Extended Relaxation Time approach (ERTA) to Relativistic non-resistive Magnetohydrodynamics

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Based on *S. Singh, M. Kurian, V. Chandra*, ***Phy. Rev. D 110, 014004 (2024)***
Work in Progress with *S. Bhadury, M. Kurian* and *V. Chandra*

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Heavy Ion Collision

- The QGP phase can be modelled in the framework of relativistic hydrodynamics which is a relativisitc generalization of fluid mechanics.

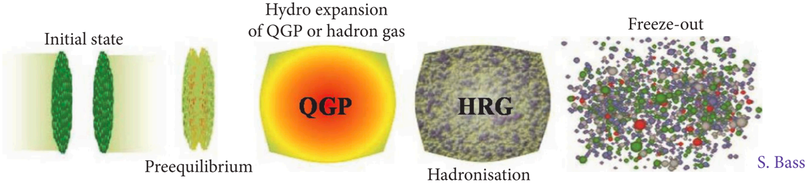


Figure: Various stages of Ultrarelativistic heavy-ion collisions

Equations of motion

- Using the energy momentum and number current conservation laws, the fluid dynamical equations of motion are given by:

$$\dot{\varepsilon} + (\varepsilon + P)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0, \tag{1}$$
$$(\epsilon + P)\dot{u}^{\mu} - \nabla^{\mu}P + \Delta^{\mu}_{\nu}\partial_{\gamma}\pi^{\gamma\nu} = 0, \tag{2}$$
$$\dot{n} + n\theta + \partial_{\mu}n^{\mu} = 0 \tag{3}$$

- ϵ, n and P are related to each other via the equation of state.
- The evolution equations for $\pi^{\mu\nu}$ and n^{μ} is needed which can be derived from entropy-current analysis, requiring $\partial_{\mu}S^{\mu} \geq 0$.
- The evolution equation for $\pi^{\mu\nu}$ that we need to ensure the second law of thermodynamics is guaranteed is:

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_{\pi}} = 2\beta_{\pi}\sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + 2\pi_{\gamma}^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_{\gamma}^{\langle\mu}\sigma^{\nu\rangle\gamma}$$
$$- \tau_{\pi n}n^{\langle\mu}\dot{u}^{\nu\rangle} + \lambda_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha + l_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle}. \tag{4}$$

- The transport coefficients must be determined from a microscopic theory.

Kinetic Theory Approach

- With $f_0(x,p)$ being the local equilibrium distribution, the deviation of equilibrium can be written as $\delta f = f - f_0$.

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int \mathrm{dP} p^\alpha p^\beta \delta f \tag{5}$$

- To get the form of δf, the evolution of $f(x,p)$ is needed via the Boltzmann equation.

$$p^\mu \partial_\mu f = \mathcal{C}[f] \tag{6}$$

- The collision term which encodes the details about various collisional processes happening in the system can be approximated using the relaxation time approximation (RTA) as:

$$\mathcal{C}[f] = -\frac{(u \cdot p)}{\tau_R}(f - f_0) \tag{7}$$

- The relaxation time τ_R is momentum independent for RTA since a momentum dependent $\tau_R(p)$ leads to violation of conservation laws with Landau matching condition:

$$\int \mathrm{dP}\mathcal{C}[f] \neq 0 \neq \int \mathrm{dP} p^\mu \mathcal{C}[f] \tag{8}$$

Extended relaxation time

- The collision kernel for extended relaxation time reads [1]:

$$C[f] = -\frac{(u \cdot p)}{\tau_R(x,p)}(f - f_0^*) \tag{9}$$

Where f_0^* is the equilibrium distribution function in the "thermodynamic" frame.

$$f_0^* = e^{-\beta^*(u^* \cdot p) + \alpha^*} \tag{10}$$

- Where $u^{*\mu}$,β^* and α^* are related with the usual variables by:

$$u^{\mu *} = u^\mu + \delta u^\mu, \qquad \mu^* = \mu + \delta\mu, \qquad T^* = T + \delta T, \tag{11}$$

δu^μ, $\delta\mu$ and δT must be determined using the matching conditions, $\epsilon = \epsilon_0$, $n = n_0$ and the Landau frame condition $u_\mu T^{\mu\nu} = \epsilon u^\nu$.

- This lets us use a momentum dependent relaxation time to determine the transport coefficients. The form of momentum dependent $\tau_R(x,p)$ is taken as:

$$\tau_R(x,p) = \frac{\kappa}{T}\left(\frac{u \cdot p}{T}\right)^\ell \tag{12}$$

[1] D. Dash, S. Bhadury, S. Jaiswal, A. Jaiswal, ***Physics Letters B, 831 (2022)***

Shear stress evolution within ERTA

- The δf with the necessary counter terms inside $u^{*\mu}$, β^* and μ^* leads to:

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \tau_{\pi n} n^{\langle\mu}\dot{u}^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu}\nabla^{\nu\rangle}\alpha + l_{\pi n}\nabla^{\langle\mu} n^{\nu\rangle}. \quad (13)$$

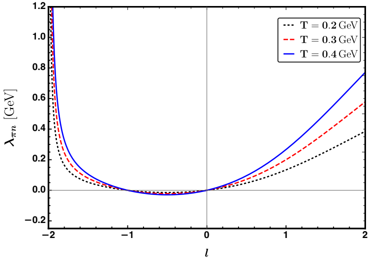
- This evolution equation is second order in the gradient expansion of the hydrodynamic fields
- One of the central results of the current work is that in the massless MB limit, **the evolution of the number diffusion n^μ is coupled to the evolution of the shear stress tensor** whereas they are decoupled in the RTA limit [2].
- We also have the number evolution equation given by:

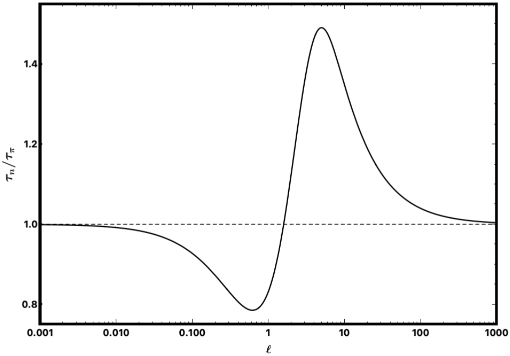
$$\dot{n}^{\langle\mu\rangle} + \frac{n^\mu}{\tau_n} = \beta_V \nabla^\mu \alpha - \lambda_{V\pi}\pi^{\mu\lambda}\nabla_\lambda \alpha - \tau_{V\pi}\pi_\lambda^\mu \dot{u}^\lambda - \delta_{VV} n^\mu \theta + l_{V\pi}\Delta_\alpha^\mu \partial_k \pi^{\alpha k} - \lambda_{VV}\sigma_\lambda^\mu n^\lambda - \lambda_\omega \omega_\lambda^\mu n^\lambda. \quad (14)$$

- All these transport coefficients depends on the momentum dependent parameter ℓ via the thermodynamic integrals.

[2] A. Jaiswal, B. Friman, K. Redlich, ***Phy Lett B 751 (2015)***

Some results..





Plot for one of these transport coefficients against ℓ as an example to show their dependence on the momentum dependence parameter ℓ. The ratio of relaxation times for number diffusion mode to shear mode is also shown.

Exact calculations of transport coefficients

- Some recent studies [3],[4] analytically extracted a full set of eigenvalues and eigenfunctions of the relativistic linearized Boltzman collision operator for $\lambda\phi^4$ theory.

$$\hat{L}\phi_k = \frac{g}{2}\int dK' dP dP' f_{0k'}(2\pi)^5 \delta^{(4)}(k + k' - p - p')(\phi_p + \phi_{p'} - \phi_k - \phi_{k'}). \tag{15}$$

- Where the eigenfunctions and their eigenvalues are given by:

$$\hat{L}L_{n\mathrm{k}}^{(2m+1)} k^{\langle\mu_1} \dots k^{\mu_\ell\rangle} = -\frac{g\mathcal{M}}{2}\left[\frac{n+m-1}{n+m+1} + \delta_{\ell 0}\delta_{n0}\right] L_{n\mathrm{k}}^{(2m+1)} k^{\langle\mu_1} \dots k^{\mu_m\rangle}, \tag{16}$$

- Expanding ϕ_k in terms of these eigenfunctions and keeping the terms with zero eigenvalues leads to the collision kernel being:

$$\hat{L}\phi_k = -\frac{g\mathcal{M}}{2}\left[\phi_k - c_0 - c_1 L_{1k}^{(1)} - c_0^{\mu} k_{<\mu>}\right] \tag{17}$$

- Recovering the RTA limit from the exact theory leads to the form of momentum-dependent relaxation time being:

$$\tau_R(p) = \frac{2(u\cdot p)}{g\mathcal{M}}$$

Which implies $\ell = 1$ and $\kappa = 4\pi^2/(ge^{\alpha})$ in the corresponding ERTA framework.

[3] Gabriel S. Rocha, Caio V.P. de Brito, and Gabriel S. Denicol, ***Phys. Rev. D 108, 036017 (2023)***

[4] Gabriel S. Rocha, Gabriel S. Denicol, and Jorge Noronha, ***Phys. Rev. Lett. 127, 042301 (2021)***

Comparison with self interacting $\lambda\phi^4$ theory

Coefficients	RTA results ($l = 0$)	ERTA results ($l = 1$)	$\lambda\phi^4$ results (exact)
τ_π	τ_c	$\frac{24d_g}{gn_0\beta^2}$	$\frac{72}{gn_0\beta^2}$
η	$\frac{4P\tau_c}{5}$	$\frac{16d_g}{g\beta^3}$	$\frac{48}{g\beta^3}$
κ	$\frac{n_0\tau_c}{12}$	$\frac{d_g}{g\beta^2}$	$\frac{3}{g\beta^2}$
$\delta_{\pi\pi}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
$\tau_{\pi\pi}$	$\frac{10}{7}$	2	2
$l_{\pi n}$	0	$-\frac{4}{3\beta}$	$-\frac{4}{3\beta}$
$\tau_{\pi n}$	0	$-\frac{16}{3\beta}$	$-\frac{16}{3\beta}$
$\lambda_{\pi n}$	0	$\frac{2}{3\beta}$	$\frac{5}{6\beta}$

Table: Comparison of the ERTA coefficients with exact results from $\lambda\phi^4$ theory for shear coefficients.

Comparison with self interacting $\lambda\phi^4$ theory.

Coefficients	RTA results ($l = 0$)	ERTA results ($l = 1$)	$\lambda\phi^4$ results (exact)
τ_n	τ_c	$\frac{20d_g}{gn_0\beta^2}$	$\frac{60}{gn_0\beta^2}$
τ_n/τ_π	1	$5/6$	$5/6$
β_v	$\frac{n_0}{12}$	$\frac{n_0}{20}$	$\frac{n_0}{20}$
$2\eta_0\lambda_{V\pi} + \kappa\lambda_{VV}$	$\frac{3n_0\tau_c}{20}$	$\frac{19d_gT^2}{5g}$	$\frac{57T^2}{5g}$
$\tau_{V\pi} + l_{V\pi}$	0	$\frac{\beta}{20}$	$\frac{\beta}{20}$
$l_{V\pi}$	0	$\frac{\beta}{20}$	$\frac{\beta}{40}$
δ_{VV}	1	1	1
λ_ω	-1	-1	-1

Table: Comparison of the ERTA coefficients with exact results from $\lambda\phi^4$ theory for number diffusion coefficients.

Transport coefficients in MHD

- In the non-resistive limit, $E^\mu \to 0$ and $F^{\mu\nu} \to B^{\mu\nu}$
- The Boltzmann equation in the presence of the external magnetic fields and using the extended relaxation time is given by:

$$p^\mu \partial_\mu f - qB^{\sigma\nu} p_\nu \frac{\partial f}{\partial p^\sigma} = -\frac{u \cdot p}{\tau_R(x,p)}(f - f_0^*) \tag{18}$$

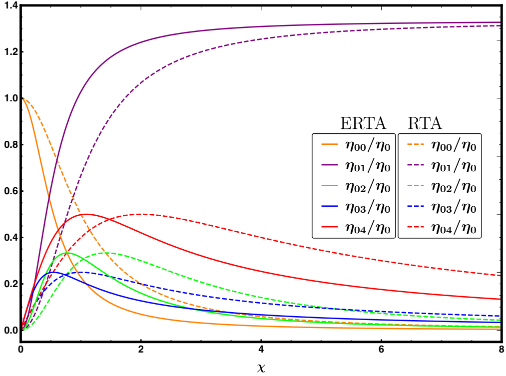
- Using the δf from above, the second order shear evolution equation is given by:

$$\begin{aligned}\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} &= 2\beta_\pi \sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} \\ &- \tau_{\pi n} n^{\langle\mu}\dot{u}^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu}\nabla^{\nu\rangle}\alpha + l_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \delta_{\pi B}\Delta^{\mu\nu}_{\eta\beta} qBb^{\gamma\eta}g^{\beta\rho}\pi_{\gamma\rho} \\ &- qB\tau_{\pi nB}\dot{u}^{\langle\mu}b^{\nu\rangle\sigma}n_\sigma - qB\lambda_{\pi nB} n_\sigma b^{\sigma\langle\mu}\nabla^{\nu\rangle}\alpha - q\tau_0\delta_{\pi nB}\nabla^{\langle\mu}\left(B^{\nu\rangle\sigma}n_\sigma\right)\end{aligned} \tag{19}$$

- With a finite magnetic field, the shear viscosity splits into five components:

$$\begin{aligned}\pi^{\mu\nu} = \Big[&2\eta_{00}\left(\Delta^{\mu\alpha}\Delta^{\nu\beta}\right) + \eta_{01}\left(\Delta^{\mu\nu} - \frac{3}{2}\Xi^{\mu\nu}\right)\left(\Delta^{\alpha\beta} - \frac{3}{2}\Xi^{\alpha\beta}\right) - 2\eta_{02}\left(\Xi^{\mu\alpha}b^\nu b^\beta\right. \\ &\left.+ \Xi^{\nu\alpha}b^\mu b^\beta\right) - 2\eta_{03}\left(\Xi^{\mu\alpha}b^{\nu\beta} + \Xi^{\nu\alpha}b^{\mu\beta}\right) + 2\eta_{04}\left(b^{\mu\alpha}b^\nu b^\beta + b^{\nu\alpha}b^\mu b^\beta\right)\Big]\sigma_{\alpha\beta}.\end{aligned} \tag{20}$$

The Navier Stokes limit (comparision with RTA MHD results)



Where, $\chi = \frac{qB\tau_0(x)}{T}$.
As we see, the momentum dependence of the ERTA has a significant effect on the various shear viscosity coefficients even in the first order.

Conclusion

- This study shows that there is a significant impact of momentum dependence of the relaxation time on the dynamics of the fluid in both with and without magnetic field.
- Incorporating these affects via the modified transport coefficients should lead to a more accurate simulation of the expanding fireball in heavy ion collisions.
- Further work can be done in recognizing the momentum dependence parameter ℓ for various theories.
- The Magnetohydrodynamics of ERTA can be studied in the resistive case as an extension of this work.

Thank you!