Modelling Finite Volume Effects in the QCD Phase Diagram

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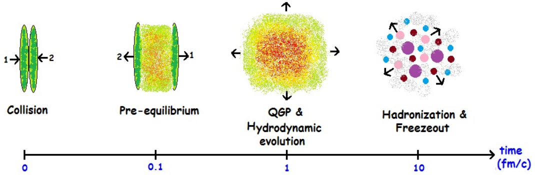
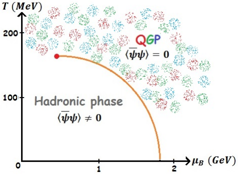
Introduction and Motivation

- Quark-Gluon Plasma (QGP) is a deconfined, chirally symmetric phase with quarks and gluons as the effective degrees of freedom. It exists at high temperatures and/or high baryon densities, e.g., in heavy-ion collision experiments (RHIC, LHC).
- Rapidly evolving finite-sized system with dimension of few fermi comparable to the typical strong interaction scale. Future experiments (NICA, FAIR) focussed towards smaller QGP size (large μ_B) where the geometry and the boundary condition are important for theoretical understanding and realistic analysis.





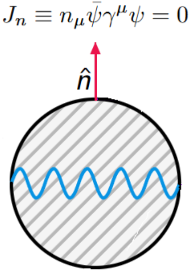
- We study the effect of the finite size of QGP on the QCD phase diagram using the NJL model in spherical geometry and MIT boundary condition.

MIT boundary condition

- Finite volume study requires a suitable boundary condition.
- Complicated dynamics: QCD medium evolution is time-dependent and shape differs event by event.
- Simplification: Spherical geometry containing deconfined QGP and MIT boundary condition on the surface.
- MIT boundary condition: normal component ($\hat{n}$) of quark current vanishes at the boundary of the spherical surface,

$$(i\hat{n}_\mu \gamma^\mu - 1)\psi(t, r, \theta, \phi)|_{r=R} = 0$$

Allowed momentum modes obtained by solving Dirac equation in curved space: $[i\gamma^\mu(\partial_\mu + \Gamma_\mu) - M]\psi_j = 0$



- Previous studies (MIT at $\mu_B = 0$):
  Chernodub *et al.*, JHEP 01, 136 (2017) (fermions in rotating cylinder)
  Zhang *et al.*, Phys. Rev. D 101, 043006 (2020) (NJL in a rotating sphere)

NJL model

- Nambu–Jona-Lasinio (NJL): Effective model to study spontaneous breaking of chiral symmetry in QCD $(SU(2)_L \times SU(2)_R \rightarrow SU(2)_V)$

$$\mathcal{L}_{NJL} = \bar{\psi}(i\gamma_\mu\partial^\mu - \hat{m})\psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\bar{\tau}\psi)^2\right] \qquad (N_f = 2, N_c = 3)$$

- In the mean-field approximation, the four-fermion interaction term generates the dress quark mass: $M = m + \sigma$ where $\sigma = -2G\langle\bar{\psi}\psi\rangle$

$$\langle\bar{\psi}\psi\rangle = -\int \frac{d^4p}{(2\pi)^4}\ Tr\left\{\frac{1}{\not{p} - M}\right\}$$

$$= -2\, N_c\, N_f \int \frac{d^3p}{(2\pi)^3}\frac{M}{E}\left(1 - \frac{1}{1 + e^{(E-\mu)/T}} - \frac{1}{1 + e^{(E+\mu)/T}}\right)$$

- Minimize effective potential to get M: $\frac{\partial\Omega(\sigma)}{\partial\sigma} = 0$

$$\Omega(\sigma) = \frac{\sigma^2}{4G} - 2\, N_c\, N_f \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3}\left[E + T\log\left(1 + e^{-(E-\mu)/T}\right) + T\log\left(1 + e^{-(E+\mu)/T}\right)\right]$$

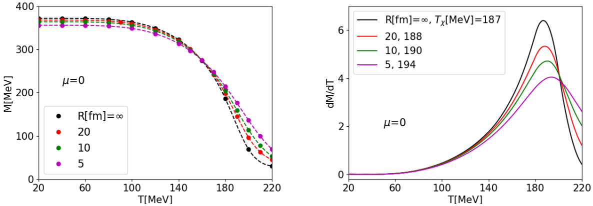
- Proper time regularization for vacuum term:

$$\frac{1}{p^2 + M^2} \longrightarrow \int_{\tau_{UV} = 1/\Lambda_{UV}^2}^{\infty} d\tau\, e^{-\tau\left(p^2 + M^2\right)},$$

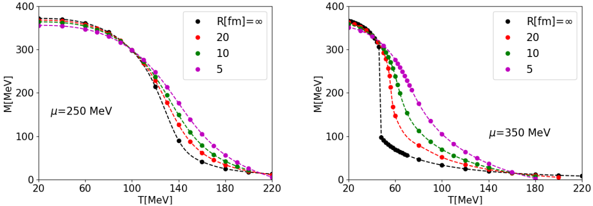
Parameter set: $m = 15$ MeV, $G = 17.2$ GeV^{-2}, $\Lambda_{UV} = 645$ MeV

Chiral transition at finite volume

- At $\mu = 0$, the chiral symmetry restoration is a crossover (smooth transition between confined and deconfined phases).
- Chiral crossover temperature (T_χ) estimated from the peak position of the derivative (dM/dT), small shift in the peak position with volume.

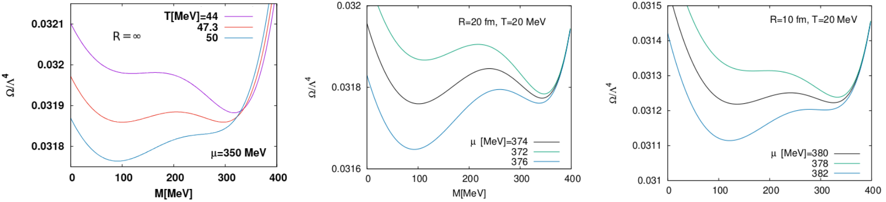


- For increasing μ, the crossover transitions become stiffer for larger volumes.



Effective potential for first-order phase transition

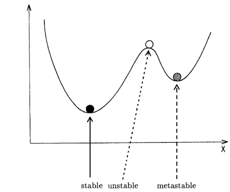
- Discontinuity indicates first-order phase transition.
- Corresponding mean-field effective potential show two-minima structure $\Longrightarrow$ equal depth signals phase transition (hadron-QGP).
- Phase transition at higher μ for smaller volumes, weakening of two-well structure and shrinking of coexistence region.

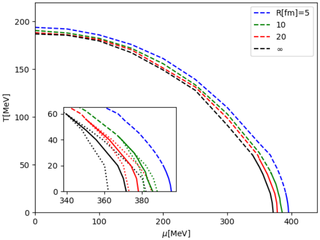


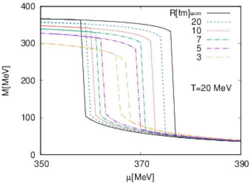
- Two-minima structure of (un)equal depths $\Longrightarrow$ Coexistence region (local minimum $\rightarrow$ metastable phase; global minimum $\rightarrow$ stable phase).

QCD phase diagram for finite-sized QGP

- At $T = 0$, phase transition (discontinuity in order parameter) for all system sizes.
- Shift of the phase transition (solid lines) towards larger μ is significant as volume is reduced.
- Shift of the crossover (dashed lines) for small μ is relatively milder.
- Coexistence region shrinks for smaller volumes and shifted towards lower T (dotted lines indicate boundary of coexistence regions $\rightarrow$ spinodal instability).



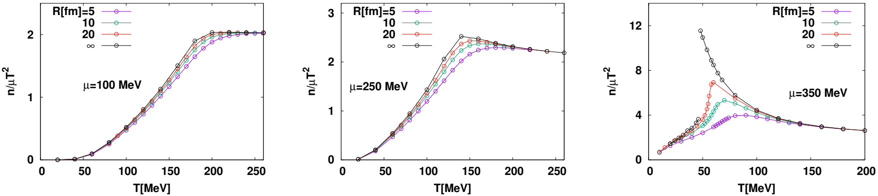




Net quark number density

- Baryon number density (n_B) and its susceptibilities are related to experimentally determined quantities (e.g., cumulants ratio), especially important at high μ_B near the first-order line and its end-point.
- Quark number density ($n = 3n_B$) shows volume dependence in the crossover/transition region.

$$n = \frac{\partial P}{\partial \mu} = 2N_c N_f \int \frac{d^3 p}{(2\pi)^3} \left(\frac{1}{1 + e^{(E-\mu)/T}} - \frac{1}{1 + e^{(E+\mu)/T}} \right)$$

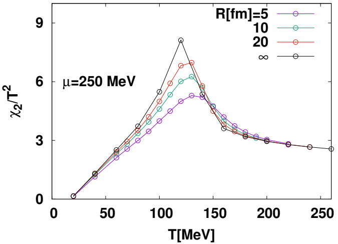


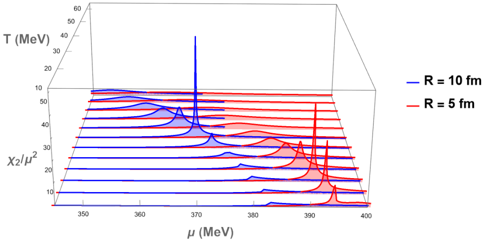
- Rise of n milder at low μ, sharper crossover for high μ for larger volumes, eventually phase (discontinuous) transition.
- Asymptotic limit $\sim \mu T^2$ (free theory) reached at sufficiently large T.

Quark number susceptibilities (χ_2)

- Non-linear susceptibilities of conserved quantities (e.g., baryon number) are sensitive for critical point search.
- Fluctuations in susceptibilities (related to cumulants) obtained by higher-order derivatives of pressure w.r.t. μ.

$$\chi_k = \frac{\partial^k P}{\partial \mu^k} = \frac{\partial^{k-1} n}{\partial \mu^{k-1}}$$

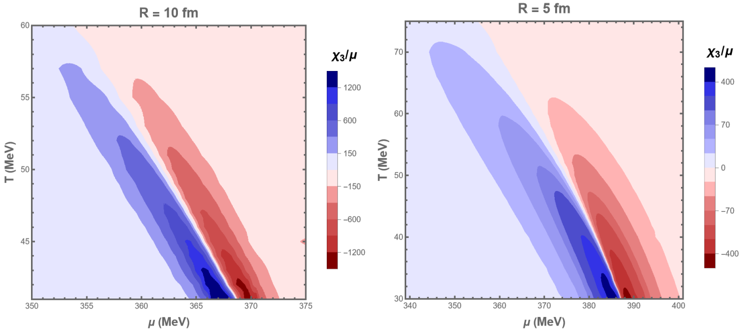




- Peak height decreases (also slightly shifted) with volume in the transition region, smoother crossover for smaller volumes.
- χ_2 peak is highest at the critical point. [$T_c \approx 42$ MeV (10 fm), $T_c \approx 20$ MeV (5 fm)], shifts T_c to lower values as volume decreases.

Quark number susceptibilities (χ_3)

- Higher order susceptibilities are crucial probes for the critical point.
- Third order susceptibility changes sign along the transition/crossover line, peaks and dips become sharper near the critical point.



- Difference in fluctuations (peak/dip for χ_3) compared to χ_2 peak is greater as volume is increased from $R = 5$ fm to $R = 10$ fm.
- Peak structure modifies due to shift in the first order endpoint with volume.

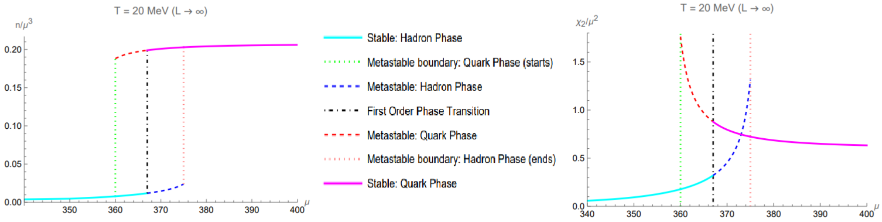
Conclusion

1. Finite volume effects on the QCD chiral phase transition is studied using spherical QGP with MIT b.c. in the mean-field NJL model framework.
2. The effect of the finite size of the QGP on the experimental observables is very crucial for low energy (small QGP system) in future experiments - especially looking for a critical point and first-order transition.
3. The phase transition line (and critical point) shows a substantial shift with volume. Reduction in the co-existence region is observed.
4. Number density and its derivatives (susceptibilities) are sensitive to QGP system size.
5. MIT b.c. shows more volume effect compared to other boundary conditions (e.g., APBC, PBC, IR cutoff)

Thank you !

Future Work

- Effect of finite volume on the dynamics of first-order QCD phase transition.
- Existence of stable quark phase (bubble nucleation) inside the metastable hadronic phase. How the finite size affect the nucleation rate? How fast is the nucleation rate compared to the QGP expansion rate? (witness metastability)
- Or does the first-order QCD transition proceed through spinodal decomposition? (ignores metastability, highly unstable and a fluctuation causes instant phase transition).

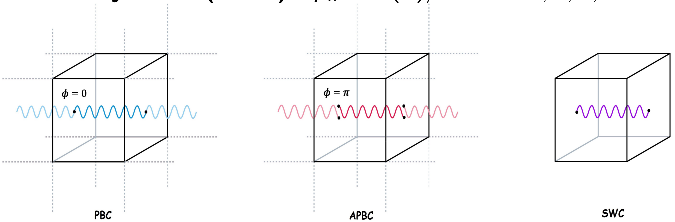


Backup: Comparison with other boundary conditions

- Cubic geometry with different boundary conditions.

  Wang *et al.*, arXiv : 1802.00258, 1806.05315

  - **Periodic (PBC)** : $p_n = 2\pi(n)/L$, $n = 0, \pm1, \pm2, ...$
  - **Anti-periodic (APBC)** : $p_n = 2\pi(n + 1/2)/L$, $n = 0, \pm1, \pm2, ...$
  - **Stationary wave (SWC)** : $p_n = \pi(n)/L$, $n = 1, 2, 3, ...$



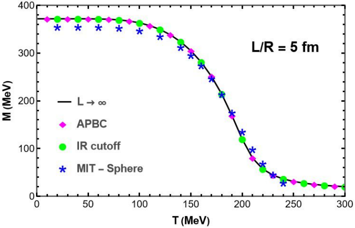
- Lower momentum (IR) cut-off $\implies p_{min} = \dfrac{\pi}{L} = \lambda$

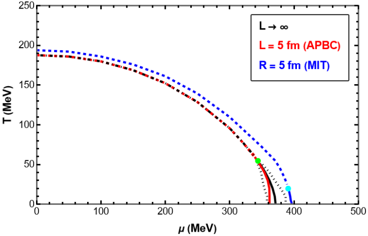
  Bhattacharyya *et al.*, Phys. Rev. D 87, 054009 (2013)

$$M = m_0 + \frac{6GM}{\pi^2}\int_{\lambda}^{\Lambda_{UV}} dp \frac{p^2}{\sqrt{p^2 + M^2}}\left(1 - \frac{1}{1 + e^{(E-\mu)/T}} - \frac{1}{1 + e^{(E+\mu)/T}}\right)$$

Backup: Comparison with other boundary conditions

- APBC/PBC and IR momentum cutoff for a $L = 5$ fm cube show very small deviation from infinite volume.
- APBC/PBC show minimal volume effect compared to spherical MIT bc ($V_{sphere} \approx 4 \times V_{cube}$), suitable for mimicking infinite volume on lattice simulation.





- The phase diagram for APBC/PBC ($R = 5$ fm) is very close to the infinite volume in the crossover region, phase transition line shows a small deviation from the infinite volume but still within the co-existence region.
- The MIT bc show a substantial shift from infinite volume across all values of T, also the coexistence region shrinks.

Backup: Comparison with other boundary conditions

- Renormalization invariant quantity (y) for (approximate) comparison between lattice and NJL model.

$$y = m\,(\bar{\psi}\psi|_L - \bar{\psi}\psi|_\infty)$$

- Mean-field NJL model qualitatively captures the essential features of the volume dependence of the fireball.