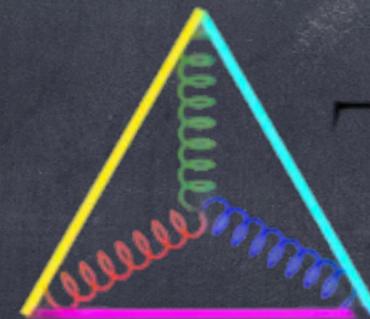# Diffusion coefficient matrix for multiple conserved charges: a Kubo approach

Sourav Dey

Based on: J. High Energ. Phys. 2024, 192 (2024)

In Collaboration with Dr. Amaresh Jaiswal, Dr. Hiranmaya Mishra

Motivation:

- Non-relativistical diffusion: Fick's Law $\quad \vec{J}_a = -K_{aa}\vec{\nabla}n_a(t,\vec{x})$

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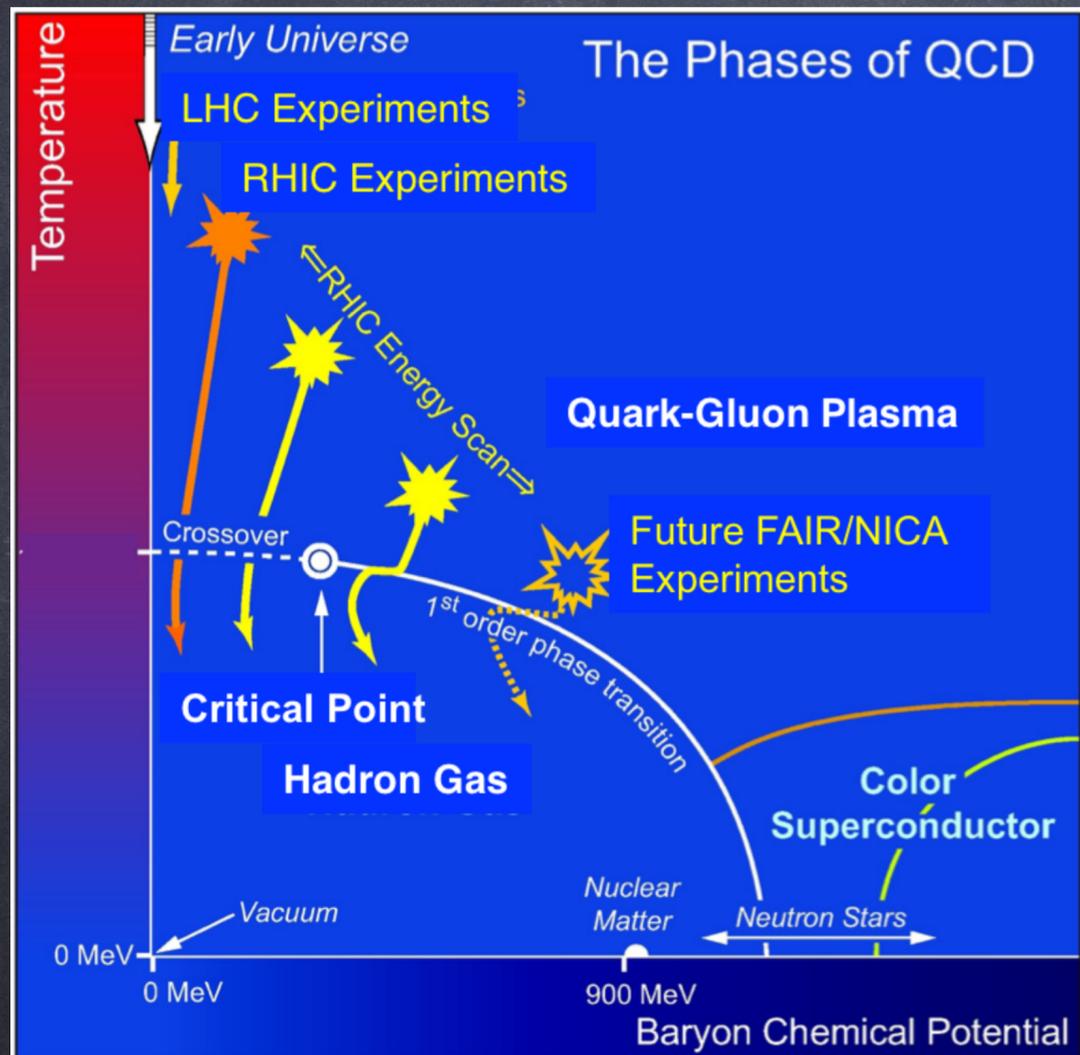
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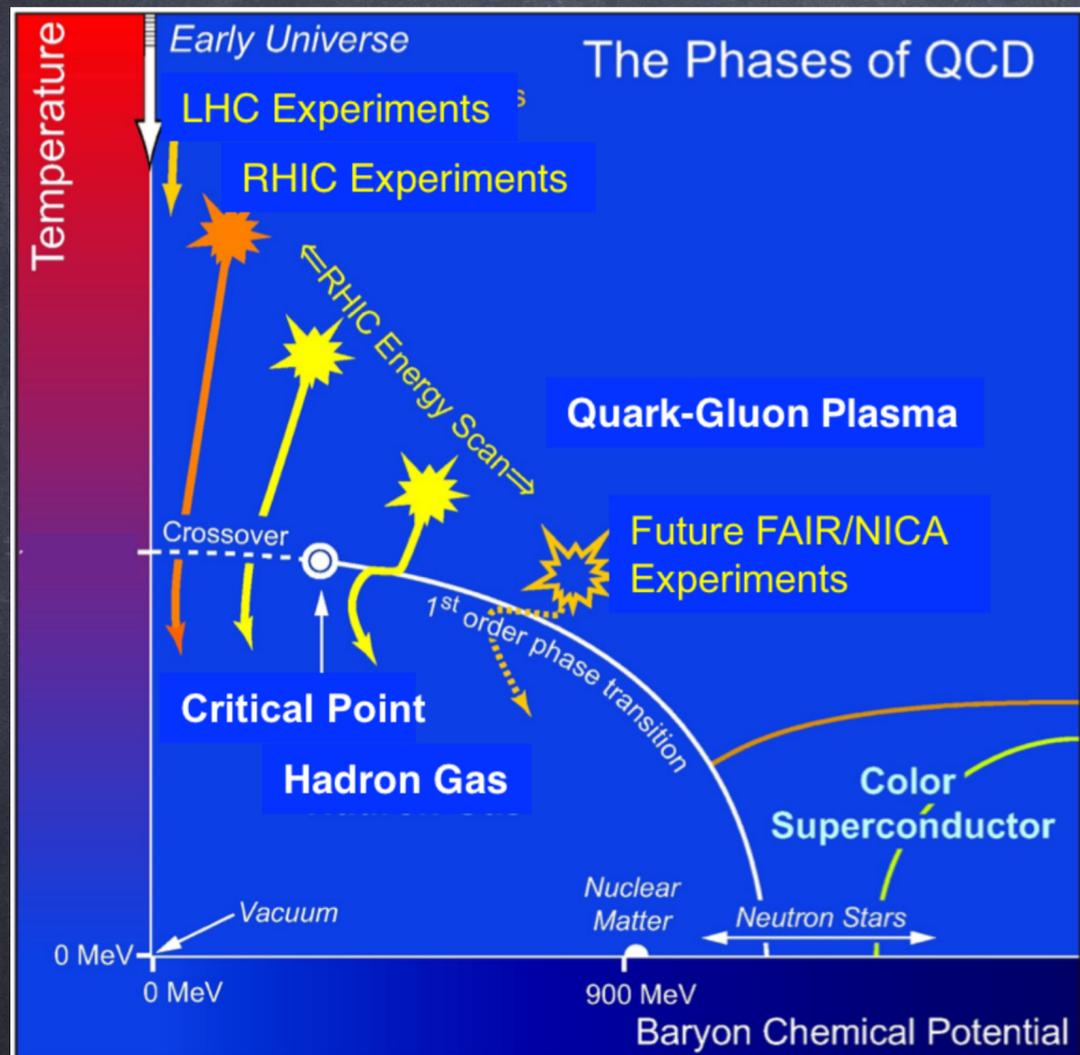- Relativistic Setup: $J_a^i = K_{aa}\nabla^i(\mu_a/T)$ $\longrightarrow$ $J_a^i = K_{ab}\nabla^i(\mu_b/T)$

M.M.Aggarwal et al. (STAR), arrive:1007.2613

B.Mohanty (STAR) J.Phys. G 38, 124023(2011)

J.T.Mitchell (PHENIX), Null. Phys. A904-905, 903c(2013)

Ref:Nayak,Tapan K.J.Phys,:Conf.Ser.1602(2020)012003

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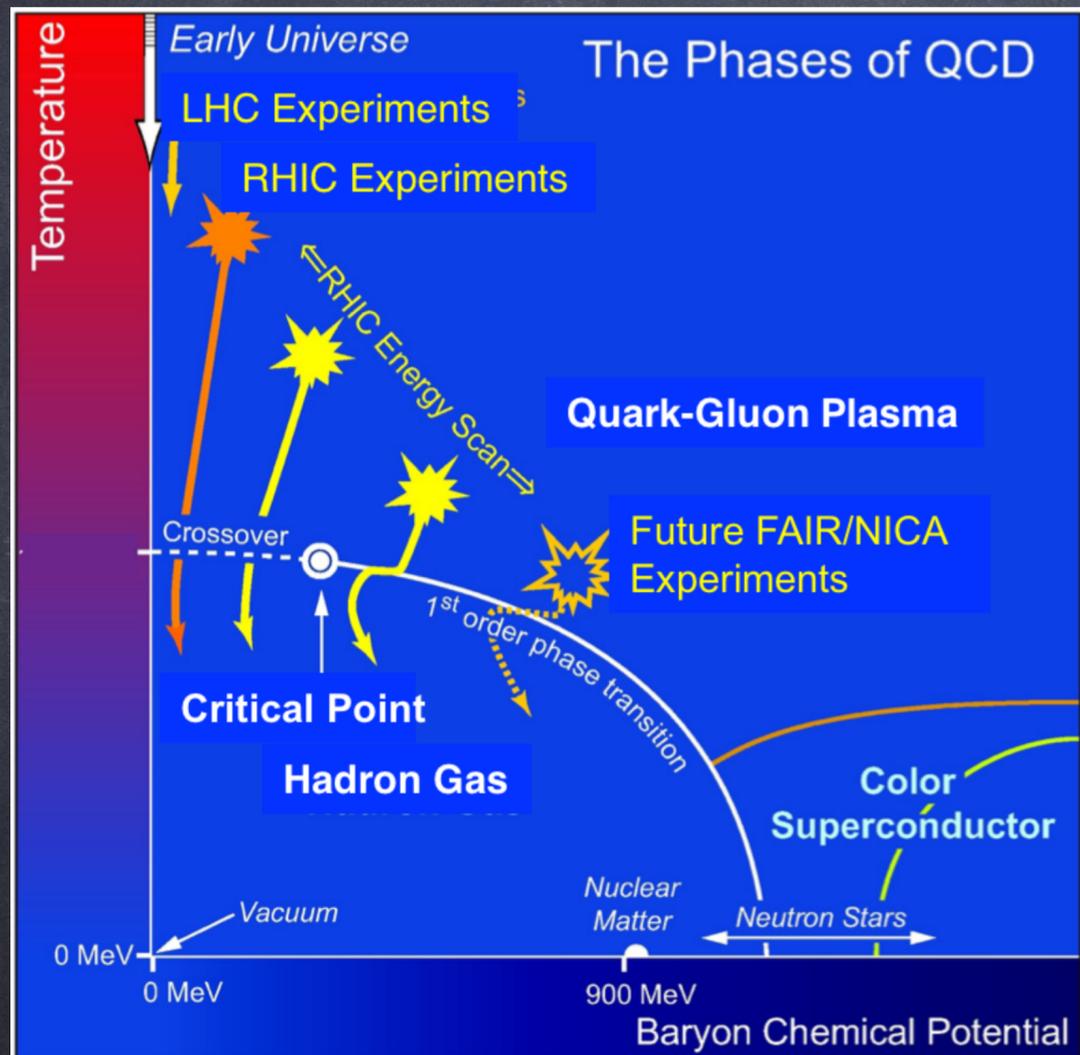$$

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M.Greif, J.A.Fotakis, G.S.Denicol and C.Greiner, Phys.Rev.Lett(2018)242301

A.Das, H.Mishra, R.Mahapatra, Phys.Rev.D 106 (2022) 1, 014013

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Zubarev's NESO method

- Provides a suitable framework for describing systems in the hydrodynamic regime with well defined thermodynamic parameters (e.g. T, μ_a).

- NESO, which characterizes the non-equilibrium state of system.

$$\hat{\varrho}(t) = \frac{1}{Q} \exp\left[-\int d^3\mathbf{x}\, \hat{\mathcal{Z}}(\mathbf{x}, t)\right]$$

$$Q = \mathrm{Tr}\left(exp\left[-\int d^3\mathbf{x}\, \hat{\mathcal{Z}}(t, \mathbf{x})\right]\right)$$

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$$\hat{\mathcal{Z}}(\mathbf{x}, t) = \varepsilon \int_{-\infty}^{t} dt_1\, e^{\varepsilon(t_1 - t)}\left[\beta^{\nu}(\mathbf{x}, t_1)\, \hat{T}_{0\nu}(\mathbf{x}, t_1) - \sum_a \alpha_a(\mathbf{x}, t_1)\, \hat{J}_a^0(\mathbf{x}, t_1)\right]$$

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$$\partial_{\mu}\hat{T}^{\mu\nu} = 0. \qquad \partial_{\mu}\hat{J}_a^{\mu} = 0$$

Where, $\quad \beta^{\mu}(t, \mathbf{x}) = \beta(t, \mathbf{x}) \, u^{\mu}(t, \mathbf{x}) \, ,$

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For example,

$$\hat{T}^{\mu\nu} = \hat{\epsilon} u^\mu u^\nu - \hat{P} \Delta^{\mu\nu} + \hat{Q}^\mu u^\nu + \hat{Q}^\nu u^\mu + \hat{\mathcal{T}}^{\mu\nu}$$

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$$\langle \hat{n}_a(t, \mathbf{x}) \rangle = n_a(t, \mathbf{x})$$

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Off-equilibrium contribution in density operator : $\quad \hat{\varrho}(t) + \delta\hat{\varrho}(t)$

$$\langle \hat{\mathcal{O}}(t, \mathbf{x}) \rangle = Tr\left(\hat{\varrho}(t)\hat{\mathcal{O}}(t, \mathbf{x}) \right)$$

$$= \langle \hat{\mathcal{O}}(t, \mathbf{x}) \rangle_0 + \delta\langle \hat{\mathcal{O}}(t, \mathbf{x}) \rangle$$

Green-Kubo relations

- The master formula for the whole of linear response theory comes from the first-order time-dependent perturbation theory in Quantum Mechanics,

  - For a system with a time-independent Hamiltonian $\hat{H}$ and a Heisenberg-picture operator $\hat{\mathcal{O}}(t, \mathbf{x})$ defined with respect to $\hat{H}$.

  - Adding a time-dependent $\delta\hat{H}(t)$ leads to a change: $\hat{\mathcal{O}}(t, \mathbf{x}) + \delta\hat{\mathcal{O}}(t, \mathbf{x})$

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Kubo-Mori-Bogoliubov (KMB) inner product

Green-Kubo relations

- Casting $\hat{\mathcal{C}}(t, x) / \delta\hat{H}(t)$ in terms of Linear source/force term:

$$\delta\langle\hat{\mathcal{O}}(t, \mathbf{x})\rangle = -i\int_{-\infty}^{t} dt'\langle[\hat{\mathcal{O}}(t, \mathbf{x}), \delta\hat{H}(t', \mathbf{x})]\rangle \quad \longleftrightarrow \quad \delta\langle\hat{\mathcal{O}}(t, \mathbf{x})\rangle = \int_{-\infty}^{t} dt_1\, e^{\varepsilon(t_1 - t)}\left(\hat{\mathcal{O}}(\mathbf{x}, t), \delta\hat{\varrho}\left(t_1\right)\right)$$

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$$\hat{\mathscr{J}}_a^\mu = \hat{\mathcal{N}}_a^\mu - \frac{n_a}{h}\hat{\mathcal{Q}}^\mu$$

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$$\left\langle\hat{\mathscr{J}}^{\mu}_a(\mathbf{x},t)\right\rangle = -\sum_b \int d^4 x_1 \left(\hat{\mathscr{J}}^{\mu}_a(\mathbf{x},t), \hat{\mathscr{J}}^{\nu}_b\left(\mathbf{x}_1, t_1\right)\right)\nabla_\nu\alpha_b$$

$$= -\,\mathscr{K}^{\mu\nu}_{ab}\,\nabla_\nu\alpha_b$$

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- The KMB inner product must satisfy the Onsager reciprocal relations.

$$\mathcal{K}_{ab}^{\mu\nu} = \mathcal{K}_{ba}^{\nu\mu}$$

$$= -\mathcal{K}_{ab}^{\mu\nu}\nabla_\nu\alpha_b$$

- Orthogonality condition.

$$u_\mu\mathcal{K}_{ab}^{\mu\nu} = 0$$

- These two properties helps to decompose $\mathcal{K}^{\mu\nu}_{ab}$:

$$u_\mu \mathcal{K}^{\mu\nu}_{ab} = 0 \quad \longrightarrow \quad \mathcal{K}^{\mu\nu}_{ab} = \Delta^{\mu\nu} \kappa_{ab}$$

$$\mathcal{K}^{\mu\nu}_{ab} = \mathcal{K}^{\nu\mu}_{ba} \quad \longrightarrow \quad \kappa_{ab} = \kappa_{ba}$$

- Which leads to the expression for diffusion matrix as :

$$\kappa_{ab} = \frac{1}{3} \Delta_{\mu\nu} \mathcal{K}^{\mu\nu}_{ab} = -\frac{1}{3} \int d^4 \mathbf{x}_1 \left(\hat{\mathscr{J}}^\mu_a(\mathbf{x}, t), \hat{\mathscr{J}}_{b\mu}(\mathbf{x}_1, t_1) \right)$$

$$\kappa_{ab} = \frac{T}{3} \lim_{\omega \to 0} \frac{\rho_{\hat{\mathscr{J}}_a \hat{\mathscr{J}}_b}(\omega, \mathbf{0})}{\omega}$$

- Diffusion in particle/charge basis :

  - Each particle ('a') carries multiple charges ('A'):

    $$n_A = \sum_a q_{aA} \, n_a \text{ such that } \sum_A \mu_A \, n_A = \sum_a \mu_a \, n_a$$

  - Accordingly, the diffusion current in the conserved charge basis :

    $$\hat{\mathscr{J}}^\mu_A = \sum_a q_{aA} \, \hat{\mathscr{J}}^\mu_a = \hat{\mathscr{N}}^\mu_A - \frac{n_A}{h} \, \hat{Q}^\mu$$

    $$\kappa_{AB} = -\frac{1}{3} \int d^4\mathbf{x}_1 \left(\hat{\mathscr{J}}^\mu_A(\mathbf{x}, t), \hat{\mathscr{J}}_{B\mu}(\mathbf{x}_1, t_1) \right) = \sum_{a,b} q_{aA} \, q_{bB} \, \kappa_{ab}$$

- A toy model calculation with two interacting charged scalar fields :

  - Lagrangian:

$$\mathscr{L}(x) = \partial\phi^\dagger\partial\phi + \partial\xi^\dagger\partial\xi - V(\phi,\xi)$$

Where, $V(\phi,\xi) = m_\phi^2\,\phi^\dagger\phi + m_\xi^2\,\xi^\dagger\xi + \dfrac{g}{2}\phi^\dagger\phi\,\xi^\dagger\xi + \dfrac{\lambda_1(\phi^\dagger\phi)^2}{4} + \dfrac{\lambda_2(\xi^\dagger\xi)^2}{4}$

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$$\begin{pmatrix} \mu_\phi \\ \mu_\xi \end{pmatrix} = \begin{pmatrix} q_{\phi 1} & q_{\phi 2} \\ q_{\xi 1} & q_{\xi 2} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

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$$\mu_\phi n_\phi + \mu_\xi n_\xi = \mu_1 n_1 + \mu_2 n_2$$

$$Z(T, \mu_\phi, \mu_\xi) = C \int \mathcal{D}\phi\mathcal{D}\xi\, e^{-S_E} = Z(T, \mu_1, \mu_2)$$

- Spectral function of diffusion currents and transport coefficients:

$$
\rho_{\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b}(\omega, \mathbf{0}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d\omega'}{\pi} \sum_c \mathbf{p}^2 \left(\left[4\delta_{ab}\delta_{ca} - \frac{2}{h}\left(n_a\delta_{cb} + n_b\delta_{ca}\right)(2\omega' + \omega) + \frac{n_a n_b}{h^2}(2\omega' + \omega)^2 \right] \right.
$$

$$
\left. \times \rho_c(\omega', \mathbf{p}) \rho_c(\omega' + \omega, \mathbf{p}) \left[f_c(\omega' + \omega) - f_c(\omega') \right] \right)
$$

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$$\times \rho_c(\omega', \mathbf{p})\,\rho_c(\omega' + \omega, \mathbf{p}) \left[f_c(\omega' + \omega) - f_c(\omega') \right] \Bigg)$$

here,

$$f_a(\omega) = \frac{1}{e^{\beta(\omega - \mu_a)} - 1}$$

and

$$h = \epsilon + P$$

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$$\kappa_{ab} = \frac{T}{3} \lim_{\omega \to 0} \frac{\rho_{\hat{\mathscr{J}}_a \hat{\mathscr{J}}_b}(\omega, \mathbf{0})}{\omega}$$

- Spectral function of diffusion currents and transport coefficients:

$$\rho_{\hat{\mathscr{J}}_a \hat{\mathscr{J}}_b}(\omega, \mathbf{0}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d\omega'}{\pi} \sum_c \mathbf{p}^2 \left(\left[4\delta_{ab}\delta_{ca} - \frac{2}{h}(n_a\delta_{cb} + n_b\delta_{ca})(2\omega' + \omega) + \frac{n_a n_b}{h^2}(2\omega' + \omega)^2 \right] \right.$$

$$\left. \times \rho_c(\omega', \mathbf{p})\, \rho_c(\omega' + \omega, \mathbf{p}) \left[f_c(\omega' + \omega) - f_c(\omega') \right] \right)$$

$$\kappa_{ab} = \frac{T}{3} \lim_{\omega \to 0} \frac{\rho_{\hat{\mathscr{J}}_a \hat{\mathscr{J}}_b}(\omega, \mathbf{0})}{\omega}$$

$$= \frac{4}{3} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d\omega'}{\pi} \sum_c \mathbf{p}^2 \left(\left[\delta_{ab}\delta_{ca} - \frac{\omega'}{h}(n_a\delta_{cb} + n_b\delta_{ca}) + n_a n_b \left(\frac{\omega'}{h} \right)^2 \right] \rho_c(\omega', \mathbf{p})^2 f_c(\omega') \left[1 + f_c(\omega') \right] \right)$$
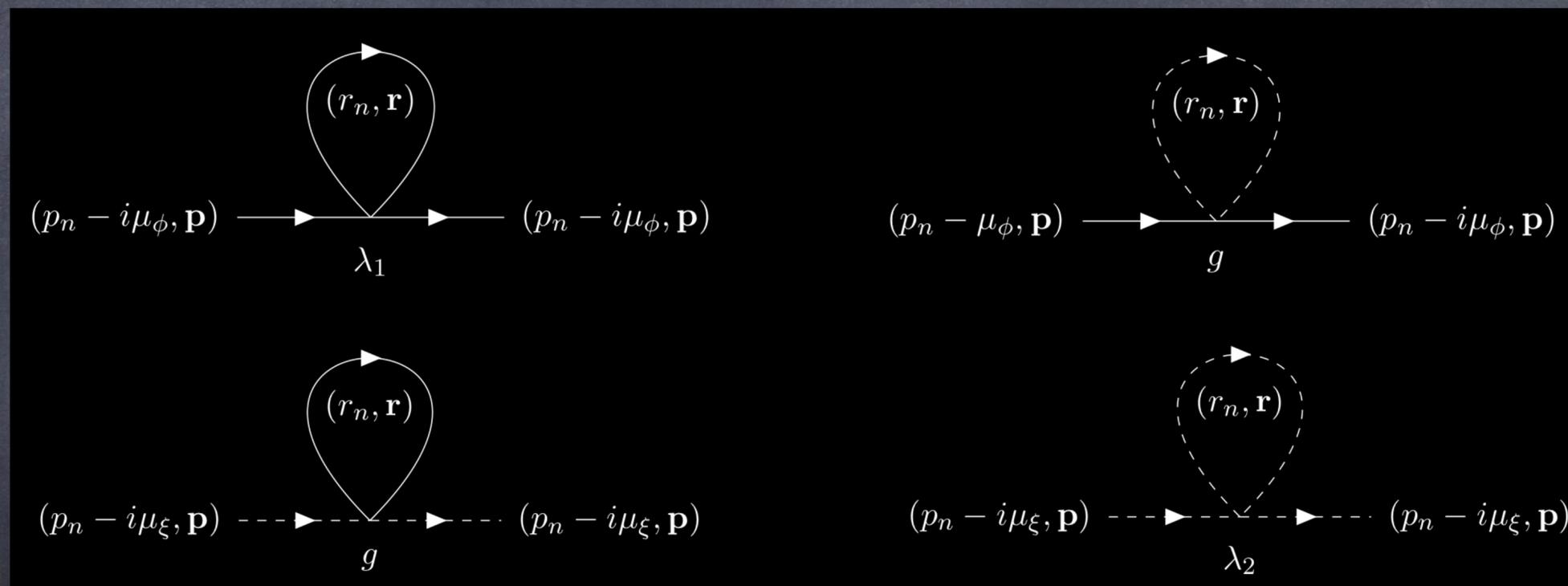
- Quasiparticle approximation with finite thermal width :

$$\rho_a(k_0, \mathbf{k}) = \frac{\text{Im}(\Sigma_a(k_0, \mathbf{k}))}{\left(k_0^2 - \mathbf{k}^2 - m_a^2 - \text{Re}(\Sigma_a(k_0, \mathbf{k}))\right)^2 + |\text{Im}(\Sigma_a(k_0, \mathbf{k}))|^2}$$

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$$\mathbf{Re}(\Sigma_a(k_0, \mathbf{k})) \sim \lambda$$

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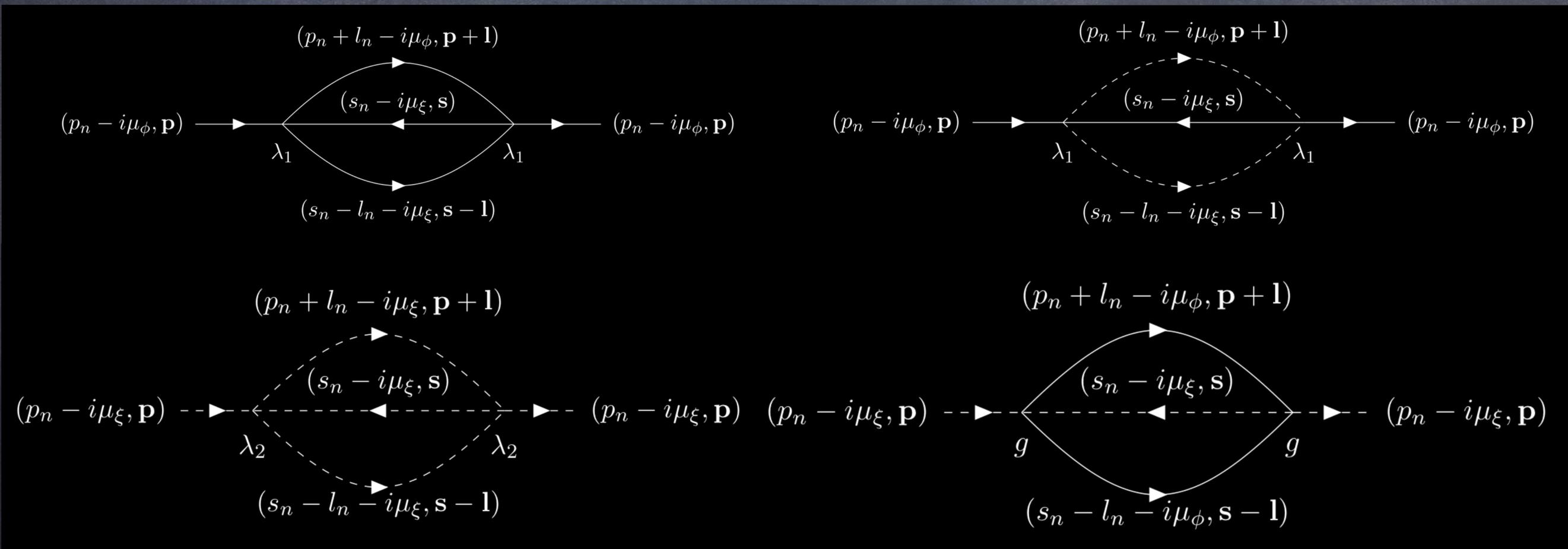
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$$\text{Re}(\Sigma_a(k_0, \mathbf{k})) \sim \lambda \qquad\qquad \text{Im}(\Sigma_a(k_0, \mathbf{k})) \sim \lambda^2$$

$$\rho_a(\omega, \mathbf{p}) \approx \frac{1}{2i} \left[\frac{1}{\left(\omega - i\Gamma_{\mathbf{p}a}\right)^2 - E_{\mathbf{p}a}^2} - \frac{1}{\left(\omega + i\Gamma_{\mathbf{p}a}\right)^2 - E_{\mathbf{p}a}^2} \right] \left[1 + \mathcal{O}(\lambda^2) \right]$$

Where, $\qquad \Gamma_{a\mathbf{k}} = \dfrac{\text{Im}(\Sigma_a(k_0, \mathbf{k}))}{2E_{a\mathbf{k}}}$

- After taking pinching pole approximation, dominating contribution $\sim \dfrac{1}{\Gamma_{\mathbf{p}a}}$:

$$\kappa_{\phi\phi} = \frac{1}{3} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p}^2 \Bigg(\frac{1}{2\Gamma_{\phi\mathbf{p}} E_{\phi\mathbf{p}}^2} \bigg[\Big(1 - \frac{n_\phi E_{\phi\mathbf{p}}}{h} \Big)^2 f_\phi(E_{\phi\mathbf{p}}) \Big(1 + f_\phi(E_{\phi\mathbf{p}}) \Big) + \Big(1 + \frac{n_\phi E_{\phi\mathbf{p}}}{h} \Big)^2$$

$$\times \bar{f}_\phi(E_{\phi\mathbf{p}}) \Big(1 + \bar{f}_\phi(E_{\phi\mathbf{p}}) \Big) \bigg] + \frac{1}{2\Gamma_{\xi\mathbf{p}} E_{\xi\mathbf{p}}^2} \Big(\frac{n_\xi E_{\xi\mathbf{p}}}{h} \Big)^2$$

$$\times \bigg[f_\xi(E_{\xi\mathbf{p}}) \Big(1 + f_\xi(E_{\xi\mathbf{p}}) \Big) + \bar{f}_\xi(E_{\xi\mathbf{p}}) \Big(1 + \bar{f}_\xi(E_{\xi\mathbf{p}}) \Big) \bigg] \Bigg)$$

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$$\kappa_{\xi\xi} = \kappa_{\phi\phi} \Big|_{\phi \leftrightarrow \xi}$$

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$\kappa_{\phi\xi} = \kappa_{\xi\phi}$

$$= \frac{1}{3}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\mathbf{p}^2 \Bigg(\frac{1}{2\Gamma_{\phi\mathbf{p}}E^2_{\phi\mathbf{p}}}\Bigg[\frac{n_\xi E_{\phi\mathbf{p}}}{h}\Big(\frac{n_\phi E_{\phi\mathbf{p}}}{h} - 1\Big)f_\phi(E_{\phi\mathbf{p}})\Big(1 + f_\phi(E_{\phi\mathbf{p}})\Big)$$

$$+ \frac{n_\xi E_{\phi\mathbf{p}}}{h}\Big(\frac{n_\phi E_{\phi\mathbf{p}}}{h} + 1\Big)\bar{f}_\phi(E_{\phi\mathbf{p}})\Big(1 + \bar{f}_\phi(E_{\phi\mathbf{p}})\Big)\Bigg] + \frac{1}{2\Gamma_{\xi\mathbf{p}}E^2_{\xi\mathbf{p}}}\Bigg[\frac{n_\phi E_{\xi\mathbf{p}}}{h}\Big(\frac{n_\xi E_{\xi\mathbf{p}}}{h} - 1\Big)$$

$$\times f_\xi(E_{\xi\mathbf{p}})\Big(1 + f_\xi(E_{\xi\mathbf{p}})\Big) + \frac{n_\phi E_{\xi\mathbf{p}}}{h}\Big(\frac{n_\phi E_{\xi\mathbf{p}}}{h} + 1\Big)\bar{f}_\xi(E_{\xi\mathbf{p}})\Big(1 + \bar{f}_\xi(E_{\xi\mathbf{p}})\Big)\Bigg]\Bigg)$$

- Translate the results for elements of the diffusion matrix into the charge basis :

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$$\left. + \frac{1}{2\Gamma_{a\mathbf{p}} E_{a\mathbf{p}}^2} \left[\left(q_{aA} + \frac{n_A E_{a\mathbf{p}}}{h} \right) \left(q_{aB} + \frac{n_B E_{a\mathbf{p}}}{h} \right) \bar{f}_a(E_{a\mathbf{p}}) \left(1 + \bar{f}_a(E_{a\mathbf{p}}) \right) \right] \right)$$

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- Relaxation time :

$$\tau_a = \frac{1}{2\Gamma_{a\mathbf{p}}} \qquad \text{A.Das, H.Mishra, R.Mahapatra, Phys.Rev.D 106 (2022) 1, 014013.}$$

- Translate the results for elements of the diffusion matrix into the charge basis for bosonic/fermionic system:

$$\kappa_{AB} = \frac{1}{3} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p}^2 \sum_a \left(\frac{\tau_{a\mathbf{p}}}{2E_{a\mathbf{p}}^2} \left[\left(q_{aA} - \frac{n_A E_{a\mathbf{p}}}{h} \right) \left(q_{aB} - \frac{n_B E_{a\mathbf{p}}}{h} \right) f_a(E_{a\mathbf{p}}) \left(1 \pm f_a(E_{a\mathbf{p}}) \right) \right] \right.$$

$$\left. + \frac{\tau_{a\mathbf{p}}}{E_{a\mathbf{p}}^2} \left[\left(q_{aA} + \frac{n_A E_{a\mathbf{p}}}{h} \right) \left(q_{aB} + \frac{n_B E_{a\mathbf{p}}}{h} \right) \bar{f}_a(E_{a\mathbf{p}}) \left(1 \pm \bar{f}_a(E_{a\mathbf{p}}) \right) \right] \right)$$

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Thermal Conductivity:
$$\kappa_T = \frac{h^2}{T^2 (\sum_a \mu_a n_a)^2} \sum_A \mu_A \kappa_{AB} \mu_B$$

S.Jain,JHEP 1003:101,2010; A.Jaiswal,Phys.Lett.B 751 (2015) 548-552

Translate the results for elements of the diffusion matrix into the charge basis for fermionic system:

$$\kappa_{AB} = \frac{1}{3}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\mathbf{p}^2 \sum_a \left(\frac{\tau_{a\mathbf{p}}}{2E_{a\mathbf{p}}^2}\left[\left(q_{aA}-\frac{n_AE_{a\mathbf{p}}}{h}\right)\left(q_{aB}-\frac{n_BE_{a\mathbf{p}}}{h}\right)f_a(E_{a\mathbf{p}})\Big(1-f_a(E_{a\mathbf{p}})\Big)\right] \right.$$

$$\left. +\frac{\tau_{a\mathbf{p}}}{E_{a\mathbf{p}}^2}\left[\left(q_{aA}+\frac{n_AE_{a\mathbf{p}}}{h}\right)\left(q_{aB}+\frac{n_BE_{a\mathbf{p}}}{h}\right)\bar{f}_a(E_{a\mathbf{p}})\Big(1-\bar{f}_a(E_{a\mathbf{p}})\Big)\right]\right)$$

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Thermal Conductivity: $\quad \kappa_T = \dfrac{h^2}{T^2(\sum_a \mu_a n_a)^2}\sum_A \mu_A \kappa_{AB} \mu_B$

for small μ $\quad \dfrac{\kappa_T}{\eta} = \dfrac{7N\pi^2}{9}\left(\dfrac{T}{\sum_a \mu_a^2}\right),\quad$ at conformal limit

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$$

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for large μ

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\frac{\kappa_T}{\eta} = \frac{2\pi^2 T}{3} \frac{\sum_i \mu_i^2}{\sum_i \mu_i^4}, \qquad \text{at conformal limit}
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Translate the results for elements of the diffusion matrix into the charge basis for bosonic/fermionic system:

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$$

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Thermal Conductivity:
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$$

$$
\mu_a \to \lambda \mu_a
$$

for large μ

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\frac{\kappa_T}{\eta} = \frac{2\pi^2 T}{3} \frac{\sum_i \mu_i^2}{\sum_i \mu_i^4} ,
$$

The Wiedmann-Franz law

Translate the results for elements of the diffusion matrix into the charge basis for bosonic/fermionic system:

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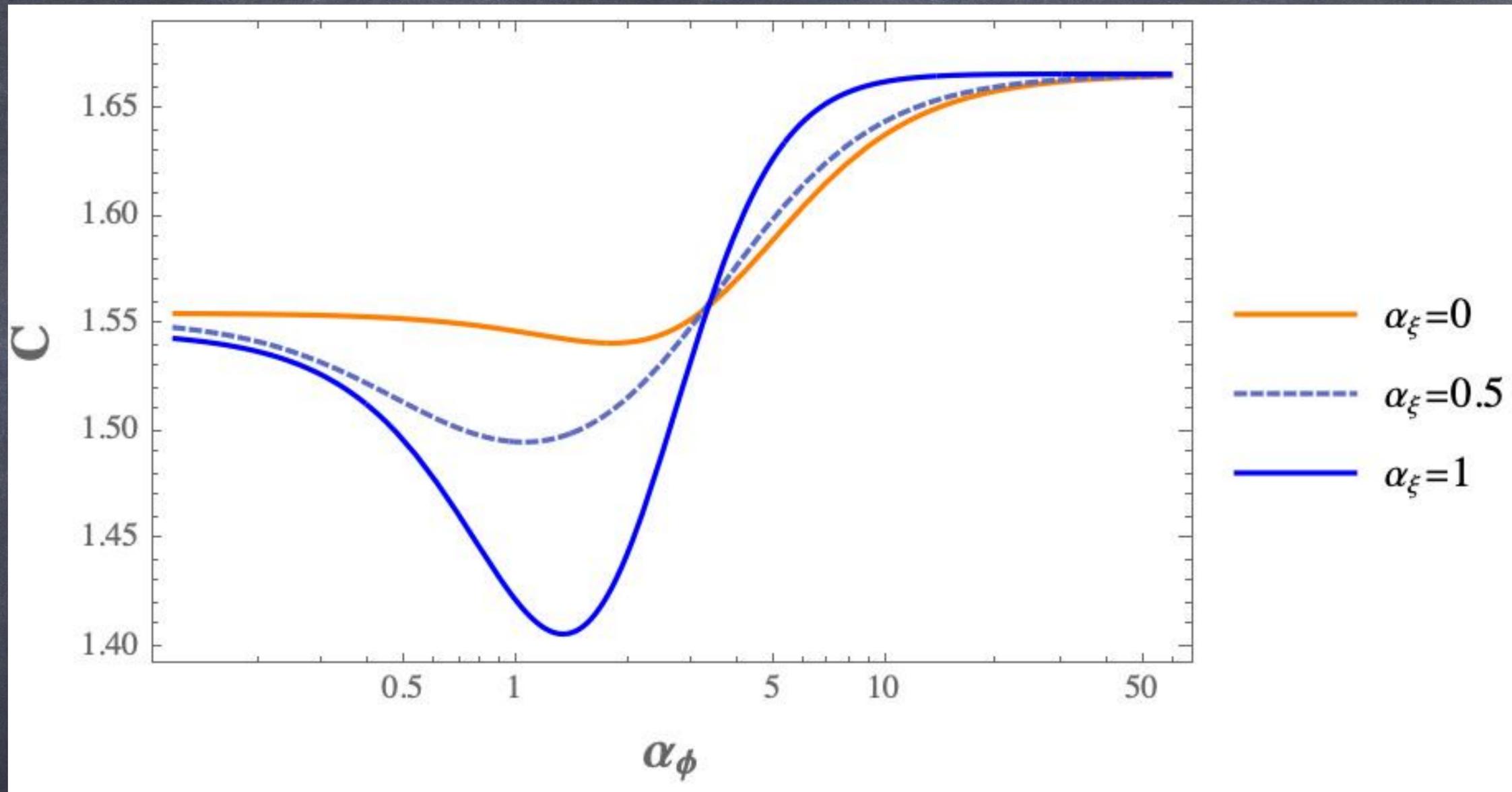**Thermal Conductivity:**

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The Wiedmann-Franz law

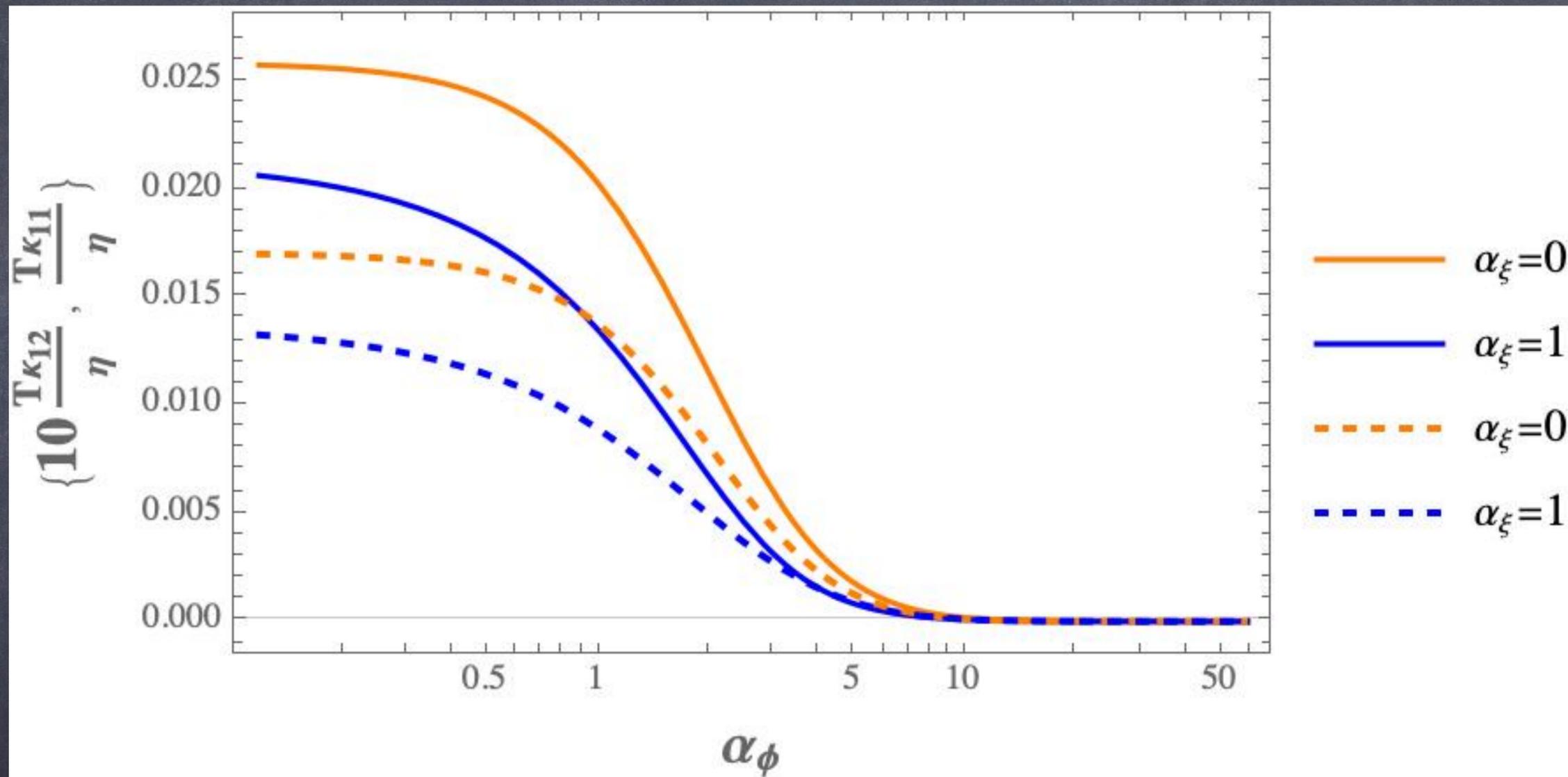$$\frac{\kappa_T}{\eta} = C \left(\frac{T}{\sum_a \mu_a^2} \right)$$

Results

Plot of the scaled thermal conductivity measure C as a function of α_ϕ

for $N = 2$.

Results

Plot of the scaled diffusion coefficients as a function of α_ϕ

for $N = 2$.

Future Works

- Such expressions are rather general and can be used for estimating the diffusion coefficients using nonperturbative methods like lattice QCD or using any effective models of strong interactions like Nambu--Jona-Lasinio (NJL) model, quasi particle models , or sigma models coupled to quarks.

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- In the context of neutron stars, it will be interesting to investigate the effects of diffusion on the oscillation of neutron stars as a result of an internal instability or an external perturbation.

T.K.jha,H.Mishra,V.Sreekanth, Phys.Rev.C82:025803,2010; K.Y. Kraav, M.E. Gusakov and E.M. Kantor, arXiv:2105.08121 [astro-ph.HE]; L. Gavassino, Class. Quantum Grav. 40 165008 (2023).

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Thank you

Backup Slides

$$\hat{\mathcal{Z}}(\mathbf{x}, t) = \varepsilon \int_{-\infty}^{t} dt_1 \, e^{\varepsilon(t_1 - t)} \left[\beta^\nu(\mathbf{x}, t_1) \, \hat{T}_{0\nu}(\mathbf{x}, t_1) - \sum_a \alpha_a(\mathbf{x}, t_1) \, \hat{J}_a^0(\mathbf{x}, t_1) \right]$$

$$\partial_\mu \hat{T}^{\mu\nu} = 0, \quad \partial_\mu \hat{J}_a^\mu = 0$$

$$\hat{\varrho}(t) = \frac{1}{\mathcal{Q}} \, exp\left[\hat{\mathcal{A}} - \hat{\mathcal{B}} \right]$$

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$$\hat{\mathcal{A}} = \int d^3\mathbf{x} \left[\beta^\nu(\mathbf{x}, t)\hat{T}_{0\nu}(\mathbf{x}, t) - \sum_a \alpha_a(\mathbf{x}, t)\hat{J}_a^0(\mathbf{x}, t) \right]$$

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$$\hat{\varrho}(t) = \frac{1}{\mathcal{Q}} \, exp\left[\hat{\mathcal{A}} - \hat{\mathcal{B}} \right] \qquad \langle \hat{\mathcal{O}}(t, \mathbf{x}) \rangle = Tr\left(\hat{\varrho}(t) \hat{\mathcal{O}}(t, \mathbf{x}) \right)$$

$$\hat{\mathcal{B}} = \int d^3\mathbf{x} \int_{-\infty}^{t} dt_1 e^{\varepsilon(t_1 - t)} \, \hat{\mathcal{C}}(\mathbf{x}, t_1)$$
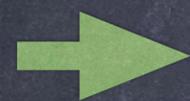
$$\hat{\mathcal{C}}(\mathbf{x}, t_1) = \hat{T}^{\mu\nu}(\mathbf{x}, t_1) \, \partial_\mu \beta_\nu(\mathbf{x}, t_1) - \sum_a \hat{J}_a^\mu(\mathbf{x}, t_1) \, \partial_\mu \alpha_a(\mathbf{x}, t_1)$$

- Casting $\hat{\mathscr{C}}(t, \mathbf{x})$ in terms of Linear gradient force term:

$$\hat{\mathscr{C}}(\mathbf{x}, t_1) = \hat{T}^{\mu\nu}\left(\mathbf{x}, t_1\right) \partial_\mu \beta_\nu \left(\mathbf{x}, t_1\right) - \sum_a \hat{J}_a^\mu \left(\mathbf{x}, t_1\right) \partial_\mu \alpha_a \left(\mathbf{x}, t_1\right)$$

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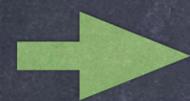
$$\hat{T}^{\mu\nu} = \hat{\epsilon} u^\mu u^\nu - \hat{P} \Delta^{\mu\nu} + \hat{Q}^\mu u^\nu + \hat{Q}^\nu u^\mu + \hat{\mathscr{T}}^{\mu\nu}$$

$$\hat{J}^\mu_a = \hat{n}_a u^\mu + \hat{\mathscr{N}}^\mu_a$$

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$$\hat{\mathscr{C}}(\mathbf{x}, t_1) = \hat{T}^{\mu\nu}\left(\mathbf{x}, t_1\right) \partial_\mu \beta_\nu\left(\mathbf{x}, t_1\right) - \sum_a \hat{J}_a^\mu\left(\mathbf{x}, t_1\right) \partial_\mu \alpha_a\left(\mathbf{x}, t_1\right)$$

$$\hat{T}^{\mu\nu} = \hat{\epsilon} u^\mu u^\nu - \hat{P} \Delta^{\mu\nu} + \hat{Q}^\mu u^\nu + \hat{Q}^\nu u^\mu + \hat{\mathscr{T}}^{\mu\nu}$$

$$\hat{J}_a^\mu = \hat{n}_a u^\mu + \hat{\mathscr{N}}_a^\mu$$

$$\langle \hat{\epsilon}(t, \mathbf{x}) \rangle = \epsilon(t, \mathbf{x})$$

$$\langle \hat{n}_a(t, \mathbf{x}) \rangle = n_a(t, \mathbf{x})$$

- Casting $\hat{\mathscr{C}}(t, \mathsf{x}) \, / \, \delta\hat{H}(t)$ in terms of Linear source/force term:

$$\delta\hat{H}(t) = -\int d^3\mathsf{x} \; \hat{\mathscr{O}}(t, \mathsf{x}) \, \hat{\mathscr{F}}(t, \mathsf{x})$$

$$\delta\langle\hat{\mathscr{O}}(t, \mathsf{x})\rangle = -i\int_{-\infty}^{t} dt' \, d^3\mathsf{x}' \langle[\hat{\mathscr{O}}(t, \mathsf{x}), \hat{\mathscr{O}}(t', \mathsf{x})]\rangle\hat{\mathscr{F}}(t', \mathsf{x})$$

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$$\delta\hat{H}(t) = -\int d^3x \, \hat{\mathcal{O}}(t, x) \, \hat{\mathscr{F}}(t, x)$$

$$\delta\langle\hat{\mathcal{O}}(t, x)\rangle = -i\int_{-\infty}^{t} dt' \, d^3x' \langle[\hat{\mathcal{O}}(t, x), \hat{\mathcal{O}}(t', x)]\rangle\hat{\mathscr{F}}(t', x)$$

$$\downarrow$$

$$\mathscr{R}(t - t', x - x')$$

- Casting $\delta\hat{H}(t)$ in terms of Linear source/force term:

$$\delta\langle\hat{\mathcal{O}}(t,\mathbf{x})\rangle = -i\int_{-\infty}^{t} dt'\, d^3\mathbf{x}'\, \mathscr{R}(t-t',\mathbf{x}-\mathbf{x}')\, \hat{\mathscr{F}}(t',\mathbf{x})$$

$$\delta\langle\hat{\mathcal{O}}_a(t,\mathbf{x})\rangle = -i\int_{-\infty}^{t} dt'\, d^3\mathbf{x}'\, \mathscr{R}_{ab}(t-t',\mathbf{x}-\mathbf{x}')\, \hat{\mathscr{F}}_b(t',\mathbf{x})$$

- Casting $\delta \hat{H}(t)$ in terms of Linear source/force term:

$$\delta \langle \hat{\mathcal{O}}(t, \mathbf{x}) \rangle = -i \int_{-\infty}^{t} dt' \, d^3\mathbf{x}' \, \mathcal{R}(t - t', \mathbf{x} - \mathbf{x}') \, \hat{\mathcal{F}}(t', \mathbf{x})$$

$$\delta \langle \hat{\mathcal{O}}_a(t, \mathbf{x}) \rangle = -i \int_{-\infty}^{t} dt' \, d^3\mathbf{x}' \, \mathcal{R}_{ab}(t - t', \mathbf{x} - \mathbf{x}') \, \hat{\mathcal{F}}_b(t', \mathbf{x})$$

$$\delta \left\langle \hat{\mathcal{O}}^{\mu_1 \mu_2 \cdots \mu_n}(\mathbf{x}, t) \right\rangle = \int d^4 x_1 \left(\hat{\mathcal{O}}^{\mu_1 \mu_2 \cdots \mu_n}(\mathbf{x}, t), \hat{\mathcal{C}}\left(\mathbf{x}_1, t_1\right) \right)$$

- The master formula for the whole of linear response theory comes from the first-order time-dependent perturbation theory in Quantum Mechanics,

  - For a system with a time-independent Hamiltonian $\hat{H}$ and a Heisenberg-picture operator $\hat{\mathcal{O}}(t, \mathbf{x})$ defined with respect to $\hat{H}$.

  - Adding a time-dependent $\delta\hat{H}(t)$ leads to a change: $\hat{\mathcal{O}}(t, \mathbf{x}) + \delta\hat{\mathcal{O}}(t, \mathbf{x})$

$$\delta\langle\hat{\mathcal{O}}(t, \mathbf{x})\rangle = -i\int_{-\infty}^{t} dt' \langle[\hat{\mathcal{O}}(t, \mathbf{x}), \delta\hat{H}(t', \mathbf{x})]\rangle$$

- Thermodynamical + fluid dynamical relations needs to use:

  - With a proper definition of the Temperature and chemical potential

  $$S(t) = \int d^3x \, s(t, x) = - \ln(\hat{\varrho}(t))$$

  - Extensivity + First law of thermodynamics

  $$T(t, x) s(t, x) = \epsilon(t, x) + P(t, x) - \sum_a \mu_a(t, x) \, n_a(t, x)$$

  $$T(t, x) Ds(t, x) = D\epsilon(t, x) - \sum_a \mu_a(t, x) \, Dn_a(t, x) \quad \text{Where, } D = u^\mu \partial_\mu$$

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  - Conservation Equation at zeroth order

$$TD\epsilon = -\left(\epsilon + P\right)\theta \, , \quad Du_\alpha = \frac{1}{\epsilon + P}\nabla_\alpha P \, , \quad Dn_a = -n_a\,\theta$$

$$\text{Where, } \nabla_\mu = \Delta_\mu^\nu \partial_\nu \, , \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \theta = \partial_\mu u^\mu$$

- The master formula for the whole of linear response theory comes from the first-order time-dependent perturbation theory in Quantum Mechanics,

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Kubo-Mori-Bogoliubov (KMB) inner product

$$\delta\left\langle \hat{\mathcal{O}}^{\mu_1\mu_2\ldots\mu_n}(\mathbf{x}, t) \right\rangle = \int d^3\mathbf{x}_1 \int_{-\infty}^{t} dt_1\, e^{\varepsilon(t_1 - t)} \left(\hat{\mathcal{O}}^{\mu_1\mu_2\ldots\mu_n}(\mathbf{x}, t), \hat{\mathscr{C}}\left(\mathbf{x}_1, t_1\right) \right)$$

- Substituting these equations

  - One can write $\hat{\mathscr{C}}(t, \mathbf{x})$ at first order in gradient

$$\hat{\mathscr{C}} = -\beta\,\theta\,\hat{P}* + \beta\,\hat{\mathscr{T}}^{\mu\nu}\sigma_{\mu\nu} - \sum_a \hat{\mathscr{J}}^{\mu}_a \nabla_\mu \alpha_a$$

$$\hat{P}* = \hat{P} - \gamma\,\hat{\epsilon} - \sum_a \delta_a \hat{n}_a$$

$$\sigma_{\mu\nu} \equiv \frac{1}{2}\left[\Delta^\alpha_\mu \Delta^\beta_\nu + \Delta^\alpha_\nu \Delta^\beta_\mu - (2/3)\Delta^{\alpha\beta}\Delta_{\mu\nu}\right]\nabla_\alpha u_\beta$$

- Substituting these equations

  - One can write $\hat{\mathscr{C}}(t,x)$ at first order in gradient

$$\hat{\mathscr{C}} = -\beta\theta\hat{P}* + \beta\hat{\mathscr{T}}^{\mu\nu}\sigma_{\mu\nu} - \sum_a \hat{\mathscr{J}}_a^\mu \nabla_\mu \alpha_a$$

$$\hat{\mathscr{J}}_a^\mu = \hat{\mathscr{N}}_a^\mu - \frac{n_a}{h}\hat{\mathscr{Q}}^\mu$$

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- Substituting these equations

  - One can write $\hat{\mathscr{C}}(t,x)$ at first order in gradient

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  - Dissipative correction to hydro variables:

  $$\delta\left\langle\hat{\mathcal{O}}^{\mu_1\mu_2\ldots\mu_n}(\mathbf{x},t)\right\rangle = \int d^3\mathbf{x}_1 \int_{-\infty}^{t} dt_1\, e^{\varepsilon(t_1-t)}\left(\hat{\mathcal{O}}^{\mu_1\mu_2\ldots\mu_n}(\mathbf{x},t), \hat{\mathscr{C}}\left(\mathbf{x}_1,t_1\right)\right)$$

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- Substituting these equations

  - One can write $\hat{\mathcal{C}}(t, \mathbf{x})$ at first order in gradient

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$$\left\langle \hat{\mathcal{J}}_a^\mu(\mathbf{x}, t) \right\rangle = -\sum_b \int d^4 x_1 \left(\hat{\mathcal{J}}_a^\mu(\mathbf{x}, t), \hat{\mathcal{J}}_b^\nu\left(\mathbf{x}_1, t_1\right) \right) \nabla_\nu \alpha_b$$

- Dissipative correction to hydro variables at first order :

$$\left\langle \hat{\mathcal{J}}_a^\mu(\mathbf{x}, t) \right\rangle = \int d^4 x_1 \left(\hat{\mathcal{J}}_a^\mu(\mathbf{x}, t), \hat{\mathcal{C}}\left(\mathbf{x}_1, t_1\right) \right)$$

$$\left\langle \hat{\mathcal{J}}_a^\mu(\mathbf{x}, t) \right\rangle = -\sum_b \int d^4 x_1 \left(\hat{\mathcal{J}}_a^\mu(\mathbf{x}, t), \hat{\mathcal{J}}_b^\nu\left(\mathbf{x}_1, t_1\right) \right) \nabla_\nu \alpha_b$$

$$\downarrow$$

$$\mathcal{K}_{ab}^{\mu\nu}$$

$$\left\langle \hat{\mathscr{J}}_a^\mu(\mathbf{x},t) \right\rangle = \int d^4x_1 \left(\hat{\mathscr{J}}_a^\mu(\mathbf{x},t), \hat{\mathscr{C}}\left(\mathbf{x}_1,t_1\right) \right)$$

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$$u^\mu \hat{Q}_\mu = u^\mu \hat{N}_{a\mu} = 0$$

$$u_\mu \mathscr{K}_{ab}^{\mu\nu} = 0$$

$$\kappa_{ab} = \frac{T}{3} \lim_{\omega \to 0} \frac{\rho_{\hat{\mathscr{J}}_a \hat{\mathscr{J}}_b}(\omega,\mathbf{0})}{\omega}$$

- The computation of the dissipative diffusion coefficients boils down to the evaluation of the KMB product :

  - KMB for two Hermitian operator can be represented as :

$$\left(\hat{\mathcal{O}}_a(\mathbf{x}, t), \hat{\mathcal{O}}_b\left(\mathbf{x}_1, t_1\right) \right) = -\frac{1}{\beta} \int_{-\infty}^{t_1} dt' G^R_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}\left(\mathbf{x} - \mathbf{x}_1, t - t'\right)$$

  - For the diffusion current – current response function :

$$\kappa_{ab} = \frac{T}{3} \frac{\partial}{\partial \omega} \mathbf{Im}\left(G^R_{\hat{\mathcal{J}}^\mu_a, \hat{\mathcal{J}}_{\mu b}} \right)(\mathbf{0}, \omega) \Bigg|_{\omega \to 0} \, .$$

$$\kappa_{AB} = \frac{T}{3} \frac{\partial}{\partial \omega} \mathbf{Im}\left(G^R_{\hat{\mathcal{J}}^\mu_A, \hat{\mathcal{J}}_{\mu B}} \right)(\mathbf{0}, \omega) \Bigg|_{\omega \to 0} = \sum_{ab} q_{aA} q_{bB} \, \kappa_{ab}$$

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Where,

$$G_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}^R \left(\mathbf{x} - \mathbf{x}_1, t - t' \right) = -i\theta(t - t') \left\langle \left[\hat{\mathcal{O}}_a(\mathbf{x}, t), \hat{\mathcal{O}}_b \left(\mathbf{x}_1, t' \right) \right] \right\rangle_l$$

- The computation of the dissipative diffusion coefficients boils down to the evaluation of the KMB product :

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  - Relation between response function and retarded Green's function :

$$I\left[\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b\right] = \lim_{\epsilon \to 0} \int d^3 \mathbf{x}_1 \int_{-\infty}^{t} dt' \, e^{-\epsilon(t-t')} \left(\hat{\mathcal{O}}_a(\mathbf{x}, t), \hat{\mathcal{O}}_b\left(\mathbf{x}_1, t'\right) \right)$$

$$= - T \lim_{\omega \to 0} \lim_{k \to 0} \partial_\omega Im\left(G^R_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(k, \omega)\right)$$

- Some Useful connection between Green's Functions :

- Spectral density/ function :

$$\rho_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\omega, \mathbf{k}) = \frac{1}{2} \int d^4x \, e^{iK \cdot X} \left\langle \left[\hat{\mathcal{O}}_a(t, \mathbf{x}), \hat{\mathcal{O}}_b^\dagger(0, \mathbf{0}) \right] \right\rangle$$

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Retarded Green's function ⟵⟶ Spectral function :

$$G^R_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\omega, \mathbf{k}) = -i \int_0^\infty dt \int d^3\mathbf{x} \, e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}} \left\langle \theta(t) \left[\hat{\mathcal{O}}_a(\mathbf{x}, t), \hat{\mathcal{O}}_b^\dagger(0, \mathbf{0}) \right] \right\rangle_l$$

$$= \int_{-\infty}^{\infty} \frac{dp^0}{\pi} \frac{\rho_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(P)}{p^0 - \omega - i0^+}$$

- Some Useful connection between Green's Functions :

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$$= \int_{-\infty}^\infty \frac{dp^0}{\pi} \frac{\rho_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(P)}{p^0 - \omega - i0^+}$$

Sokhotski-Plemelj theorem

$$\mathbf{Im}\left(G^R_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\omega, \mathbf{k}) \right) = \rho_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\omega, \mathbf{k}) \qquad \longleftarrow \qquad \frac{1}{\Delta \pm i0^+} = P\left(\frac{1}{\Delta}\right) \mp i\pi\delta(\Delta)$$

- Some Useful connection between Green's Functions :

  - Spectral density/ function :

$$\rho_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\omega, \mathbf{k}) = \frac{1}{2} \int d^4x \, e^{iK \cdot X} \left\langle \left[\hat{\mathcal{O}}_a(t, \mathbf{x}), \hat{\mathcal{O}}_b^{\dagger}(0, \mathbf{0}) \right] \right\rangle$$

  - Response function ⟵⟶ Spectral function :

$$I\left[\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b \right] = -T \lim_{\omega \to 0} \lim_{k \to 0} \partial_{\omega} Im\left(G^{R}_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(k, \omega) \right)$$

$$= -T \lim_{\omega \to 0} \frac{\rho_{\hat{\mathcal{O}}_a \hat{\mathcal{O}}_b}(\omega, \mathbf{0})}{\omega}$$

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$$\kappa_{ab} = \frac{T}{3} \lim_{\omega \to 0} \frac{\rho_{\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b}(\omega, \mathbf{0})}{\omega}$$

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$$\kappa_{ab} = \frac{T}{3} \lim_{\omega \to 0} \frac{\rho_{\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b}(\omega, \mathbf{0})}{\omega} \qquad \kappa_{AB} = \sum_{ab} q_{aA} q_{bB} \, \kappa_{ab}$$

- Finite temperature and chemical potential :

- Fields in Euclidean space :

$$\phi_a(\tau, \mathbf{x}) = \sum_P \!\!\!\!\!\!\fint e^{i(p_n + i\mu_a)\tau - i\mathbf{p}\cdot\mathbf{x}} \quad \text{and} \quad \phi_a^\dagger(\tau, \mathbf{x}) = \sum_P \!\!\!\!\!\!\fint e^{-i(p_n + i\mu_a)\tau - i\mathbf{p}\cdot\mathbf{x}}$$

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  - The two-point Eucledian correlation function can be written as :

$$G_{a,b}^E(X) = \left\langle \phi_a(X)\, \phi_b^\dagger(0) \right\rangle_E$$

$$= \delta_{ab} \sum_P\!\!\!\!\!\!\int \frac{e^{i(p_n + i\mu_a)\tau - i\mathbf{p}\cdot\mathbf{x}}}{(p_n + i\mu_a)^2 + \mathbf{p}^2 + m_a^2}$$

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$$= \delta_{ab} \int_{-\infty}^{\infty} \frac{dk^0}{\pi} \frac{\rho_a(k^0, \mathbf{k})}{k^0 - i[k_n - i\mu_a]}$$

- Finite temperature and chemical potential :

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  $$\phi_a(\tau, \mathbf{x}) = \sum_P e^{i(p_n + i\mu_a)\tau - i\mathbf{p}\cdot\mathbf{x}} \quad \text{and} \quad \phi_a^\dagger(\tau, \mathbf{x}) = \sum_P e^{-i(p_n + i\mu_a)\tau - i\mathbf{p}\cdot\mathbf{x}}$$

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  For free propagator,

  $$\rho_a^f(k_0, \mathbf{k}) = \frac{k_0}{|k_0|}\pi\delta(k_0^2 - E_{a\mathbf{k}}^2)$$

- Finite temperature and chemical potential :

  - Fields in Euclidean space :

  $$\phi_a(\tau, \mathbf{x}) = \sum_P e^{i(p_n + i\mu_a)\tau - i\mathbf{p}\cdot\mathbf{x}} \text{ and } \phi_a^\dagger(\tau, \mathbf{x}) = \sum_P e^{-i(p_n + i\mu_a)\tau - i\mathbf{p}\cdot\mathbf{x}}$$

  - The two-point Eucledian correlation function can be written as :

  $$G_{a,b}^E(X) = \left\langle \phi_a(X)\, \phi_b^\dagger(0) \right\rangle_E$$

  $$= \delta_{ab} \int_{-\infty}^{\infty} \frac{dk^0}{\pi} \frac{\rho_a(k^0, \mathbf{k})}{k^0 - i[k_n - i\mu_a]}$$

  With interaction, RSF :

  $$\rho_a(k_0, \mathbf{k}) = \frac{\text{Im}(\Sigma_a(k_0, \mathbf{k}))}{\left(k_0^2 - \mathbf{k}^2 - m_a^2 - \text{Re}(\Sigma_a(k_0, \mathbf{k}))\right)^2 + |\text{Im}(\Sigma_a(k_0, \mathbf{k}))|^2}$$

- Finite temperature and chemical potential :

  - Fields in Euclidean space :

$$\phi_a(\tau, \mathbf{x}) = \sum_{P} \!\!\!\!\!\!\!\! \int e^{i(p_n + i\mu_a)\tau - i\mathbf{p}\cdot\mathbf{x}} \text{ and } \phi_a^\dagger(\tau, \mathbf{x}) = \sum_{P} \!\!\!\!\!\!\!\! \int e^{-i(p_n + i\mu_a)\tau - i\mathbf{p}\cdot\mathbf{x}}$$

  - The two-point Eucledian correlation function can be written as :

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$$\rho_a(k_0, \mathbf{k}) \equiv \frac{1}{2i}\left[G_{a,a}^E(i(k_n - i\mu) \rightarrow k^0 + i0^+, \mathbf{k}) - G_{a,a}^E(i(k_n - i\mu) \rightarrow k^0 - i0^+, \mathbf{k}) \right]$$

Field theory with finite temperature and chemical potential :

The two-point Eucledian correlation function can be written as :

$$G^E_{\hat{\mathcal{O}}_a, \hat{\mathcal{O}}_b}(X) = \left\langle \hat{\mathcal{O}}_a(X)\, \hat{\mathcal{O}}_b(0) \right\rangle_E$$

$$= \delta_{ab} \int_{-\infty}^{\infty} \frac{dk^0}{\pi} \frac{\rho_a(k^0, \mathbf{k})}{k^0 - ik_n}$$

$$\rho_{\hat{\mathcal{O}}_a, \hat{\mathcal{O}}_b}(k_0, \mathbf{k}) \equiv \frac{1}{2i} \left[G^E_{\hat{\mathcal{O}}_a, \hat{\mathcal{O}}_b}(ik_n \to k^0 + i0^+, \mathbf{k}) - G^E_{\hat{\mathcal{O}}_a, \hat{\mathcal{O}}_b}(ik_n \to k^0 - i0^+, \mathbf{k}) \right]$$

- One needs to perform Wick rotation $(t \to -i\tau)$:

$$\hat{Q}_i^E = \hat{T}_{\tau i} = i \sum_a \left(\partial_\tau \phi_a^\dagger \partial_i \phi_a + \partial_i \phi_a^\dagger \partial_\tau \phi_a \right), \quad \hat{\mathcal{N}}_{a\,i}^E = -i \left(\partial_i \phi_a^\dagger \phi_a - \phi_a^\dagger \partial_i \phi_a \right)$$

- One needs to perform Wick rotation $(t \to -i\tau)$:

$$\hat{Q}_i^E = \hat{T}_{\tau i} = i \sum_a \left(\partial_\tau \phi_a^\dagger \partial_i \phi_a + \partial_i \phi_a^\dagger \partial_\tau \phi_a \right), \quad \hat{\mathcal{N}}_{ai}^E = -i \left(\partial_i \phi_a^\dagger \phi_a - \phi_a^\dagger \partial_i \phi_a \right)$$

- The two-point Euclidean correlation function can be written as :

$$G_{\hat{Q}_i,\hat{Q}_j}^E(X) = \left\langle \hat{Q}_i(X)\, \hat{Q}_j(0) \right\rangle_E , \qquad\qquad G_{\hat{\mathcal{N}}_i,\hat{\mathcal{N}}_j}^E(X) = \left\langle \hat{\mathcal{N}}_i(X)\, \hat{\mathcal{N}}_j(0) \right\rangle_E$$

- One needs to perform Wick rotation $(t \to -i\tau)$:

$$\hat{Q}_i^E = \hat{T}_{\tau i} = i \sum_a \left(\partial_\tau \phi_a^\dagger \partial_i \phi_a + \partial_i \phi_a^\dagger \partial_\tau \phi_a \right), \quad \hat{\mathcal{N}}_{ai}^E = -i \left(\partial_i \phi_a^\dagger \phi_a - \phi_a^\dagger \partial_i \phi_a \right)$$

- The two-point Euclidean correlation function can be written as :

$$G_{\hat{Q}_i, \hat{Q}_j}^E(X) = \left\langle \hat{Q}_i(X)\, \hat{Q}_j(0) \right\rangle_E, \qquad G_{\hat{\mathcal{N}}_i, \hat{\mathcal{N}}_j}^E(X) = \left\langle \hat{\mathcal{N}}_i(X)\, \hat{\mathcal{N}}_j(0) \right\rangle_E$$

$$G_{\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b}^E(X) = \Delta^{ij} \left\langle \hat{\mathcal{J}}_{ai}(X) \hat{\mathcal{J}}_{bj}(0) \right\rangle_l = - \left[\left\langle \hat{\mathcal{N}}_{ai}(X) \hat{\mathcal{N}}_{bi}(0) \right\rangle_l - \frac{n_b}{h} \left\langle \hat{\mathcal{N}}_{ai}(X)\hat{Q}_i(0) \right\rangle_l \right.$$

$$\left. - \frac{n_a}{h} \left\langle \hat{Q}_i(X)\hat{\mathcal{N}}_{bj}(0) \right\rangle_l + \frac{n_a n_b}{h^2} \left\langle \hat{Q}_i(X)\hat{Q}_j(0) \right\rangle_l \right]$$

- One needs to perform Wick rotation $(t \to -i\tau)$:

$$\hat{Q}_i^E = \hat{T}_{\tau i} = i \sum_a \left(\partial_\tau \phi_a^\dagger \partial_i \phi_a + \partial_i \phi_a^\dagger \partial_\tau \phi_a \right), \quad \hat{\mathcal{N}}_{ai}^E = -i\left(\partial_i \phi_a^\dagger \phi_a - \phi_a^\dagger \partial_i \phi_a \right)$$

- The two-point Euclidean correlation function can be written as :

$$G_{\hat{Q}_i,\hat{Q}_j}^E(X) = \left\langle \hat{Q}_i(X)\,\hat{Q}_j(0) \right\rangle_E , \qquad G_{\hat{\mathcal{N}}_i,\hat{\mathcal{N}}_j}^E(X) = \left\langle \hat{\mathcal{N}}_i(X)\,\hat{\mathcal{N}}_j(0) \right\rangle_E$$

$$G_{\hat{\mathcal{J}}_a\hat{\mathcal{J}}_b}^E(X) = \Delta^{ij}\left\langle \hat{\mathcal{J}}_{ai}(X)\hat{\mathcal{J}}_{bj}(0) \right\rangle_l = -\left[\left\langle \hat{\mathcal{N}}_{ai}(X)\hat{\mathcal{N}}_{bi}(0) \right\rangle_l - \frac{n_b}{h}\left\langle \hat{\mathcal{N}}_{ai}(X)\hat{Q}_i(0) \right\rangle_l \right.$$

$$\left. - \frac{n_a}{h}\left\langle \hat{Q}_i(X)\hat{\mathcal{N}}_{bj}(0) \right\rangle_l + \frac{n_a n_b}{h^2}\left\langle \hat{Q}_i(X)\hat{Q}_j(0) \right\rangle_l \right]$$

$$\rho_{\hat{\mathcal{J}}_a\hat{\mathcal{J}}_b}(k_0,\mathbf{k}) \equiv \frac{1}{2i}\left[G_{\hat{\mathcal{J}}_a\hat{\mathcal{J}}_b}^E(ik_n \to k^0 + i0^+, \mathbf{k}) - G_{\hat{\mathcal{J}}_a\hat{\mathcal{J}}_b}^E(ik_n \to k^0 - i0^+, \mathbf{k}) \right]$$

Spectral function of diffusion currents and transport coefficients:

$$\rho_{\mathscr{J}_a \mathscr{J}_b}(\omega, \mathbf{l}) = \delta_{ab} \rho_{\mathscr{N}_a \mathscr{N}_a}(\omega, \mathbf{l}) - \frac{n_b}{h} \rho_{\mathscr{N}_a Q}(\omega, \mathbf{l}) - \frac{n_a}{h} \rho_{\mathscr{N}_b Q}(\omega, \mathbf{l}) + \frac{n_a n_b}{h^2} \rho_{QQ}(\omega, \mathbf{l}),$$

$$\rho_{\hat{\mathscr{J}}_a \hat{\mathscr{J}}_b}(\omega, \mathbf{0}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d\omega'}{\pi} \sum_c \mathbf{p}^2 \left(\left[4\delta_{ab}\delta_{ca} - \frac{2}{h}(n_a \delta_{cb} + n_b \delta_{ca})(2\omega' + \omega) + \frac{n_a n_b}{h^2}(2\omega' + \omega)^2 \right] \right.$$

$$\left. \times \rho_c(\omega', \mathbf{p}) \rho_c(\omega' + \omega, \mathbf{p}) \Big[f_c(\omega' + \omega) - f_c(\omega') \Big] \right)$$

Where,
$$f_a(\omega) = \frac{1}{e^{\beta(\omega - \mu_a)} - 1}$$

Spectral function of diffusion currents and transport coefficients:

$$\rho_{\mathcal{J}_a \mathcal{J}_b}(\omega, \mathbf{l}) = \delta_{ab} \rho_{\mathcal{N}_a \mathcal{N}_a}(\omega, \mathbf{l}) - \frac{n_b}{h} \rho_{\mathcal{N}_a Q}(\omega, \mathbf{l}) - \frac{n_a}{h} \rho_{\mathcal{N}_b Q}(\omega, \mathbf{l}) + \frac{n_a n_b}{h^2} \rho_{QQ}(\omega, \mathbf{l}),$$

$$\rho_{\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b}(\omega, \mathbf{0}) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \int \frac{d\omega'}{\pi} \sum_c \mathbf{p}^2 \left(\left[4\delta_{ab}\delta_{ca} - \frac{2}{h}(n_a \delta_{cb} + n_b \delta_{ca})(2\omega' + \omega) + \frac{n_a n_b}{h^2}(2\omega' + \omega)^2 \right] \right.$$
$$\left. \times \rho_c(\omega', \mathbf{p}) \rho_c(\omega' + \omega, \mathbf{p}) \Big[f_c(\omega' + \omega) - f_c(\omega') \Big] \right)$$

$$\kappa_{ab} = \frac{T}{3} \lim_{\omega \to 0} \frac{\rho_{\hat{\mathcal{J}}_a \hat{\mathcal{J}}_b}(\omega, \mathbf{0})}{\omega} = \frac{4}{3} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \int \frac{d\omega'}{\pi} \sum_c \mathbf{p}^2 \left(\left[\delta_{ab}\delta_{ca} - \frac{\omega'}{h}(n_a \delta_{cb} + n_b \delta_{ca}) + n_a n_b \left(\frac{\omega'}{h} \right)^2 \right] \right.$$
$$\left. \times \rho_c(\omega', \mathbf{p})^2 f_c(\omega') \Big[1 + f_c(\omega') \Big] \right)$$

- Translate the results for elements of the diffusion matrix into the charge basis for bosonic/fermionic system:

$$\kappa_{AB} = \frac{1}{3} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p}^2 \sum_a \left(\frac{\tau_{a\mathbf{p}}}{2E_{a\mathbf{p}}^2} \left[\left(q_{aA} - \frac{n_A E_{a\mathbf{p}}}{h} \right) \left(q_{aB} - \frac{n_B E_{a\mathbf{p}}}{h} \right) f_a(E_{a\mathbf{p}}) \Big(1 \pm f_a(E_{a\mathbf{p}}) \Big) \right] \right.$$

$$\left. + \frac{\tau_{a\mathbf{p}}}{E_{a\mathbf{p}}^2} \left[\left(q_{aA} + \frac{n_A E_{a\mathbf{p}}}{h} \right) \left(q_{aB} + \frac{n_B E_{a\mathbf{p}}}{h} \right) \bar{f}_a(E_{a\mathbf{p}}) \Big(1 \pm \bar{f}_a(E_{a\mathbf{p}}) \Big) \right] \right)$$

$$\frac{\kappa_T}{\eta} = \frac{\pi^2 T}{18 \sum_a \mu_a^2} \left[1 + \frac{90}{T^2 \pi^2} \sum_i \mu_i^2 \big(2N_c - 1 \big) \right] \quad \text{for small } \mu$$

$$\frac{\kappa_T}{\eta} = \frac{2\pi^2}{3} T \frac{\sum_i \mu_i^2}{\sum_i \mu_i^4} \quad \text{for large } \mu \qquad\qquad \frac{\kappa_T}{\eta} = \frac{\pi^2 T}{18 \sum_a \mu_a^2}$$

S.Jain,JHEP 1003:101,2010; A.Jaiswal,Phys.Lett.B 751 (2015) 548-552