{q_1q_2q_3},$$
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- Fix $\{A_1, B_1, B_2\}$ by imposing a symmetric anomalous dimension matrix γ [Buras, Weisz (1990)]

$$\mu \frac{\mathrm{d}C_i}{\mathrm{d}\mu} = \gamma_{ij} C_j, \qquad \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}, \quad \text{with } \gamma_{11} = \gamma_{22}, \quad \gamma_{12} = \gamma_{21}$$

- This condition yields

$$A_2 = -4, \qquad B_1 = -\frac{45936}{125}, \qquad B_2 = -\frac{115056}{115}$$

Calculation setup

	$b \to c\bar{u}d$	$b \to c\bar{c}s$
Generate diagrams with `QGRAF` [Nogueira (1993)]	1308 diagrams	1308 diagrams
Mapping on integral families with `TAPIR` [Gerlach, Herren, Lang (2022)] and `EXP` [Harlander, Seidensticker, Steinhauser (1998-1999)]	42 families	49 families
Reduction to master integrals with `Kira` [Klappert, Lange, Maierhöfer, Usovitsch (2021)] — Choose good basis of master integrals, where ϵ and $\rho = m_c/m_b$ factorize, with `ImproveMasters.m` [Smirnov, Smirnov (2020)]	385 master integrals	644 master integrals

Master integrals: Expand and Match

[Fael, Lange, Schönwald, Steinhauser (2021)]

1. Set up differential equation for master integrals using IBP relations [Chetyrkin, Tkachov (1981)]

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\vec{I} = A(\epsilon, \rho) \cdot \vec{I}$$

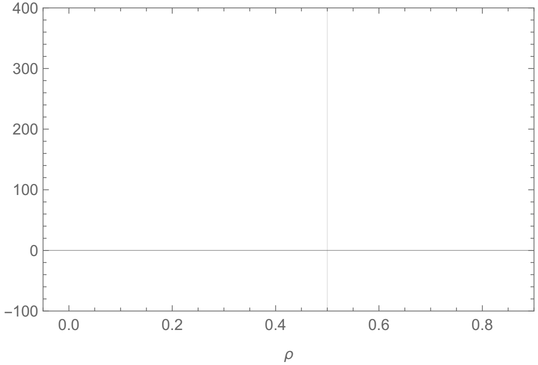
2. Make general expansion ansatz in $\rho = m_c/m_b$ around certain point ρ_0 for integral

$$I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+|\epsilon_{\min}|} \sum_{n=n_{\min}}^{n_{\max}} c[i,j,m,n]\, \epsilon^j (\rho - \rho_0)^{n/2} \log^m(\rho - \rho_0)$$

3. Insert ansatz in DEQ $\rightarrow$ linear equations for $c[i,j,m,n]$ for every power in ρ
4. Determine remaining coefficients by matching to numerical results obtained with `AMFlow` [Liu, Ma (2022)]
5. Expansions around several expansion points, match in between
   $\rightarrow$ cover $\rho \in [0, 1]$.

Master integrals: Example

1. Taylor expansion around $\rho_0 = 0.5$
2. Obtain numerical values of integrals at $\rho = 0.5$ with `AMFlow`
3. Determine expansion coefficients by matching expansion of step 1 to numerical results of step 2
4. Threshold expansion around $\rho_0 = 1/3$
5. Evaluate expansions of step 1 at $\rho = 0.4$
6. Determine expansion coefficients of expansion of step 4 by matching to numerical results of step 5
7. Repeat procedure for next expansion point



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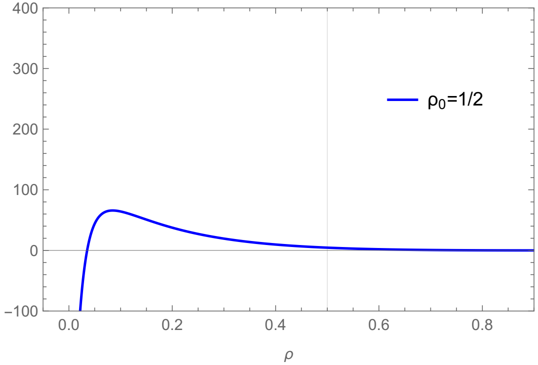

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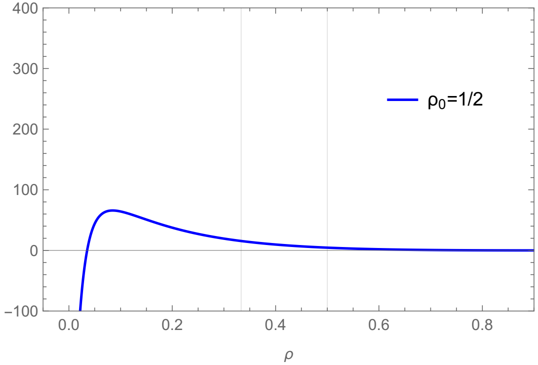

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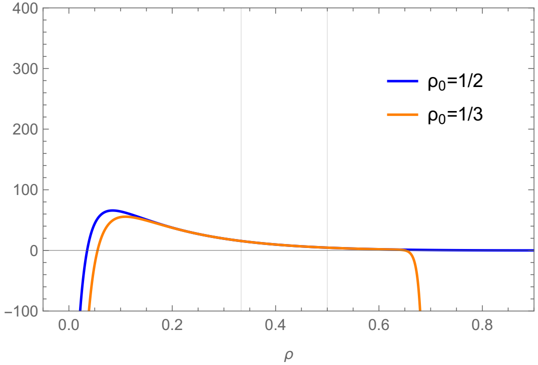

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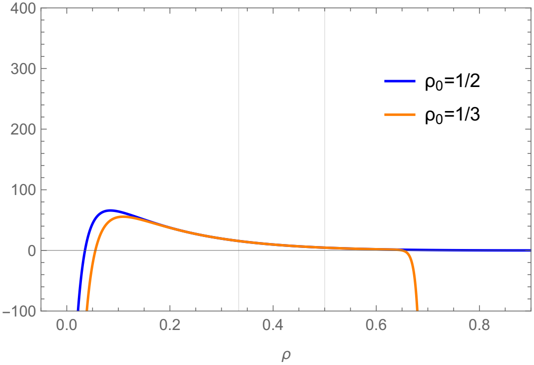

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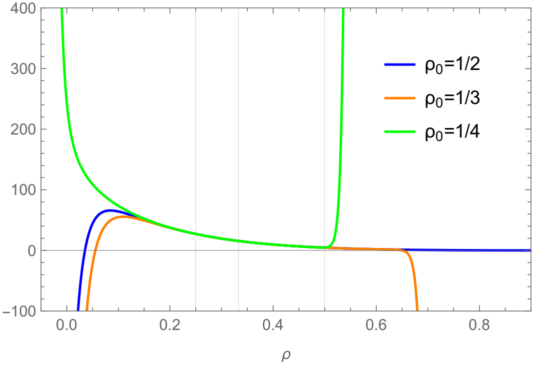

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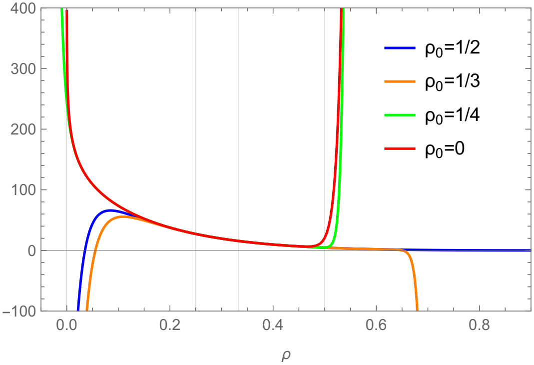

Results

$$\Gamma(B) = \Gamma_3 + \Gamma_5 \frac{\langle \mathcal{O}_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_b^3} + \cdots + 16\pi^2 \left(\Gamma_6 \frac{\langle \tilde{\mathcal{O}}_6 \rangle}{m_b^3} + \Gamma_7 \frac{\langle \tilde{\mathcal{O}}_7 \rangle}{m_b^4} + \ldots \right).$$

$$\Gamma_3 = \frac{G_f^2 m_b^5 |V_{bc}|^2}{192\pi^3} \left[C_1^2(\mu) G_{11} + C_1(\mu) C_2(\mu) G_{12} + C_2^2(\mu) G_{22} \right]$$

- with

$$G_{ij} = G_{ij}^{(0)} + \frac{\alpha_s}{\pi} G_{ij}^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 G_{ij}^{(2)}.$$

- All $G_{ij}^{(n)}$ as functions of the mass ratio $\rho = m_c/m_b$ expanded around value ρ_0 and colour factors.

decay channel	LO	NLO	NNLO
$b \to c\bar{u}d$	yes	yes	yes
$b \to u\bar{u}d$	yes	yes	yes
$b \to u\bar{c}s$	yes	yes	yes
$b \to c\bar{c}s$	yes	yes	yes

On-shell mass schemes [Preliminary!]

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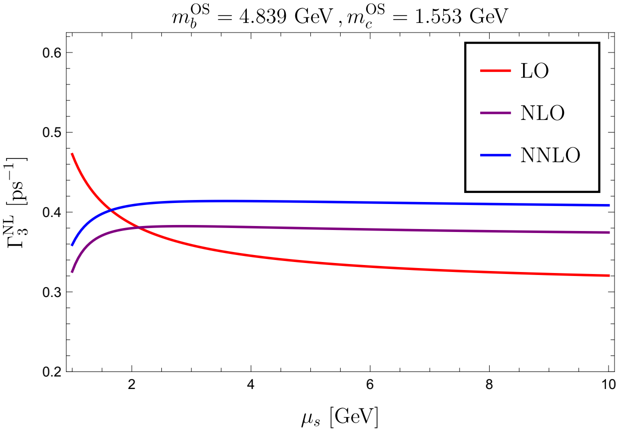
- good behaviour on renormalization scheme.
- QCD corrections sizable but not very large.

BUT:

- pole mass are not well defined!
- strong dependence on the quark masses

$$\Gamma_3 = \frac{G_f^2 m_b^5 |V_{bc}|^2}{192\pi^3}[\dots]$$



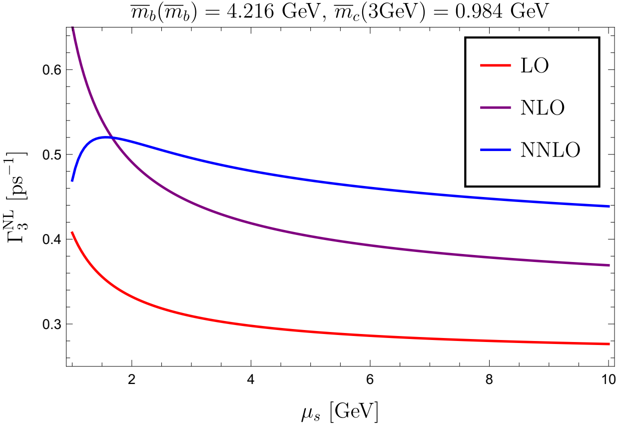
[Egner, Fael, Lenz, Piscopo, Rusov, Schönwald, Steinhauser (in preparation)]

$\overline{\mathrm{MS}}$ mass schemes [Preliminary!]

- from on-shell masses to $\overline{\mathrm{MS}}$ masses:

$$m_b^{\mathrm{OS}} = \overline{m}_b(\mu_b)(1 + \alpha_s^1\delta_{m_b}^{(1)} + \alpha_s^2\delta_{m_b}^{(2)})$$
$$m_c^{\mathrm{OS}} = \overline{m}_c(\mu_c)(1 + \alpha_s^1\delta_{m_c}^{(1)} + \alpha_s^2\delta_{m_c}^{(2)})$$

- large dependence on renormalization scale
- large QCD corrections, at $\mu_s = m_b$:
  NLO: 41% of LO
  NNLO: 22% of LO



[Egner, Fael, Lenz, Piscopo, Rusov, Schönwald, Steinhauser (in preparation)]

Kinetic mass schemes [Preliminary!]

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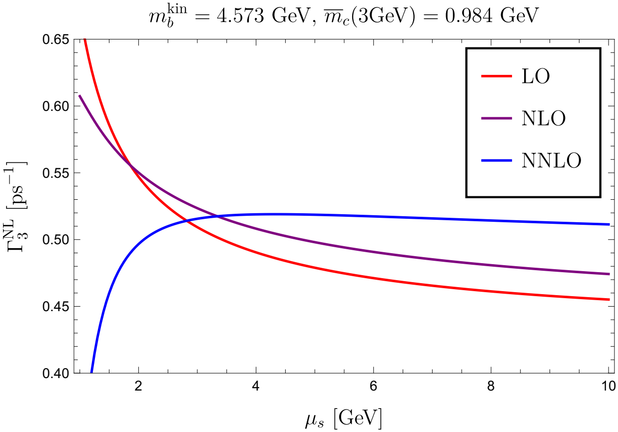
- from on-shell masses to $\overline{\mathrm{MS}}$ masses:

$$m_b^{\mathrm{OS}} = m_b^{\mathrm{kin}}(\mu^{\mathrm{cut}})(1 + \alpha_s^1 \delta_{m_b}^{(1)} + \alpha_s^2 \delta_{m_b}^{(2)})$$
$$m_c^{\mathrm{OS}} = \overline{m}_c(\mu_c)(1 + \alpha_s^1 \delta_{m_c}^{(1)} + \alpha_s^2 \delta_{m_c}^{(2)})$$

and

$$\mu_\pi^2(0) = \mu_\pi^2(\mu^{\mathrm{cut}}) - [\mu_\pi^2(\mu^{\mathrm{cut}})]_{\mathrm{pert}}$$
$$\rho_D^3(0) = \rho_D^3(\mu^{\mathrm{cut}}) - [\rho_D^3(\mu^{\mathrm{cut}})]_{\mathrm{pert}}$$

- NNLO result shows flat behaviour
  $\rightarrow$ small μ_s uncertainty
  $\rightarrow$ preferred mass scheme!



[Egner, Fael, Lenz, Piscopo, Rusov, Schönwald, Steinhauser (in preparation)]

B meson decay widths

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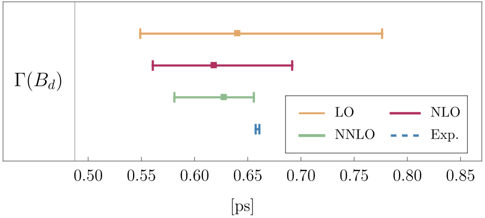
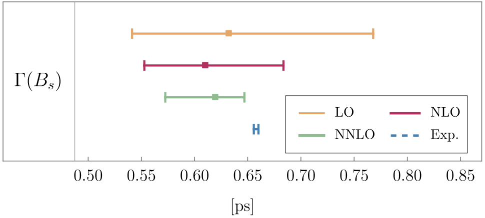
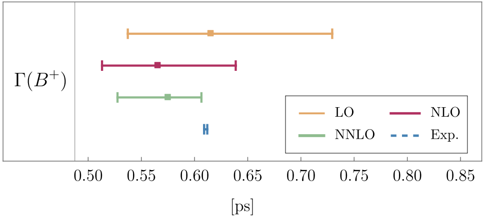
$\Gamma_5^{SL,(1)}$	[Alberti, Gambino, Nandi (2014)] [Mannel, Pivovarov, Rosenthal (2015)]	$\Gamma_5^{NL,(1)}$	[Mannel, Moreno, Pivovarov (2023)] [Mannel, Moreno, Pivovarov (2024)]
$\Gamma_6^{SL,(1)}$	[Mannel, Moreno, Pivovarov (2022)]	$\Gamma_6^{NL,(0)}$	[Lenz, Piscopo, Rusov (2020)] [Mannel, Moreno, Pivovarov (2020)]
$\tilde{\Gamma}_6^{SL,(1)}$	[Lenz, Rauh (2013)]	$\tilde{\Gamma}_6^{NL,(1)}$	[Beneke, Buchalla, Greub, Lenz, Nierste (2002)] [Franco, Lubicz, Mescia, Tarantino (2002)]
$\Gamma_7^{SL,(0)}$	[Dassinger, Mannel, Turczyk (2007)]		
$\Gamma_8^{SL,(0)}$	[Mannel, Turczyk, Uraltsev (2010)]		

B meson decay widths [Preliminary!]







[Egner, Fael, Lenz, Piscopo, Rusov, Schönwald, Steinhauser (in preparation)]

Backup

Evanescent operators

- Fierz symmetry can be restored order by order in perturbation theory by choosing the correct evanescent operators [Buras, Weisz (1990)], [Herrlich, Nierste (1994)]

$$
\begin{aligned}
E_1^{(1),q_1q_2q_3} &= (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3} P_L b^\beta)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3} P_L q_3^\alpha) - (16 - 4\epsilon + A_1\epsilon^2) O_1^{q_1q_2q_3}, \\
E_2^{(1),q_1q_2q_3} &= (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3} P_L b^\alpha)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3} P_L q_3^\beta) - (16 - 4\epsilon + A_1\epsilon^2) O_2^{q_1q_2q_3}, \\
E_1^{(2),q_1q_2q_3} &= (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L b^\beta)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L q_3^\alpha) - (256 - 224\epsilon + B_1\epsilon^2) O_1^{q_1q_2q_3}, \\
E_2^{(2),q_1q_2q_3} &= (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L b^\alpha)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L q_3^\beta) - (256 - 224\epsilon + B_2\epsilon^2) O_2^{q_1q_2q_3}
\end{aligned}
$$

- Evanescent operators and physical operators mix under renormalization:

$$
\begin{pmatrix} O_P \\ O_E \end{pmatrix} = Z \begin{pmatrix} O_{P,B} \\ O_{E,B} \end{pmatrix}
$$

$\rightarrow$ Evanescent operators contribute to physical result at higher orders.

- $\mathcal{O}(\epsilon)$ terms $\leftrightarrow$ Fierz symmetries at NLO
- $\mathcal{O}(\epsilon^2)$ terms $\leftrightarrow$ Fierz symmetries at NNLO

Evanescent operators

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- Fix $\{A_1, B_1, B_2\}$ by imposing a symmetric anomalous dimension matrix γ [Buras, Weisz (1990)]

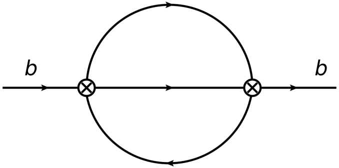
$$\mu \frac{\mathrm{d}C_i}{\mathrm{d}\mu} = \gamma_{ij} C_j, \qquad \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}, \quad \text{with } \gamma_{11} = \gamma_{22}, \quad \gamma_{12} = \gamma_{21}$$

- This condition yields

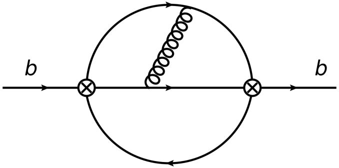
$$A_2 = -4, \qquad B_1 = -\frac{45936}{125}, \qquad B_2 = -\frac{115056}{115}$$

$$E_1^{(1),q_1q_2q_3} = (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3} P_L b^\beta)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3} P_L q_3^\alpha) - (16 - 4\epsilon - 4\epsilon^2) O_1^{q_1q_2q_3},$$
$$E_2^{(1),q_1q_2q_3} = (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3} P_L b^\alpha)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3} P_L q_3^\beta) - (16 - 4\epsilon - 4\epsilon^2) O_2^{q_1q_2q_3},$$
$$E_1^{(2),q_1q_2q_3} = (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L b^\beta)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L q_3^\alpha) - (256 - 224\epsilon - \frac{45936}{125}\epsilon^2) O_1^{q_1q_2q_3},$$
$$E_2^{(2),q_1q_2q_3} = (\bar{q}_1^\alpha \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L b^\alpha)(\bar{q}_2^\beta \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L q_3^\beta) - (256 - 224\epsilon - \frac{115056}{115}\epsilon^2) O_2^{q_1q_2q_3}$$

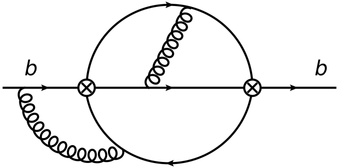
Evanescent operators



$$\{O_1, O_2, E_1^{(1)}, E_2^{(1)}, E_1^{(2)}, E_2^{(2)}\}$$
$$\times\{O_1, O_2, E_1^{(1)}, E_2^{(1)}, E_1^{(2)}, E_2^{(2)}\}$$



$$\{O_1, O_2, E_1^{(1)}, E_2^{(1)}\}$$
$$\times \{O_1, O_2, E_1^{(1)}, E_2^{(1)}\}$$



$$\{O_1, O_2\} \times \{O_1, O_2\}$$

Master integrals: Expand and Match

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Which expansion for which expansion point?

- Depends on singular points of the $A(\epsilon, \rho)$:
  - $b \to c\bar{u}d$ & $b \to u\bar{c}s$: $\{0, 1/3, 1\}$
  - $b \to c\bar{c}s$: $\{0, 1/4, 1/2, 1\}$
  - $b \to u\bar{u}d$: $\{0, 1/2, 1\}$

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\vec{I} = A(\epsilon, \rho) \cdot \vec{I}$$

$$= \frac{A'(\epsilon, \rho) \cdot \vec{I}}{\rho - 1/3} + \frac{A''(\epsilon, \rho) \cdot \vec{I}}{\rho - 1} + \dots$$

$$I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+|\epsilon_{\min}|} \sum_{n=n_{\min}}^{n_{\max}} c[i, j, m, n]\, \epsilon^j (\rho - \rho_0)^{n/2} \log^m(\rho - \rho_0)$$

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- Expansion around $\{2, 4\}$-particle threshold

$$\to \quad I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+|\epsilon_{\min}|} \sum_{n=n_{\min}}^{n_{\max}} c[i, j, m, n]\, \epsilon^j (\rho - \rho_0)^{n/2} \log^m(\rho - \rho_0)$$

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- Expansion around $\rho_0 \in \{0, 1\}$ or 3-particle threshold

$$\to \quad I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+|\epsilon_{\min}|} \sum_{n=0}^{n_{\max}} c[i,j,m,n]\, \epsilon^j (\rho - \rho_0)^{n} \log^m(\rho - \rho_0)$$

Master integrals: Expand and Match

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  - $b \to c\bar{u}d$ & $b \to u\bar{c}s$: $\{0, 1/3, 1\}$
  - $b \to c\bar{c}s$: $\{0, 1/4, 1/2, 1\}$
  - $b \to u\bar{u}d$: $\{0, 1/2, 1\}$

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\vec{I} = A(\epsilon, \rho) \cdot \vec{I} = \frac{A'(\epsilon, \rho) \cdot \vec{I}}{\rho - 1/3} + \frac{A''(\epsilon, \rho) \cdot \vec{I}}{\rho - 1} + \ldots$$

$$I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+|\epsilon_{\min}|} \sum_{n=n_{\min}}^{n_{\max}} c[i, j, m, n]\, \epsilon^j (\rho - \rho_0)^{n/2} \log^m(\rho - \rho_0)$$

- Expansion around regular point

$$\to \quad I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{n=0}^{n_{\max}} c[i, j, 0, n]\, \epsilon^j (\rho - \rho_0)^n$$