On The Convergence of RTA Gradient Expansion

Reghukrishnan G
SPS, NISER

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Overview

- Introduction - Hydrodynamics
- The Boltzmann Equation
- RTA Gradient Expansion
- Convergence
- Summary

Hydrodynamics

- A system at local thermal equilibrium can be described by macroscopic fields, energy momentum tensor and number density.
- The dynamics of the system can be obtained from conservation laws and equation of state.
- Systems out of but near equilibrium can be described by adding gradient corrections to the fields.
- For causal evolution of the system additional degrees of freedom and their evolution equations are required.

$$T^{\mu\nu}_{eq} = \epsilon U^\mu U^\nu - P\Delta^{\mu\nu}$$
$$n^\mu = nU^\mu \qquad (1)$$

$$\partial_\mu T^{\mu\nu} = 0 \qquad \epsilon = \epsilon(P) \qquad (2)$$
$$\partial_\mu n^\mu = 0$$

$$T^{\mu\nu} = \epsilon U^\mu U^\nu - P\Delta^{\mu\nu} + \Pi^{\mu\nu}$$
$$\dot{\Pi}^{\mu\nu} \sim F(\Pi, \partial^\mu U^\nu) \qquad (3)$$

The Boltzmann Equation

- The one particle phase space distribution function can describe a system at low densities.
- The Evolution of the distribution function is given by the Boltzmann Equation.
- We can obtain macroscopic fields from the moments of the distribution function.
- The dynamics is obtained from the Boltzmann Equation.

$$p^\mu \partial_\mu f = C(f,f) \tag{4}$$

$$T^{\mu\nu} = \int \frac{d^3p}{p^0} p^\mu p^\nu f$$

$$n^\mu = \int \frac{d^3p}{p^0} p^\mu f \tag{5}$$

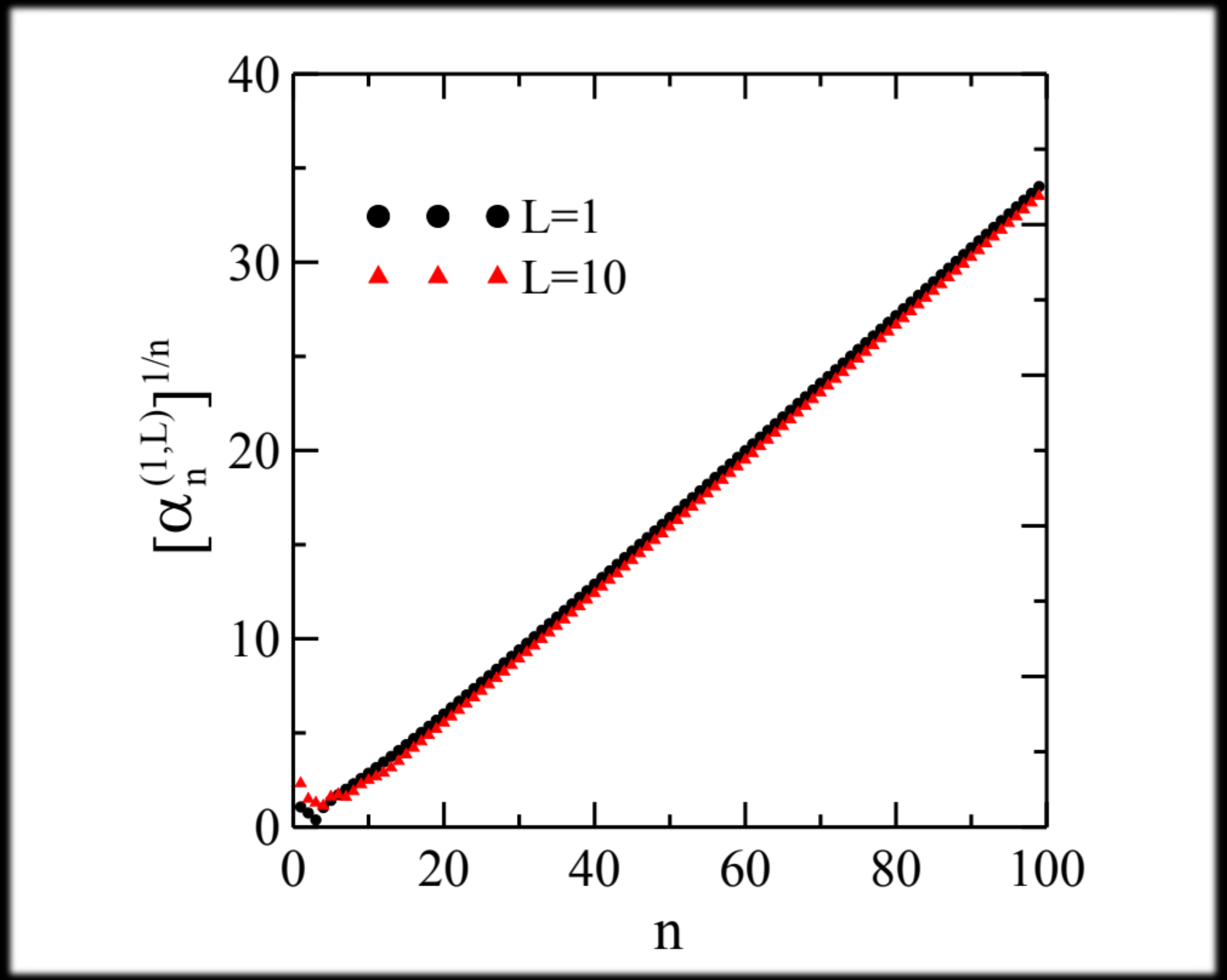
$$\dot{\Pi}^{\mu\nu} = \int \frac{d^3p}{p^0} p^\mu p^\nu \delta\dot{f} \tag{6}$$

Relaxation Time Model

- The Boltzmann Collision Kernel can be approximated by a relaxation time model.

$$p^\mu \partial_\mu f = -\frac{U \cdot p}{\tau_R}(f - f_{eq}) \qquad (7)$$

- We can obtain an expansion (Chapman–Enskog) of the distribution function in terms of equilibrium fields



Coefficient of gradient expansion vs order.
(arXiv:1608.07869v1)

$$f = f_{eq} - \frac{\tau_R}{U \cdot p} p^\mu \partial_\mu f \longrightarrow f(p,x) = f_{eq} + \sum_{n=1}^{\infty} \left[-\frac{\tau_R}{U \cdot p} p^\mu \partial_\mu \right]^n f_{eq}(p,x) \qquad (8)$$

- This expansion is **divergent**.

Solution to the RTA Kernel

- The first order partial differential equation

$$\left(p^\mu \partial_\mu + \frac{U \cdot p}{\tau_R}\right) f = \frac{U \cdot p}{\tau_R} f_{eq} \qquad (9)$$

- We can write a formal integral solution to the RTA Kernel.

$$f(t) = e^{-\xi(t,t_0)} f_0(t, t_0) + \int_{t_0}^{t} \frac{dt'}{p^0} \frac{U(t,t') \cdot p}{\tau_R(t,t')} e^{-\xi(t,t')} f_{eq}(t,t') \qquad \xi' = \int_{t'}^{t} \frac{U \cdot p}{p^0 \tau_R} dt'' \qquad (10)$$

- The first term is a damped free streaming term and the second term contains the contributions from collisions.

Gradient Expansion

- We can then obtain a gradient expansion from the integral solution.

$$f_G = \sum_{n=0}^{\infty} \left\{ 1 - e^{-\xi_0} \sum_{k=0}^{n} \frac{\xi_0^k}{k!} \right\} \left[-\frac{\tau_R}{p \cdot U} p^\mu \partial_\mu \right]^n f_{eq}(x,t) \tag{11}$$

- First sum gives the Chapman Enskog gradients and the second sum gives exponentially damped terms.

- This expansion is convergent.

Approach to Hydrodynamics

- Applicability of hydrodynamics is determined by the damped term,

$$e^{-\xi_0} \sum_{k=0}^{n} \frac{\xi_0^k}{k!} \left[-\frac{\tau_R}{p \cdot U} p^\mu \partial_\mu \right]^n f_{eq}(x, t) \qquad (12)$$

- In the RTA model, Hydrodynamic equations describe the system if the large gradients are suppressed by the damping.

Convergence of the Moments

- Moment equations in Bjorken flow $\rho_{n,l} = \frac{1}{(2\pi)^3} \int d^3p (p^0)^n \left(\frac{p^z}{p^0} \right)^{2l} f$

$$\partial_\tau \rho_{n,l} + \frac{2l+1}{\tau} \rho_{n,l} - \frac{2l-n}{\tau} \rho_{n,l+1} = -\frac{1}{\tau_R} \left(\rho_{n,l} - \rho_{n,l}^{eq} \right) \qquad (15)$$

- If the relaxation time is momentum independent, convergence is inherited by the moments.

$$\rho_{n,l}(\tau) = e^{-\hat{\xi}_l(\tau,\tau_0)} \rho_{n,l}^0(\tau) + \int_{\tau_0}^{\tau} d\tau' \frac{1}{\tau_R} e^{-\hat{\xi}_l(\tau,\tau')} \rho_{n,l}^{eq}(\tau') \qquad (16)$$

Summary

- A formal integral solution to the RTA Boltzmann equation can be obtained
- Gradient expansion of the integral solution contains non-hydrodynamic terms that ensures convergence.
- Convergence is inherited by the moments.
- Under the RTA model smallness of the free streaming component and not the gradient determines the validity of hydrodynamics.

Thank You

The Integral Solution

- Shift Operator

$$\hat{S}h_l = h_{l+1} \qquad (16)$$

- The moment Equation

$$\left[\partial_\tau + \left(\frac{2l+1}{\tau} + \frac{1}{\tau_R} - \frac{2l-n}{\tau}\hat{S}\right)\right]\rho_{n,l}(\tau) = \frac{1}{\tau_R}\rho^{eq}_{n,l}(\tau) \qquad (17)$$

The Integral Solution

- The modified Damping factor

$$\exp \hat{\xi}_l(\tau, \tau') = \int_{\tau'}^{\tau} \left(\frac{2l+1}{\tau} + \frac{1}{\tau_R} - \frac{2l-n}{\tau} \hat{S} \right) \quad (18)$$

- The formal solution
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$$\rho_{n,l}(\tau) = e^{-\hat{\xi}_l(\tau,\tau_0)} \rho_{n,l}^{0}(\tau) + \int_{\tau_0}^{\tau} d\tau' \frac{1}{\tau_R} e^{-\hat{\xi}_l(\tau,\tau')} \rho_{n,l}^{eq}(\tau') \quad (19)$$

$$\rho_{n,l}^{F}(\tau) = \sum_{k=0}^{\infty} e^{-\xi_0} K_{n,l,k}(\tau, \tau_0) \rho_{n,l+k}^{0}(\tau) \qquad \rho_{n,l}^{G}(\tau) = \sum_{k=0}^{\infty} \int_{\tau_0}^{\tau} d\tau' \frac{e^{-\xi'}}{\tau_R} K_{n,l,k}(\tau, \tau') \rho_{n,l+k}^{eq}(\tau') \quad (20)$$

Explicit form of the solution

- The solution can be explicitly written as

$$\rho_{n,l}(\tau) = \sum_{k=0}^{\infty} e^{-\xi_0} K_{n,l,k}(\tau,\tau_0)\rho^0_{n,l+k}(\tau) + \sum_{k=0}^{\infty} \int_{\tau_0}^{\tau} d\tau' \frac{e^{-\xi'}}{\tau_R} K_{n,l,k}(\tau,\tau')\rho^{eq}_{n,l+k}(\tau') \qquad (21)$$

- where

$$K_{n,l,k}(\tau,\tau') = \frac{\left(l - \frac{n}{2}\right)^{(k)}}{k!} \left(\frac{\tau'}{\tau}\right)^{2l+1} \left[1 - \left(\frac{\tau'}{\tau}\right)^2\right]^k \qquad (22)$$

- This reduces to the known solution for the conformal case, (n =2, l =0, energy density)

$$\rho^G_{2,0}(\tau) = \int_{\tau_0}^{\tau} d\tau' \frac{e^{-\xi'}}{\tau_R} \left(\frac{\tau'}{\tau}\right) {}_2F_1\left[\frac{1}{2}, -\frac{1}{2}; \frac{3}{2}; \left(\frac{\tau'}{\tau}\right)^2 - 1\right] T^4 \qquad (22)$$

The Gradient Expansion

- We get the convergent gradient expansion

$$\rho_{n,l}^{G} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma(m+1,\xi_0)}{m!} \left[\tau_R \frac{\partial}{\partial \tau}\right]^m K_{n,l,k}(\tau,\tau) \rho_n^{eq}(\tau, l+k) \qquad (23)$$