Out of equilibrium physics in initial stages and near QCD critical point

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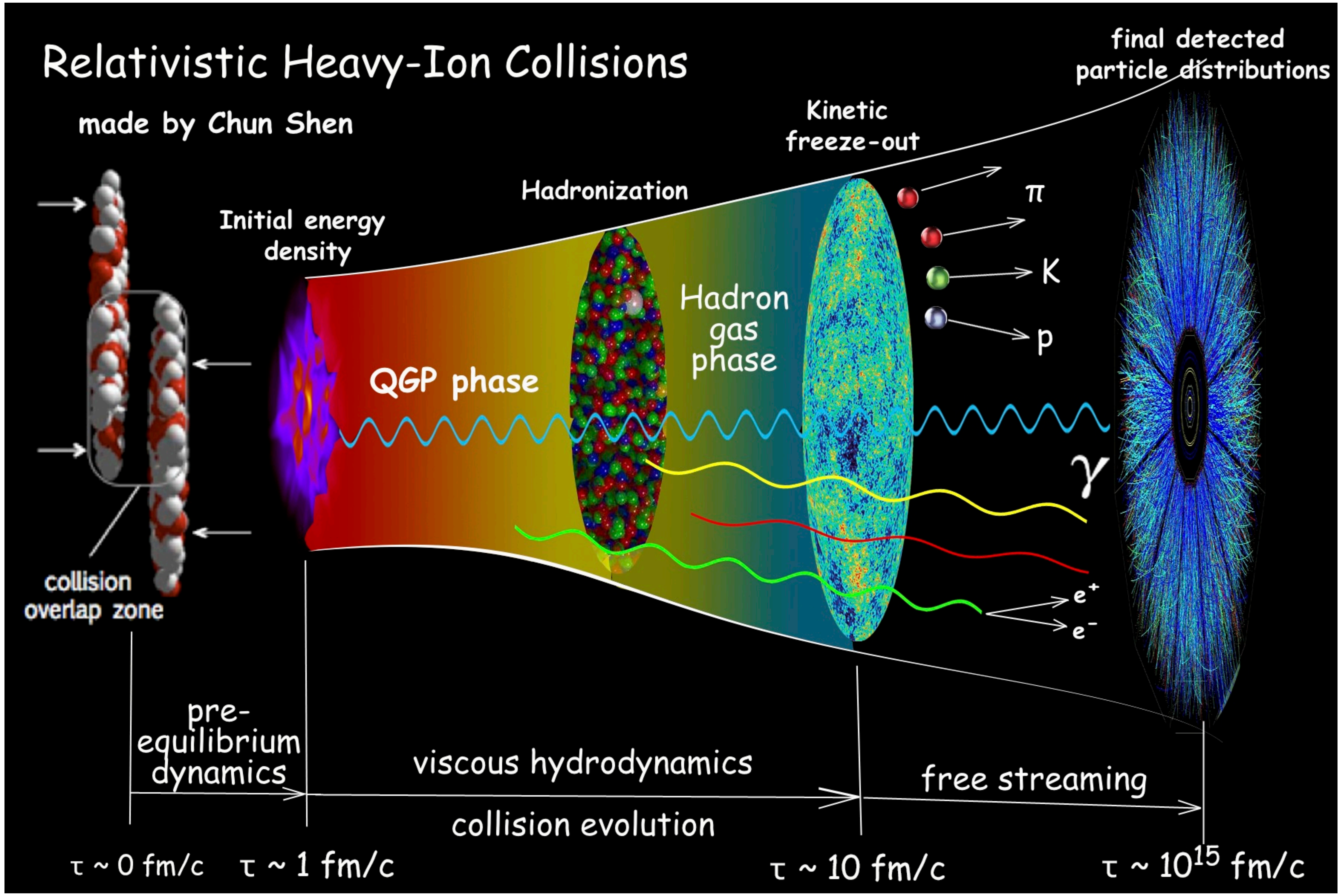
Outline

- Part I: Out-of-equilibirum dynamics in early stages of heavy-ion collisions
  - Competition between interactions that try to establish local thermal equilibrium and rapid expansion of the medium which forbids it.
- Part II: Out-of-equilibirum dynamics near a critical point
  - Even if a dynamic system is in local thermal equilibrium, it will fall out of equilibrium as a critical point is approached (critical slowing down).
- In both these cases, suitable extensions of hydrodynamic-like theories may be useful to model the dynamics.

The ‘standard model’ of heavy-ion collisions

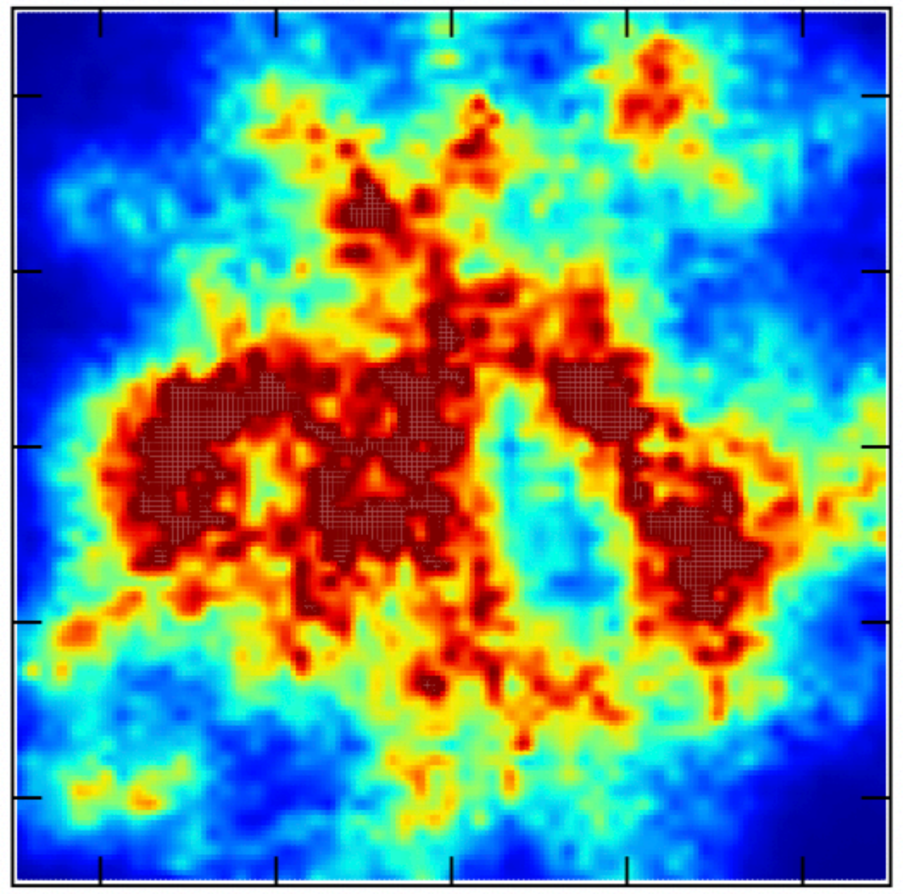


- Collision of highly Lorentz-contracted nuclei.
- Deposition of kinetic energy, liberation of quarks, gluons: formation of quark-gluon plasma.
- Many interesting questions on QGP pertain to dynamics.
  - How the plasma flows: transport coefficients $\eta/s, \zeta/s$
- How flow is reached: isotropization, hydrodynamization, thermalization

Need a dynamical description of the plasma

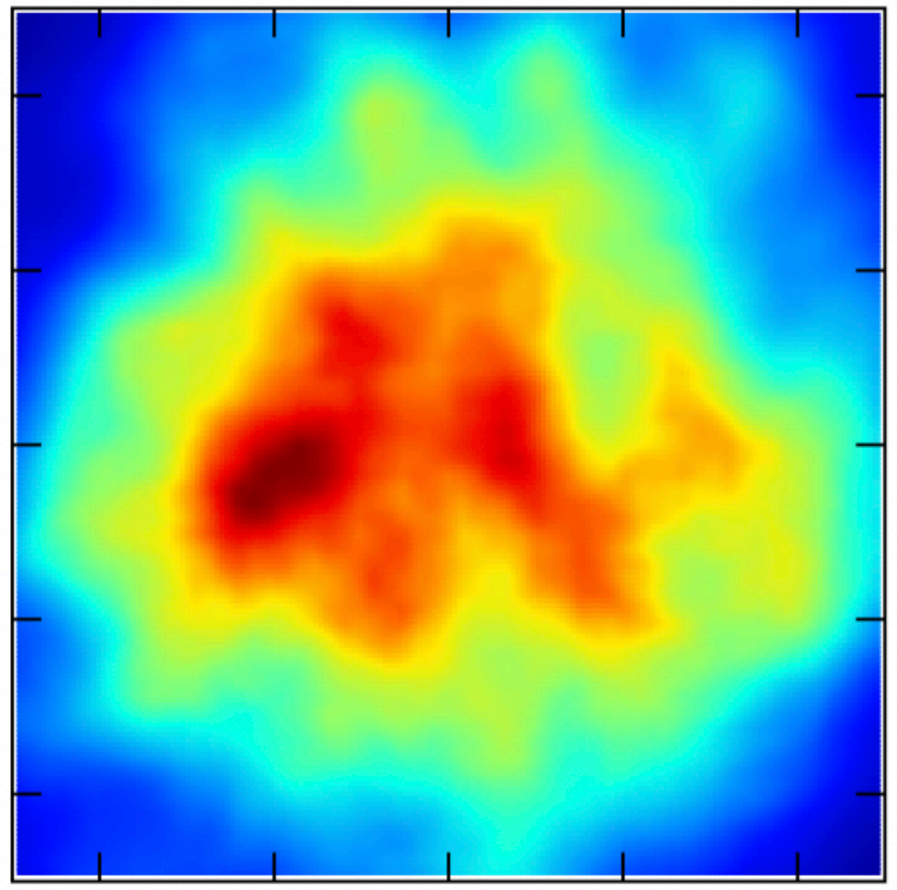
"Hydrodynamization"

Mazeliauskas & Berges, Heller et al, Romatschke, Schenke et al, Kurkela and Wiedemann



$\tau \sim 0\,\mathrm{fm/c}$

Keegan et al, JHEP (2016)



$\tau \sim 1\,\mathrm{fm/c}$ (Typical starting time for hydro)

For hydrodynamics to apply, system must be close to local thermal equilibrium

- The system formed is initially far from local thermal equilibrium; characterized by large spatial gradients
    - Key question: Is the system weakly coupled or strongly coupled?
- If weakly-coupled, can be described in terms of quasi-particles using kinetic theory. (This talk)
- If strongly coupled, quasi-particle description does not hold; approaches such as holography needed. $\mathcal{N} = 4$ super Yang-Mills theory used. Chester, Yaffe, Heller, Janik, van Der Schee, and others

Kinetic theory

- Models microscopic behavior of constituents; collisions/scattering. No assumption of local thermal equilibrium. Thus, applicable both near and far from local equilibrium.

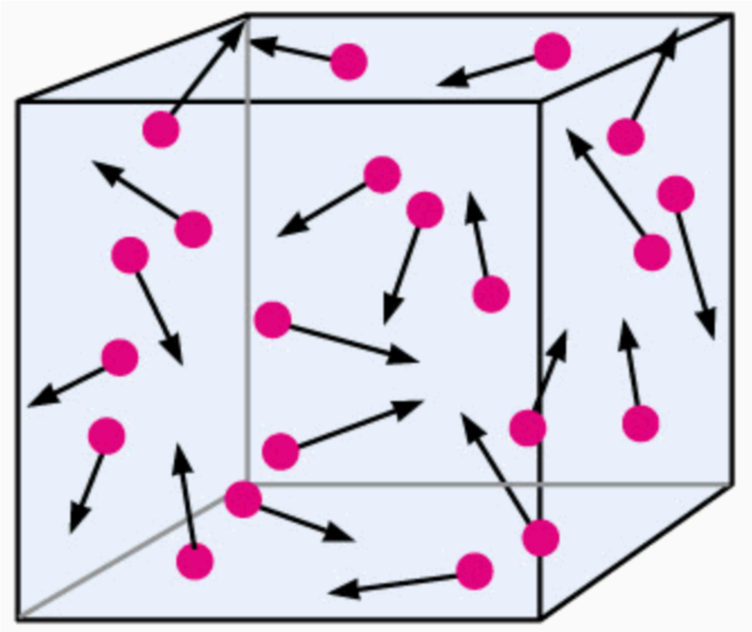
Assumption: mean-free path and relaxation timescales long compared to interaction timescales.

- Evolution of $f(t, \vec{x}, \vec{p})$ governed by Boltzmann equation:

$$E_p \partial_t f + \vec{p} \cdot \vec{\nabla} f = \mathscr{C}[f]$$

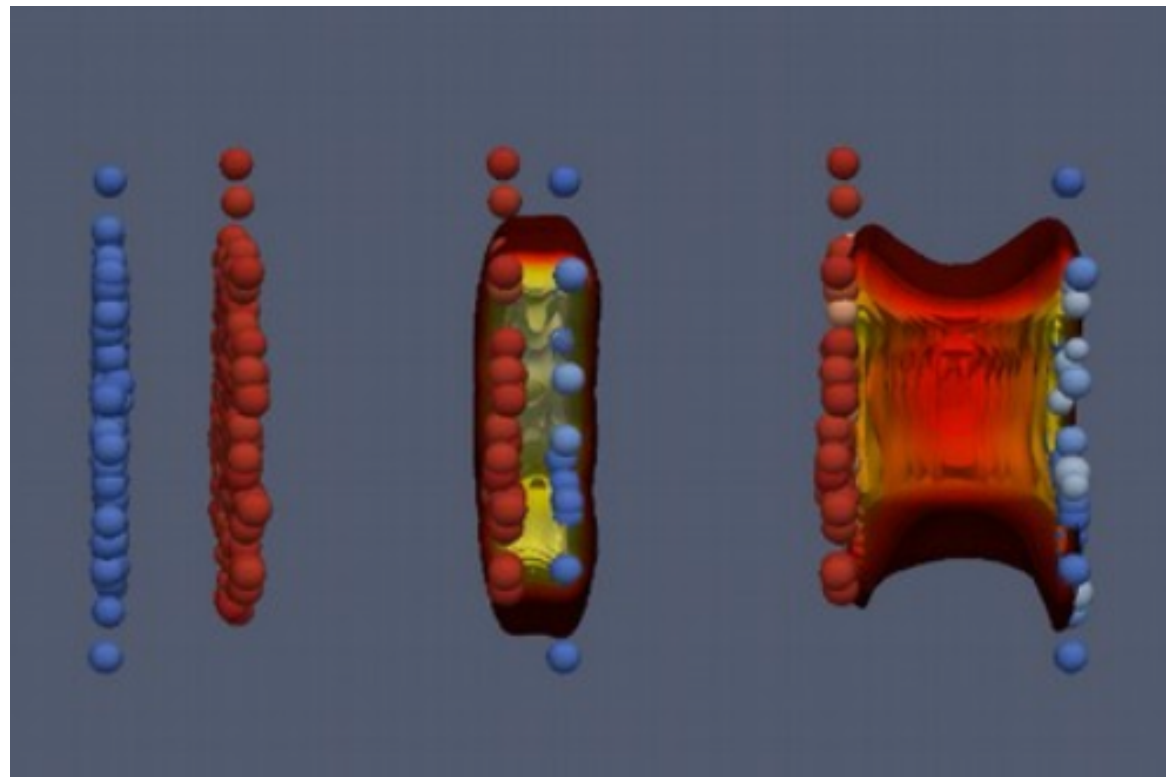
(describes free-streaming)

interactions; scatterings



- Conserved macroscopic quantities $(T^{\mu\nu}, J^{\mu})$ related to $f(x,p)$

$$T^{\mu\nu}(x) = \int_p p^\mu \, p^\nu \, f(x,p) \qquad J^\mu(x) = \int_p p^\mu \, f(x,p)$$



- A useful model for early-time dynamics in HIC: Bjorken flow

$v^x = v^y = 0, v^z = z/t$ Simplified stresses, $T^{\mu\nu} = \mathrm{diag}(e, P_T, P_T, P_L)$

Kinetic theory

- Models microscopic behavior of constituents; collisions/scattering. Unlike hydro, does not assume local thermal equilibrium. Applicable both near and far from local equilibrium.

Assumption: mean-free path and relaxation timescales long compared to interaction timescales.

- Evolution of $f(t, \vec{x}, \vec{p})$ governed by Boltzmann equation: $p^\mu \partial_\mu f = \mathscr{C}[f]$

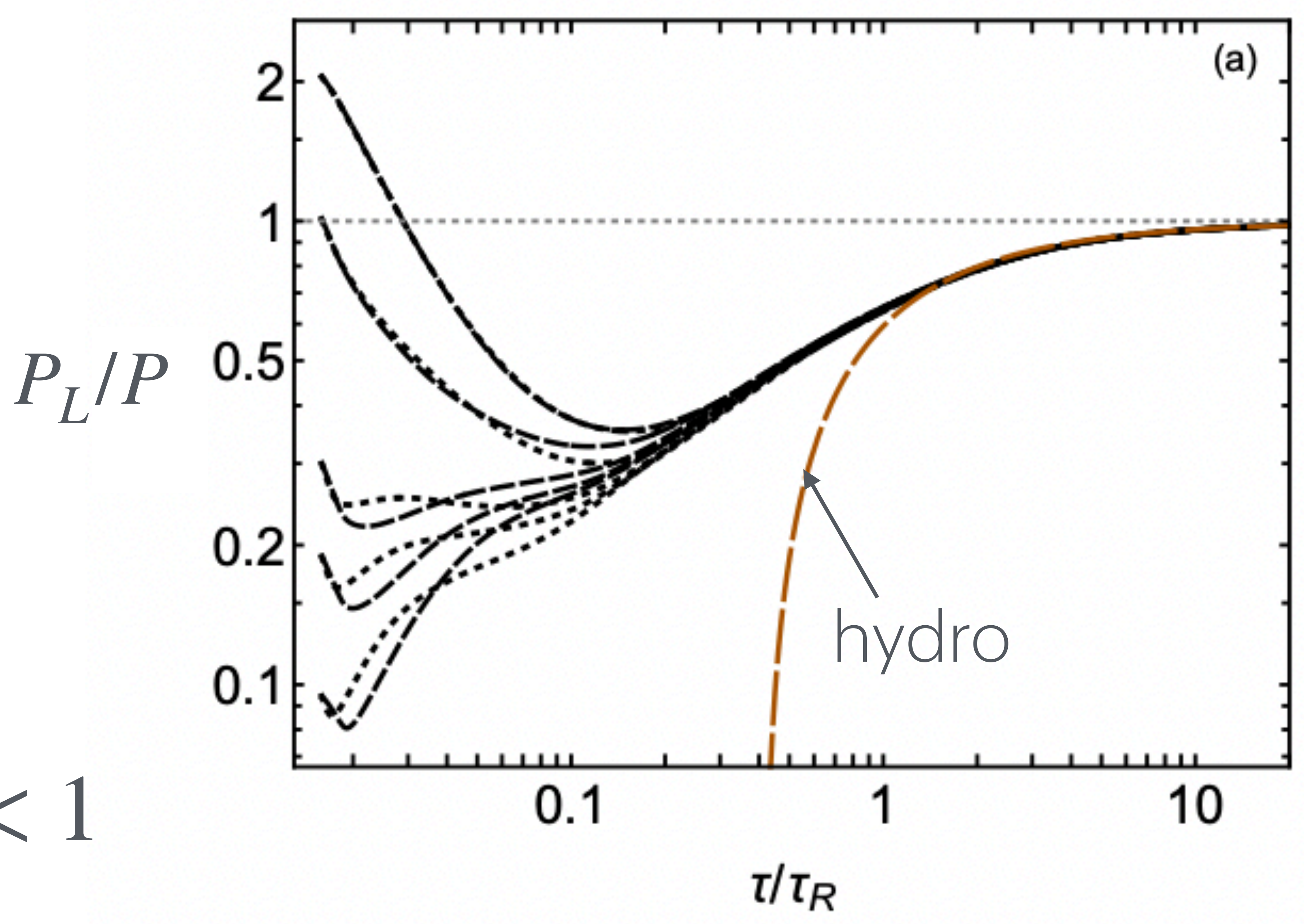
- QCD kinetic theory in Bjorken flow

$$\partial_\tau f_{g,q}(\mathbf{p},\tau) - \frac{p_z}{\tau}\partial_{p_z} f_{g,q}(\mathbf{p},\tau) = -\underbrace{\mathcal{C}^{2\leftrightarrow 2}_{g,q}[f]}_{\text{Elastic scattering}} - \underbrace{\mathcal{C}^{1\leftrightarrow 2}_{g,q}[f]}_{\text{Inelastic scattering}},$$

Arnold, Moore, Yaffe, JHEP (2003)

Kurkela, Mazeliauskas, Paquet, Schlichting, Teaney

Almaalol et al PRL (2020)



- Effective longitudinal pressure P_L drops rapidly at $\tau/\tau_R \ll 1$

Kinetic theory: Toy model

- Many features of early stages can be captured in a toy model (relaxation-time approximation)

$$\frac{\partial f}{\partial t} - \frac{p^z}{t}\frac{\partial f}{\partial p^z} = -\frac{1}{\tau_R}\left(f - f_{eq}\right)$$

Acts like an external force

$\Longrightarrow$ shrinks momentum distribution along p^z

Isotropizes momenta

Longitudinal pressure

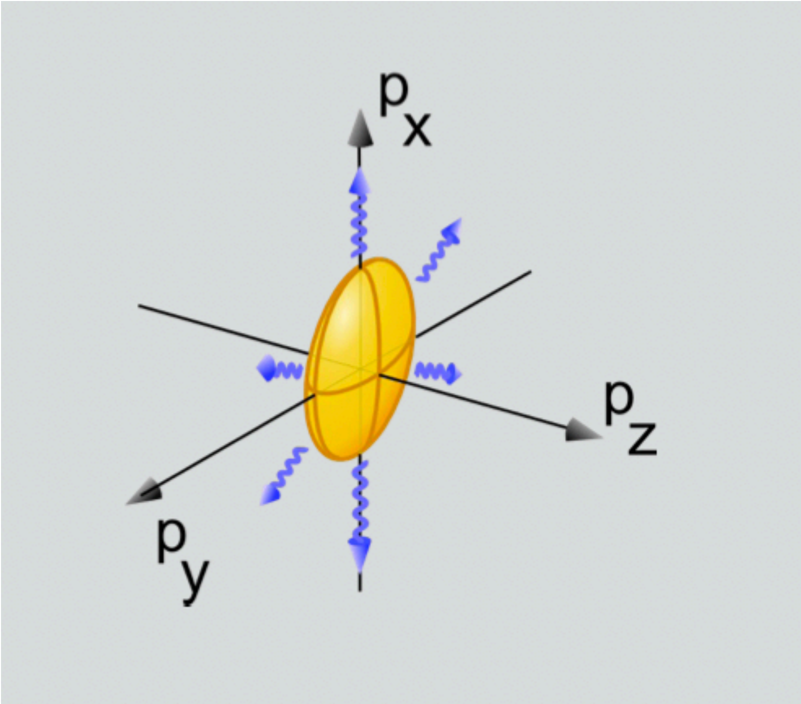
$$P_L = \int \frac{p_z^2}{E_p} f$$

Transverse pressure

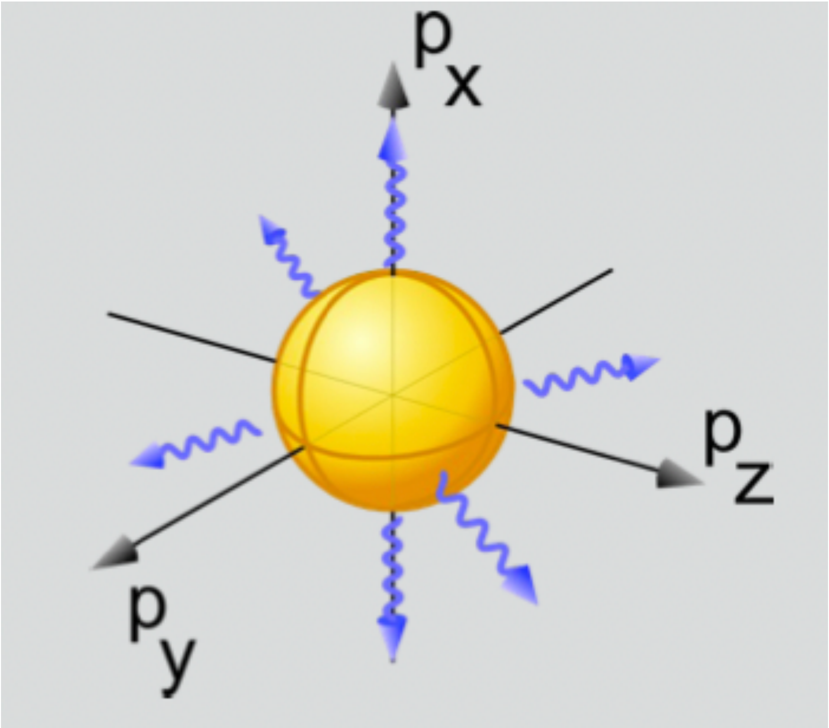
$$P_T = \frac{1}{2}\int \frac{p_T^2}{E_p} f$$

Competition between

Expansion $\Longrightarrow$



Collisions $\Longrightarrow$



Kinetic theory describes transition from collision-less regime to hydro regime (dominated by collisions)

Kinetic theory using moments

Blaizot and Yan, PLB (2018)

- Too much information in the full distribution function. Focus on particular moments of $f(\tau, \vec{p})$

$$\mathscr{L}_n(\tau) \equiv \int_p p^2 P_{2n}(\cos\theta) f(\tau, \vec{p})$$

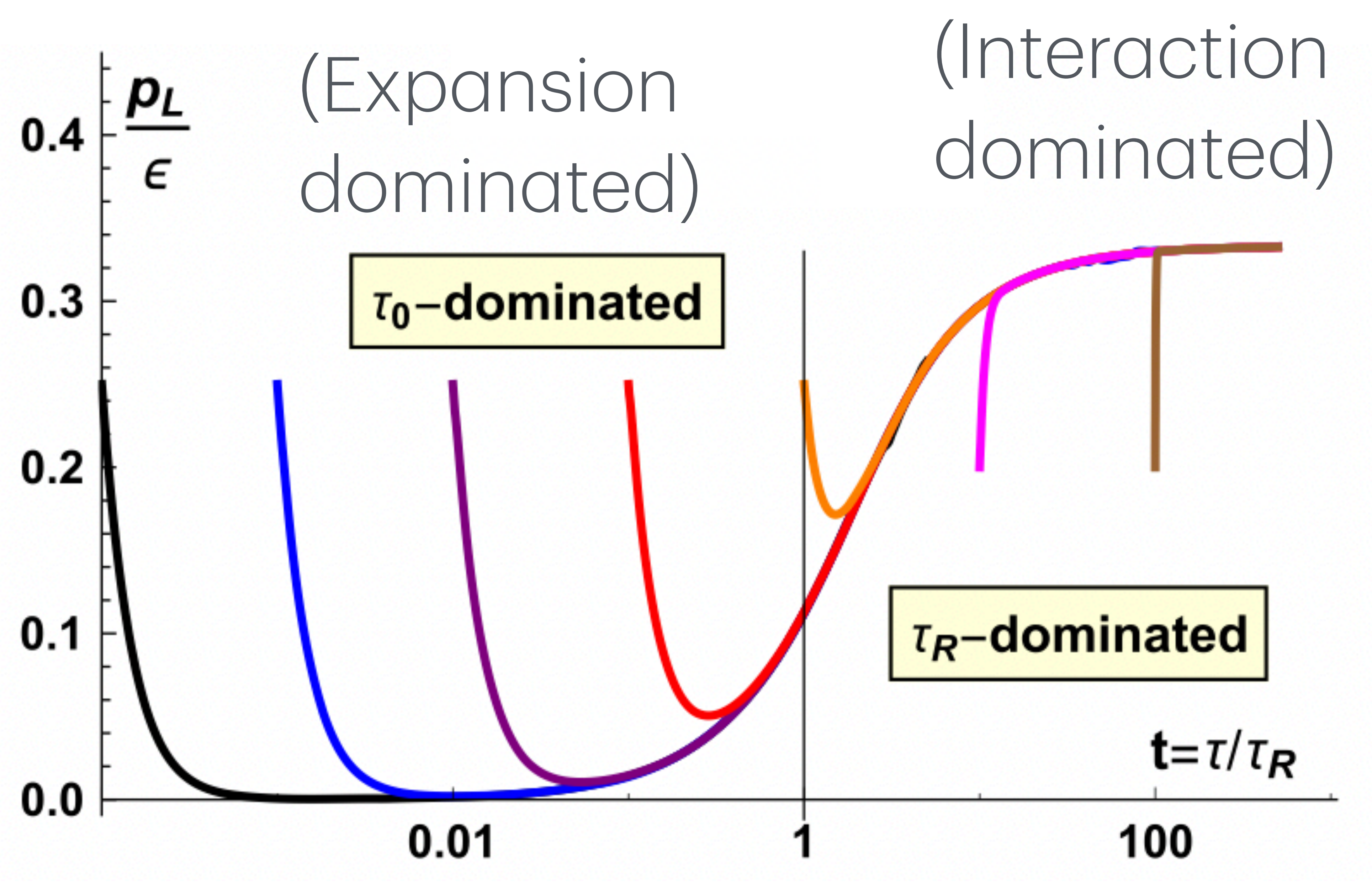
Energy-momentum tensor is described by first two moments: $\mathscr{L}_0 = e$, $\mathscr{L}_1 = P_L - P_T$

- The moments satisfy coupled equations

$$\frac{d\mathscr{L}_0}{d\tau} = -\frac{1}{\tau}\left[a_0\mathscr{L}_0 + c_0\mathscr{L}_1\right]$$

$$\frac{d\mathscr{L}_n}{d\tau} = -\frac{1}{\tau}\left[a_n\mathscr{L}_n + b_n\mathscr{L}_{n-1} + c_n\mathscr{L}_{n+1}\right] - \frac{\mathscr{L}_n}{\tau_R}$$

(Free-streaming) (Collisions)



Kurkela, van der Schee, Wiedemann, Wu, PRL (2020)

- Collisionless regime characterized by two fixed points (one stable, one unstable): Stable FP $P_L/e \to 0$.

Israel-Stewart hydro and moments

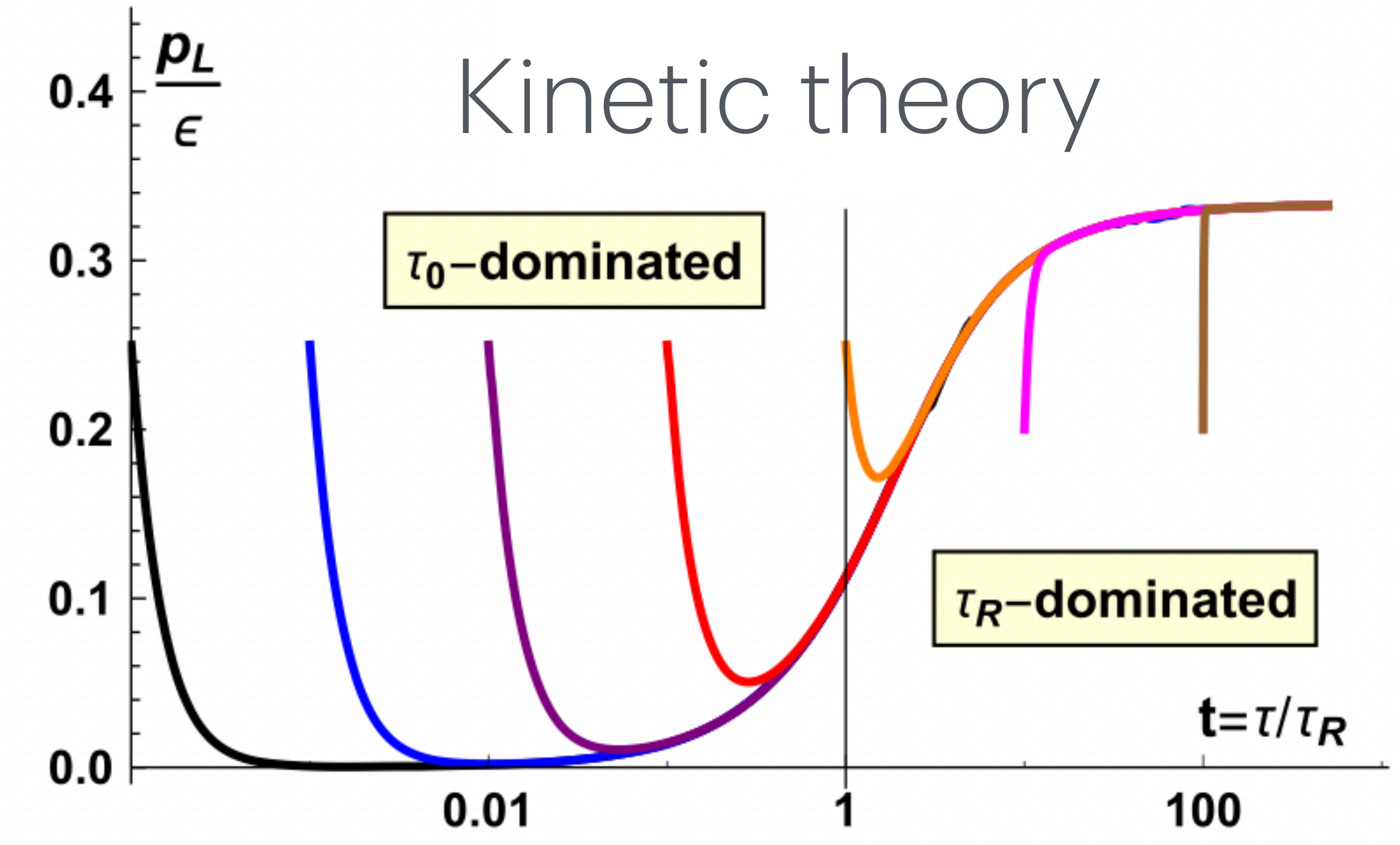
- The moments equations contains Israel-Stewart like "hydro" (ISH) (truncate at n = 1)

$$\frac{d\mathscr{L}_0}{d\tau} = -\frac{1}{\tau}\left[a_0\mathscr{L}_0 + c_0\mathscr{L}_1\right]$$

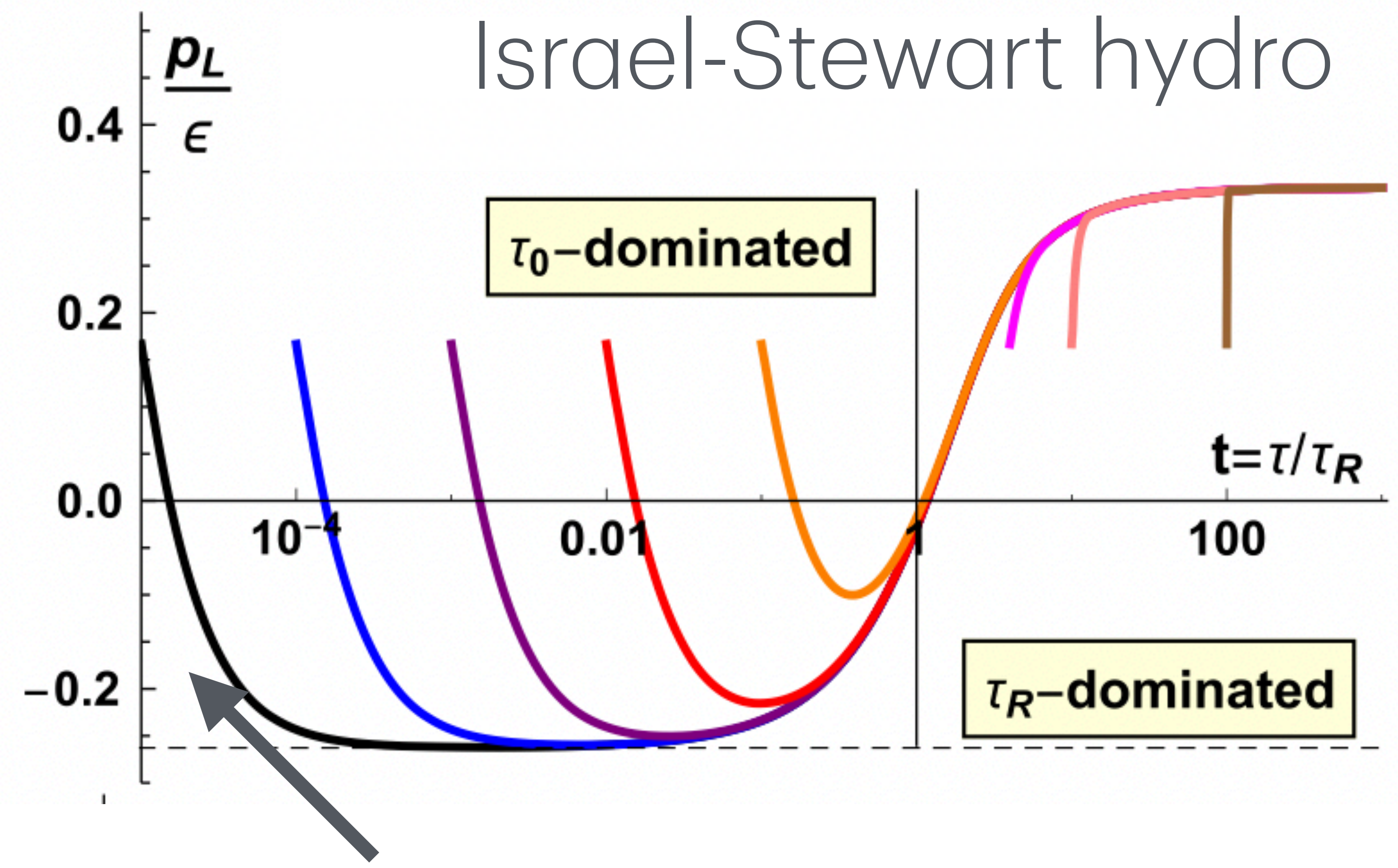
$$\frac{d\mathscr{L}_1}{d\tau} = -\frac{1}{\tau}\left[a_1\mathscr{L}_1 + b_1\mathscr{L}_0 + \cancel{c_1\mathscr{L}_2}\right] - \frac{\mathscr{L}_1}{\tau_R}$$

Free-streaming Collisions

- ISH are extensively used in heavy-ion simulations.
- ISH captures qualitative features of far-off-equilibrium dynamics.
- By modifying a coefficient to reproduce fixed point in collisionless regime, one can obtain nice matching with kinetic theory. Blaizot and Yan



Kurkela, van der Schee, Wiedemann, Wu, PRL (2020)



(Negative pressure!)

The Maximum-Entropy framework

C.C., Heinz, Schaefer, PRC 108 (2023), 034907

- How to formulate a (3+1)-d far-from-equilibrium macroscopic theory? Transverse gradients will also initiate flow. Fixed points not known apriori, should work irrespective of symmetries of flow.
- To evolve components of $T^{\mu\nu} = e\, u^\mu\, u^\nu - (P + \Pi)\, \Delta^{\mu\nu} + \pi^{\mu\nu}$

$$\dot{e} = -\,(e + P + \Pi)\, \nabla_\mu u^\mu + \pi^{\mu\nu}\, \nabla_{(\mu} u_{\nu)}$$ (energy density evolution)

$$(e + P + \Pi)\, \dot{u}^\mu = \nabla^\mu P + \cdots$$ (velocity evolution)

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_R} = 2\,\eta\, \nabla^{\langle\mu}\, u^{\nu\rangle} - \frac{4}{3}\, \pi^{\mu\nu}\, \nabla_\mu u^\mu \cdots - 2\, \rho^{\mu\nu\alpha\beta}_{(-2)}\, \nabla_\alpha u_\beta$$ (shear evolution)

Denicol, Niemi, Molnar, Rischke PRD (2012), Jaiswal, Bhalerao, Pal (2014)

Similar eq. for bulk pressure

- Need an evolution equation for $\rho^{\mu\nu\alpha\beta}_{(-2)}$. This leads to an infinite tower of coupled equations.

Requires truncation, i.e., to construct $f(x,p)$ using knowledge of $T^{\mu\nu}$.

The maximum-entropy distribution

E. Jaynes, Phys. Rev. 106, 620 (1957)

The least biased distribution that uses all of, and only the information provided by $T^{\mu\nu}$ is the one that maximizes the non-equilibrium entropy

$$s[f] = -\int dP\ (u\cdot p)\ \left(f\ \log(f) - f\right)$$

subject to constraints that f(x,p) satisfies,

$$\int dP\ (u\cdot p)^2\ f = e, \quad -\frac{1}{3}\int dP\, p_{\langle\mu\rangle}\, p^{\langle\mu\rangle}\, f = P + \Pi, \quad \int dP\, p^{\langle\mu}\, p^{\nu\rangle}\, f = \pi^{\mu\nu}$$

Introduce Lagrange multipliers $(\Lambda, \lambda_\Pi, \gamma_{\langle\mu\nu\rangle})$ corresponding to constraints and solve for the functional derivative $\dfrac{\delta s[f]}{\delta f} = 0$

C.C., Heinz, Schaefer, PRC 108 (2023), 034907,
Everett, C.C., Heinz, PRC (2021), 064902

$$\boxed{f_{\mathrm{ME}} = \exp\left[-\Lambda\ (u\cdot p) + \frac{\lambda_\Pi}{u\cdot p}\, p_{\langle\alpha\rangle}p^{\langle\alpha\rangle} - \frac{\gamma_{\langle\alpha\beta\rangle}}{u\cdot p}\, p^{\langle\alpha}p^{\beta\rangle}\right]}$$

Features of Max-Entropy distribution

In the fluid rest-frame $$f_{\mathrm{ME}} = \exp\left[-\Lambda\, E_p + \frac{\lambda_\Pi}{E_p}\,\vec{p}^{\,2} - \frac{\gamma_{\langle ij\rangle}}{u\cdot p}\, p^{\langle i}p^{j\rangle}\right]$$

Plays role similar to an inverse temperature

Isotropic deviation from equilibrium

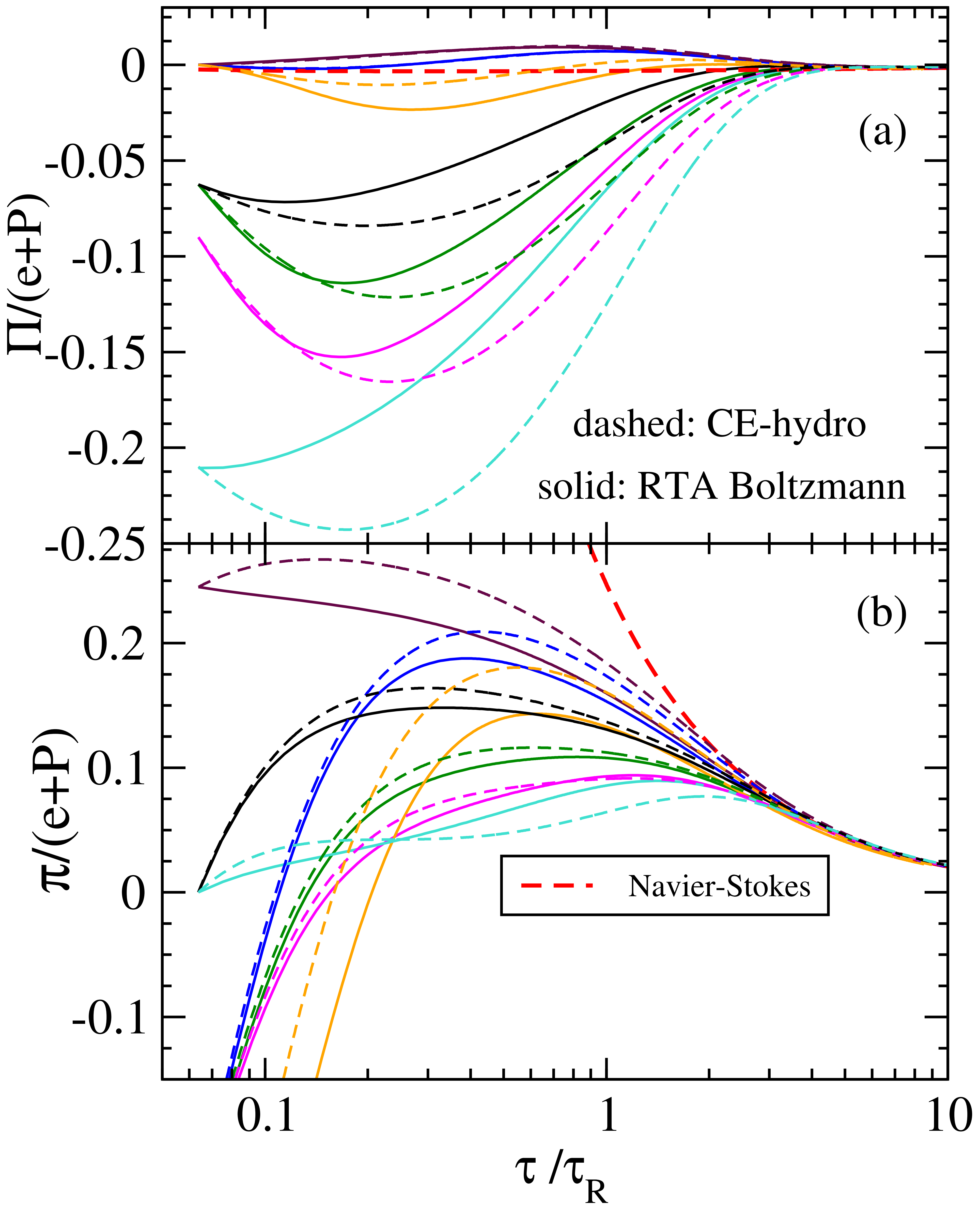
Anisotropic deviation from equilibrium

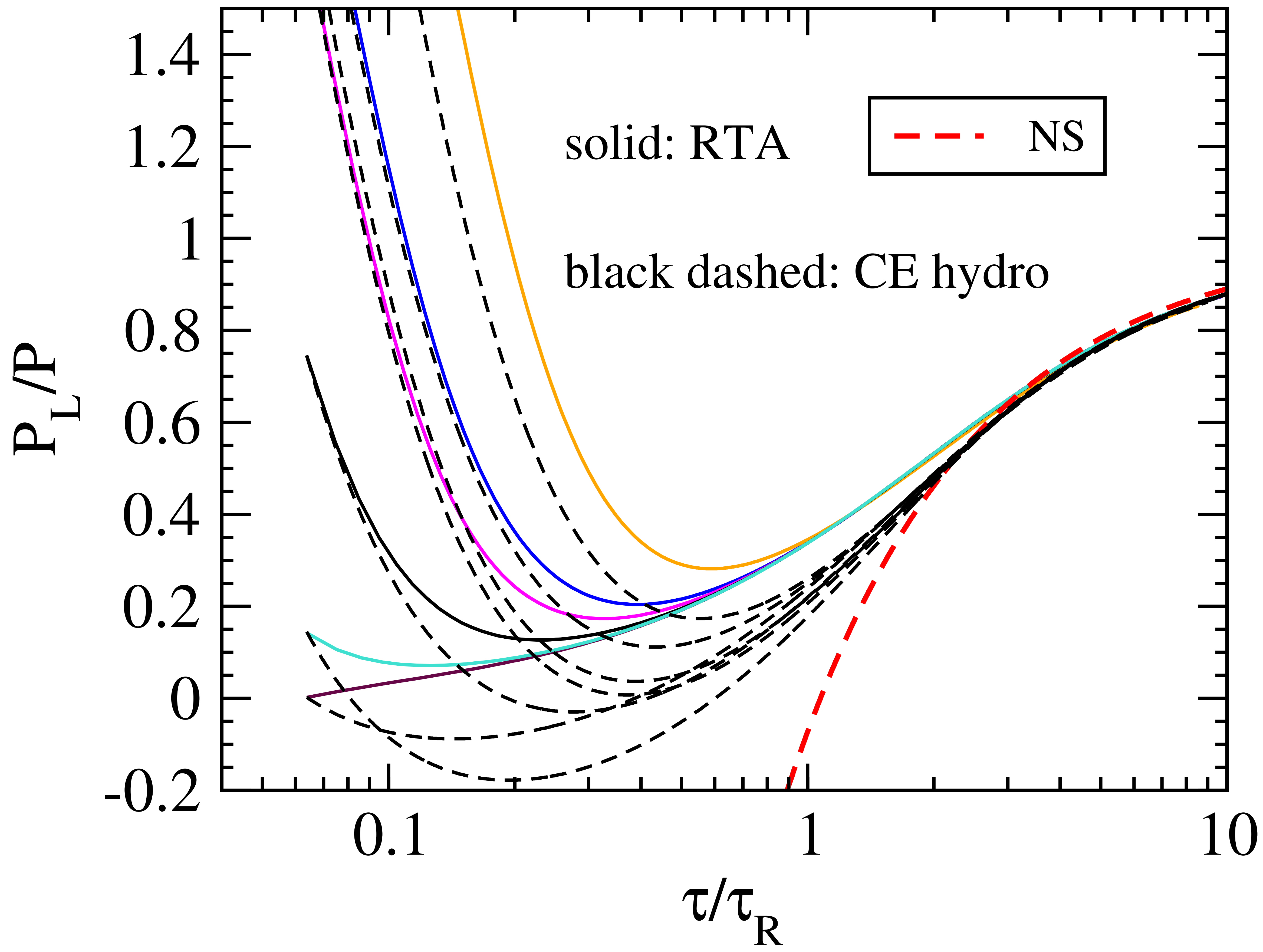
- Positive-definite for all momenta
- Non-linear dependence on shear and bulk stresses
- Reduces to the Chapman-Enskog δf in the limit of small viscous stresses.
- Ensuing dynamical framework consistent with the second-law.

See also, "Maximum-entropy freezeout" by Pradeep and Stephanov, PRL (2023)

Standard Israel-Stewart hydro

S. Jaiswal, C.C., et al, PRC 105, 024911 (2022)

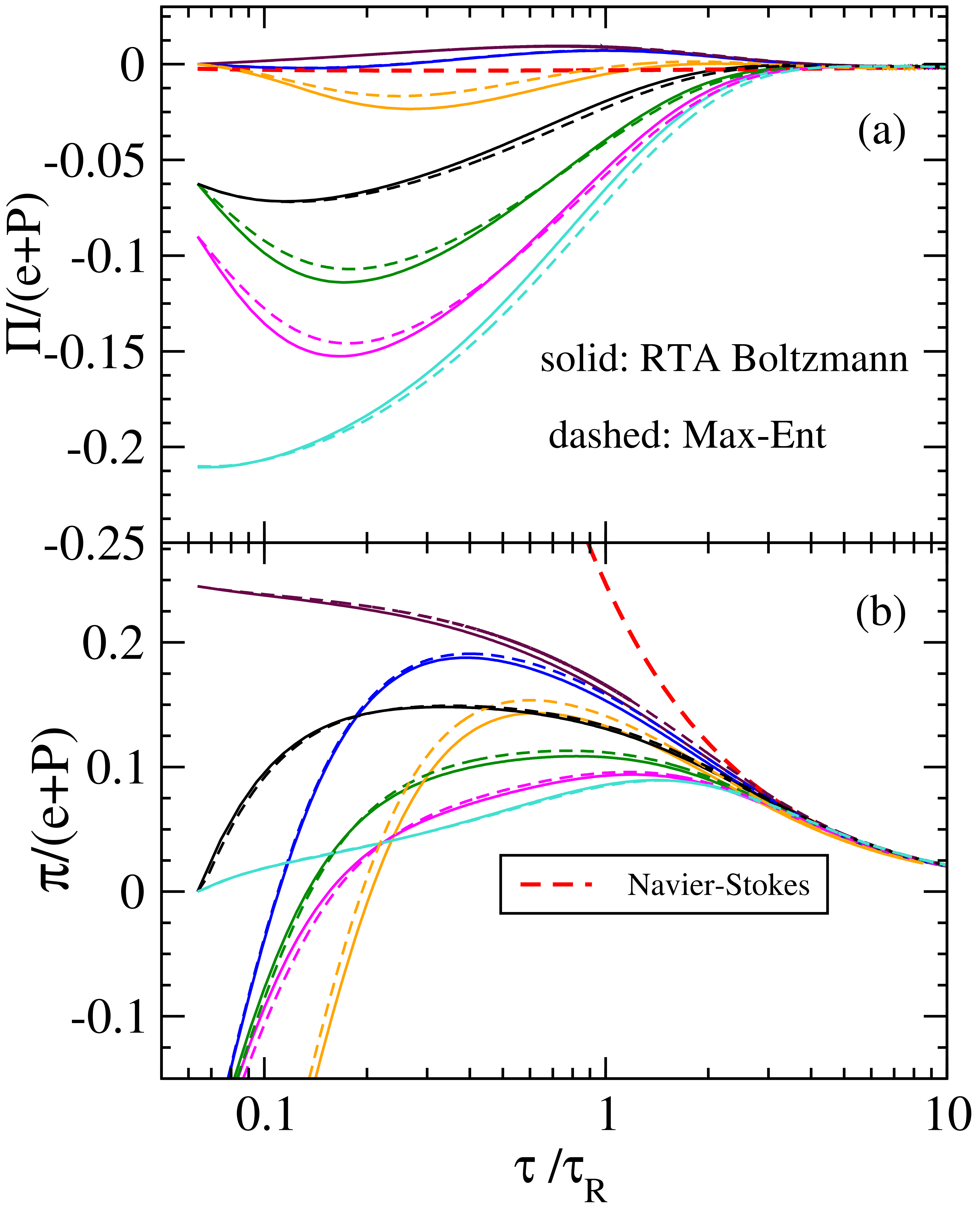


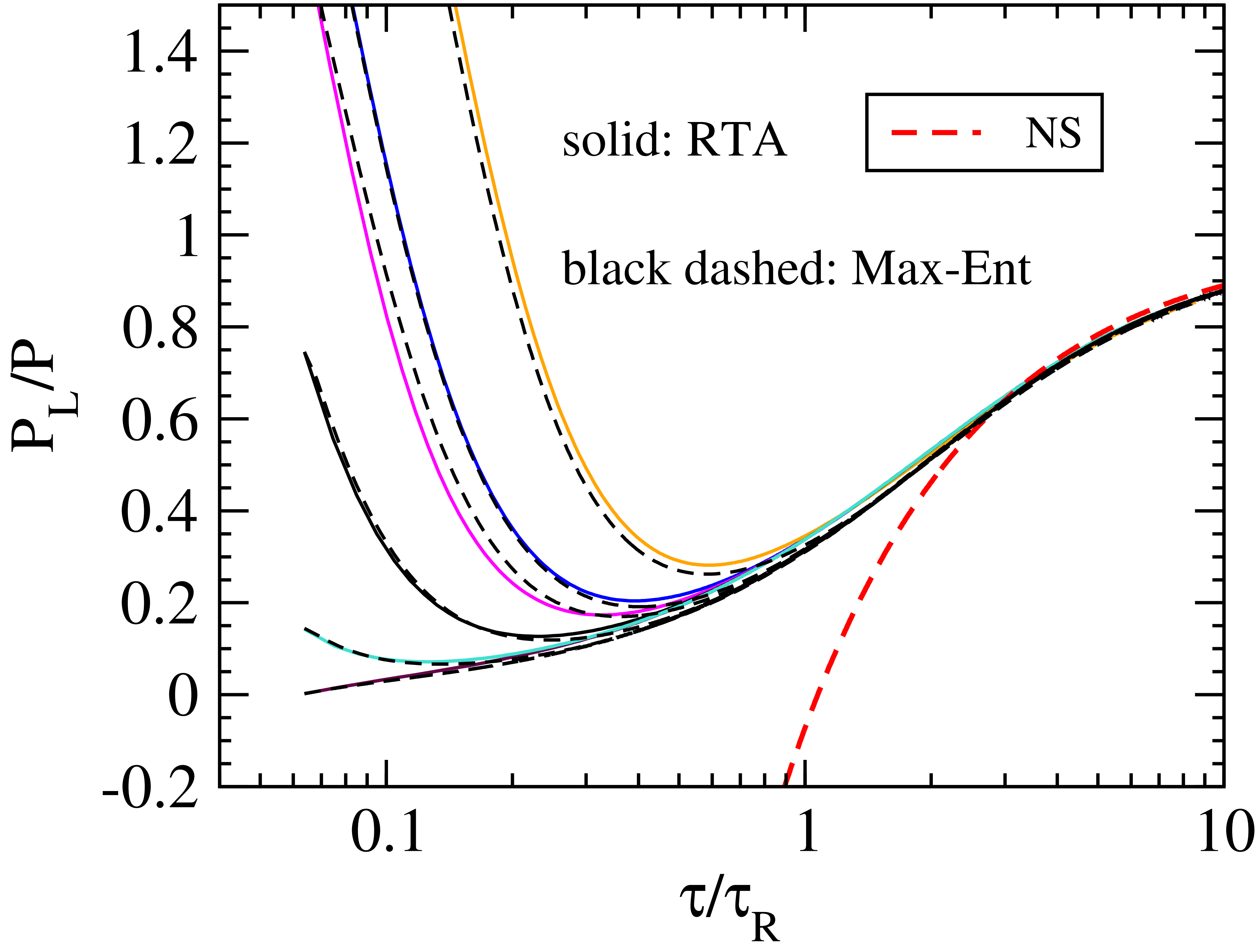


- Standard hydro is not in good agreement with kinetic theory at large Knudsen numbers.
- Does not describe early time universality accurately

Maximum-Entropy framework

C.C., Heinz, Schaefer, PRC 108 (2023), 034907





- Max-Ent is in good agreement with kinetic theory even at large Knudsen numbers.

- Accurately describes early time universality.

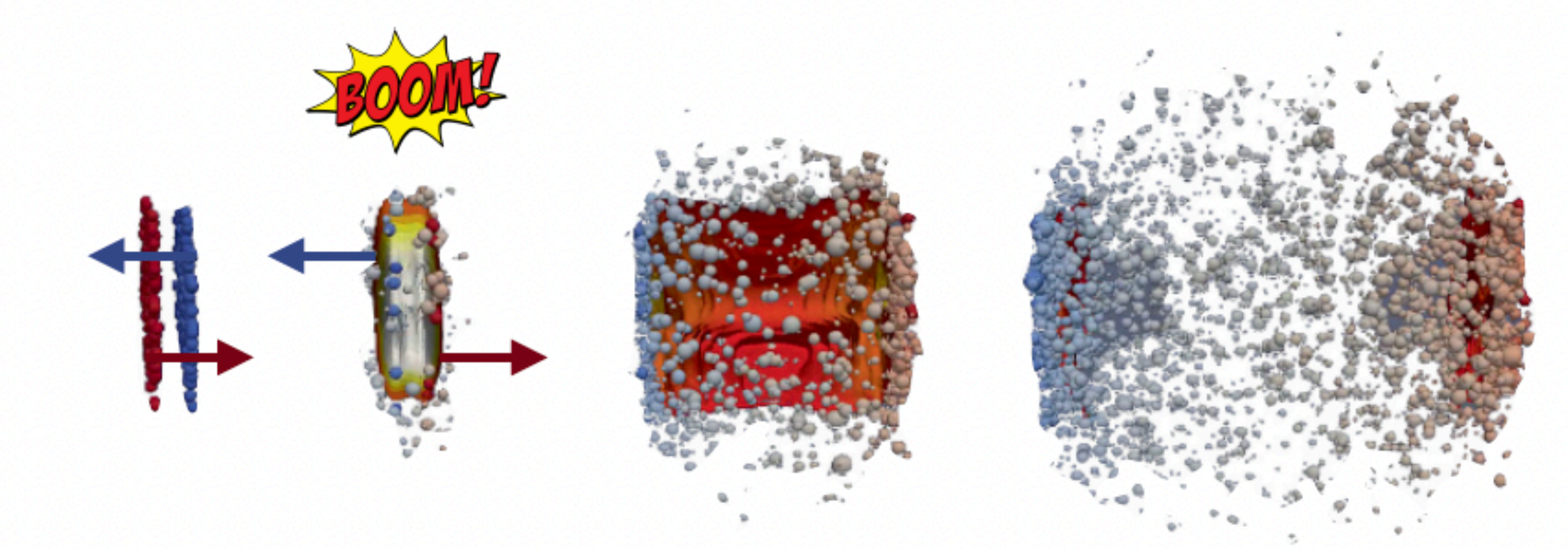
Summary: Part I (out of equilibrium dynamics in initial stages of HIC)

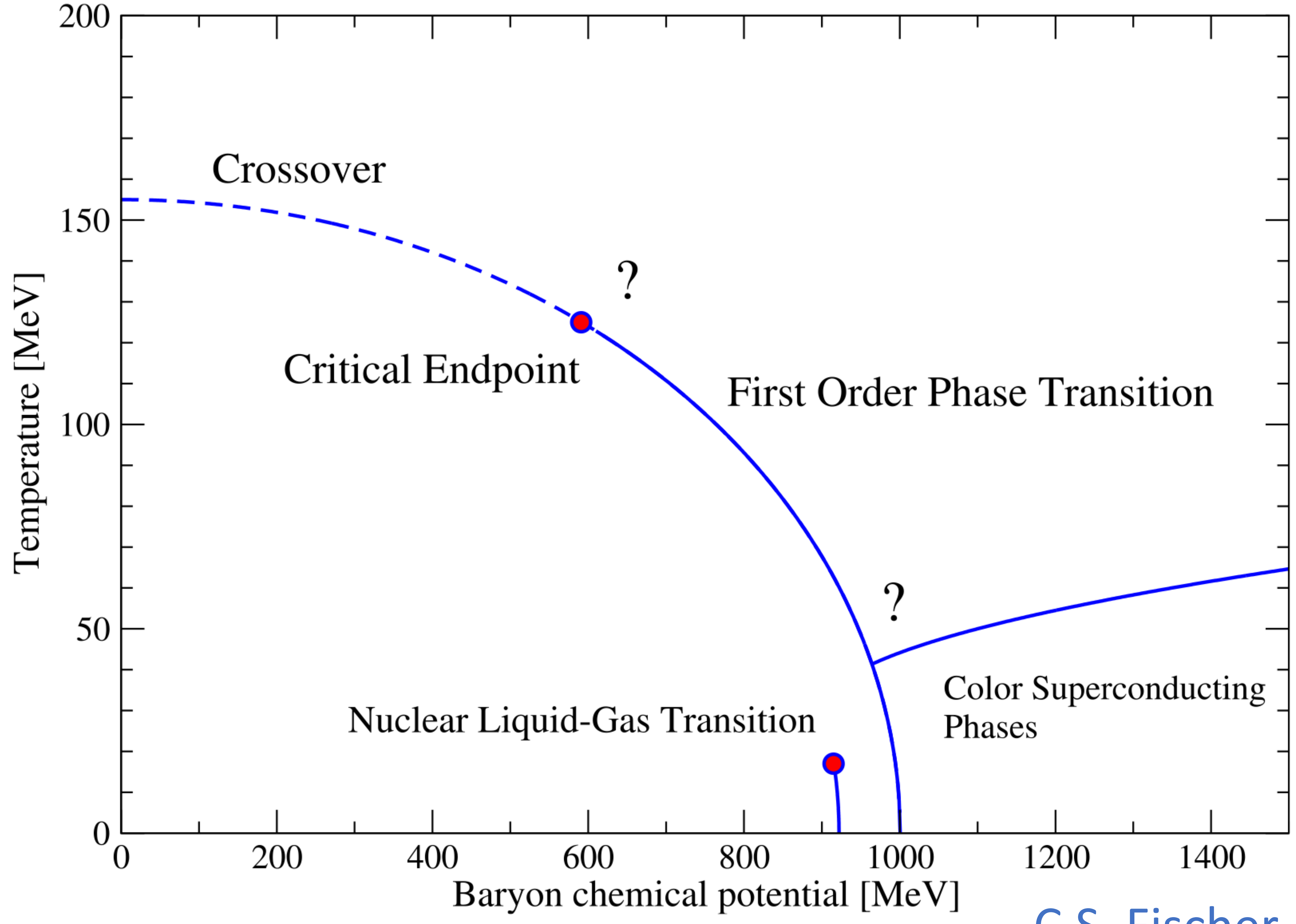
- If the pre-hydrodynamic evolution admits a kinetic theory description, Israel-Stewart like "hydro" frameworks may capture certain aspects of the macroscopic dynamics even far-from-equilibrium.

- The framework of maximum-entropy may serve as a proxy for kinetic theory as far as describing evolution of $(T^{\mu\nu}, J^{\mu})$ is concerned. Need for (3+1)-d simulations to test this expectation.

Part II: Out-of-equilibrium dynamics near a critical point

Out-of-equilibrium dynamics near critical point



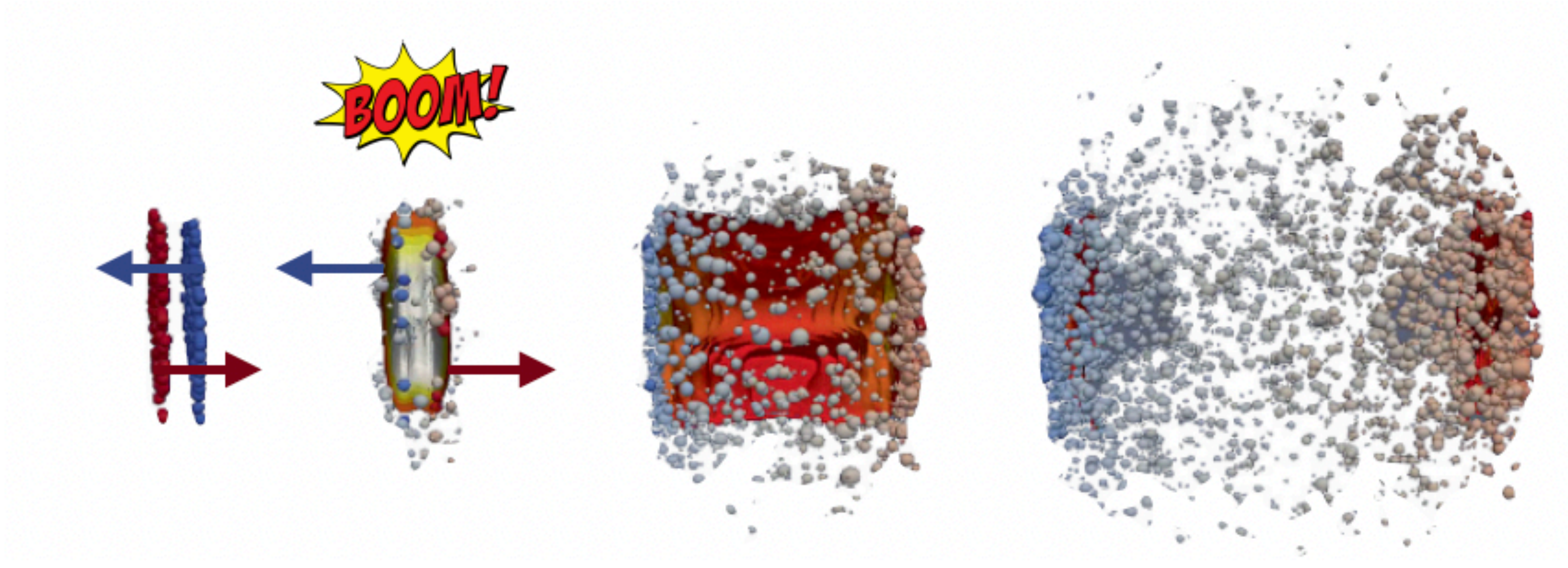


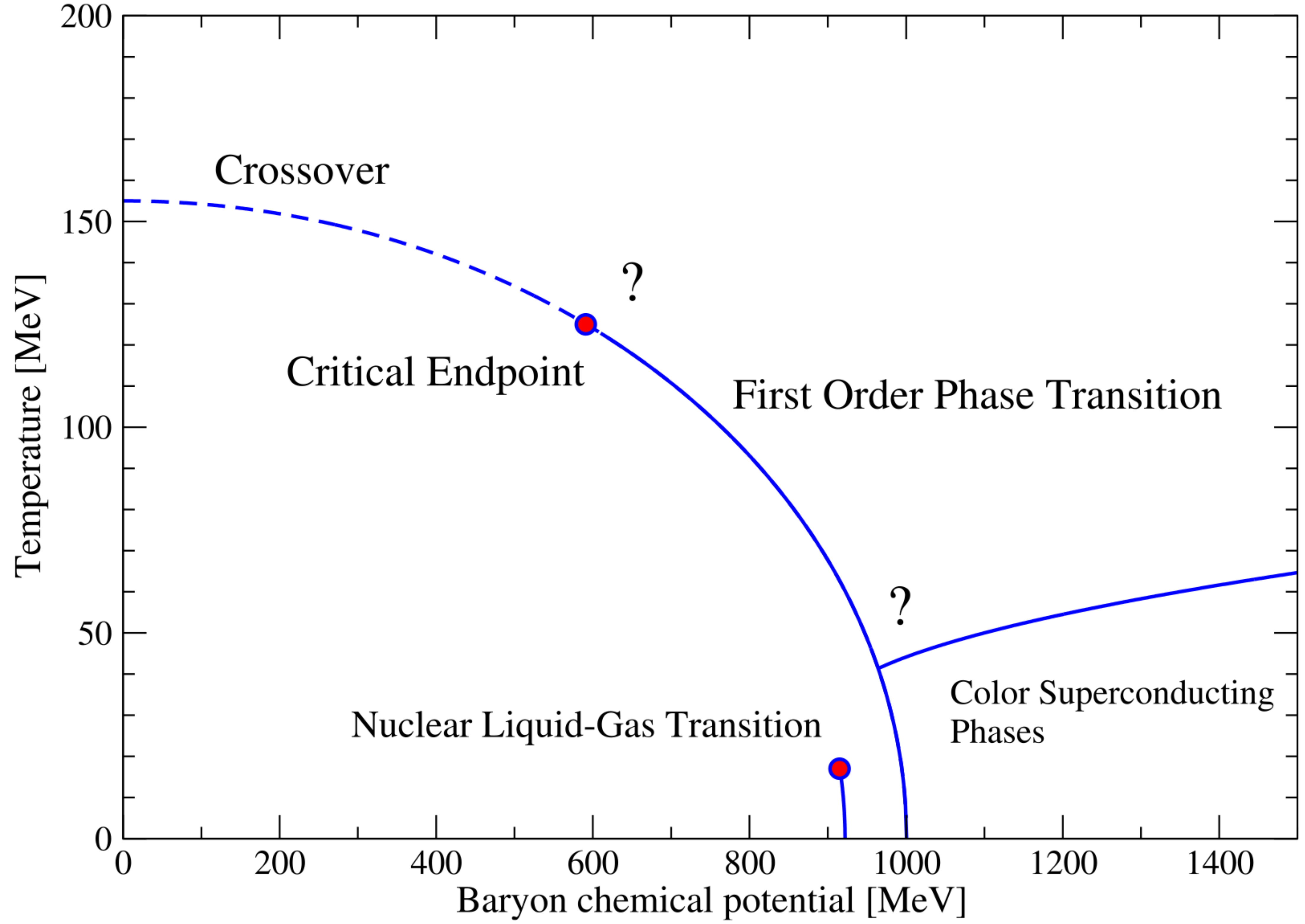
C.S. Fischer, Prog. Part. Nucl. Phys. 105, 1 (2019)

- Long-term goal of BES: Identify signatures of a possible critical end point of QCD using heavy-ion collisions. Talks by B. Mohanty, A. Pandav
- Near a critical point, fluctuations become dominant. But fluctuations not equilibrated as fireball is rapidly expanding. Talk by M. Pradeep
- Need for a dynamical theory of critical fluctuations.
- Fluid dynamics should still be applicable, but with appropriate modifications:
  - Inclusion of thermal fluctuations, slow dynamics of order parameter, and criticality in equation of state.

Talk by J. Goswami

Critical Dynamics





- Dynamics of critical fluctuations are universal.
- Hence, study QCD critical dynamics using the simplest system from the same dynamic universality class.
- Universality class depends on
  - Order parameter being conserved/non-conserved.
  - Coupling of order parameter to other slow modes, eg, hydrodynamic modes.
- QCD critical point shares the same static universality class as the 3d Ising Model

The basic idea

- The properties of a fluid are defined by slow, macroscopic degrees of freedom: conserved densities, i.e., densities of energy, momentum, or any conserved charge.
- If a fluid is near a critical point, the dynamics of its order parameter becomes slow (critical slowing down). Must be included in the hydrodynamic description.

Hohenberg & Halperin

- The macroscopic fields fluctuate as they couple to microscopic degrees of freedom.
- The theory to be solved is then stochastic hydrodynamics coupled to an order parameter.
  - Such theories are classified by Hohenberg & Halperin: purely relaxational dynamics (Model A), critical diffusion (Model B), critical anti-ferromagnet (Model G), critical diffusion coupled to Navier-Stokes (Model H).

relevant to QCD

Rajagopal and Wilczek

Son and Stephanov

Previous works

- Use framework of non-critical stochastic hydro and include criticality in EOS and transport coefficients.
  - Deterministic approaches: The above framework can be used in linearized regime to write deterministic eqs for n-point equal time functions: Hydro+, Hydro++, hydro-kinetics.

    Stephanov, Yin, X. An, Akamatsu, Teaney, Mazeliaukas, F. Yan, H. U. Yee, Martinez, Schaefer…

  - Extend them to critical regime by replacing susceptibilities and relaxation-rates by their critical expectations. Numerical studies of one-dimensional expanding systems.

    M. Nahrgang et al., G. Pihan et al. , M. Bluhm, L. Du, Heinz and others

- Not many studies of direct simulation of critical fluid dynamics. A novel approach to simulate stochastic dynamics based on Metropolis has been recently formulated.

  Florio, Grossi, Soloviev, Teaney, Schaefer, Skokov, Basar, Bhambure, Singh, Newhall et al

Stochastic dynamics: deterministic approach

- Hydro equations are conservation eqs: $\partial_\mu T^{\mu\nu} = 0,\ \partial_\mu J^\mu = 0$ Stephanov, Yin, X. An, Basar, Akamatsu, Teaney, Mazeliaukas, F. Yan, H. U. Yee, Martinez, Schaefer…

$$\partial_t \psi = -\nabla \cdot \mathrm{Flux}[\psi]$$

- Stochastic variables $\tilde{\psi} = (\tilde{T}^{0i}, \tilde{J}^0)$ are local operators coarse-grained (over cells b: $(l \ll b \ll L)$)

$$\partial_t \tilde{\psi} = -\nabla \cdot \left(\mathrm{Flux}[\psi] + \mathrm{Noise}\right)$$

Landau-Lifshitz

- Now, variables are one-point and two-point functions:

$\psi = \langle \tilde{\psi} \rangle$ and $G = <\tilde{\psi}\tilde{\psi}> - <\tilde{\psi}><\tilde{\psi}>$ (Equal time correlation)

- Due to non-linearities fluxes depend on G

$\partial_t \psi = -\nabla \cdot \mathrm{Flux}[\psi, G]$ (Conservation) $\partial_t G = L[G;\psi]$ (Relaxation)

- Typically, the slowest hydro mode is included $G = \langle \delta m(x_1)\, \delta m(x_2) \rangle$ where $m = s/n$. Approach used in expanding systems Akamatsu et al, Rajagopal, Ridgway, Weller, Yin, M. Nahrgang et al., G. Pihan et al. , M. Bluhm, L. Du, Heinz and others

Stochastic dynamics: numerical approach

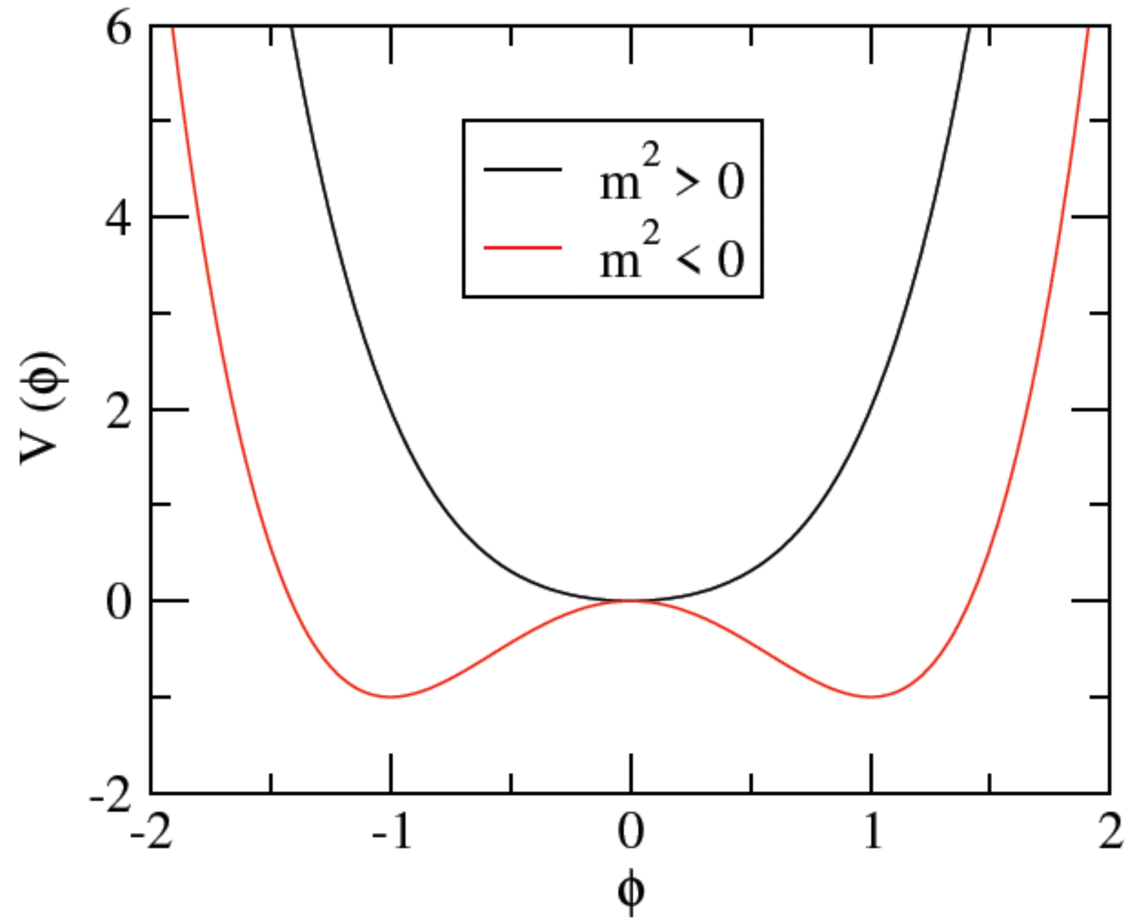
- First: critical diffusion of a conserved order parameter (Model B)
  - Simulation of diffusive dynamics using a Metropolis algorithm
  - Dynamic scaling in Model B
- Second: Coupling of the conserved order parameter to hydrodynamic modes (Model H)
  - Modification to dynamic scaling behavior compared to Model B
  - Effective shear viscosity of the fluid

Model B

- Consider the Ising model. Coarse grain the spin (microscopic) degrees of freedom to obtain an order parameter $\phi(x)$ (magnetization density).

- The statics of the system near the critical point (small ϕ) is governed by an effective free-energy functional (Ginzburg-Landau)

$$F[\phi] = \int d^3x \left[\frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4\right]$$



- Dynamics: If the order parameter is conserved, its evolution may be modeled as

$$\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0,$$ the current $$\vec{j} = -\Gamma\,\vec{\nabla}\frac{\delta F}{\delta\phi} + \vec{\xi}$$

Diffusion Noise

Noise ensures fluctuation-dissipation

$$\langle \xi^i(t,\vec{x})\,\xi^j(t',\vec{x}')\rangle = 2\,\Gamma\,T\,\delta^{ij}\,\delta(t-t')\,\delta^3(\vec{x}-\vec{x}')$$

Metropolis step

- Choose trial updates at $\vec{x}$ and $\vec{x}+\hat{\mu}$ (conserves ϕ)

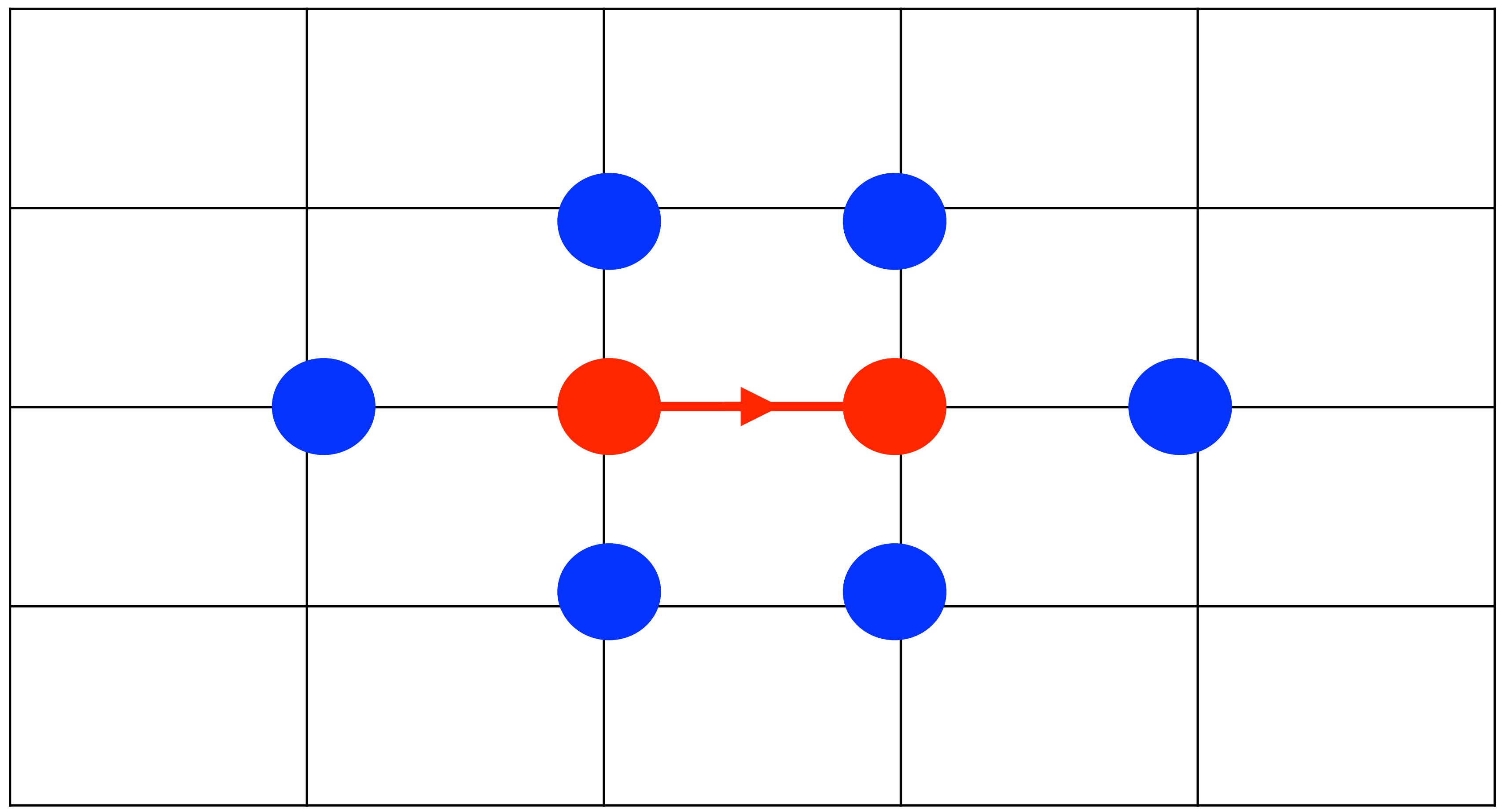
$$\phi^{\text{trial}}(t+\Delta t,\vec{x}) = \phi(t,\vec{x}) - q_\mu, \quad \phi^{\text{trial}}(t+\Delta t,\vec{x}+\hat{\mu}) = \phi(t,\vec{x}+\hat{\mu}) + q_\mu$$

$$q_\mu = \sqrt{2\,\Gamma\,T\,\Delta t}\;\xi_\mu$$

- Compute the change in free energy due to these updates

$$F[\phi] = \int d^3x \left[\frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4\right]$$

- Accept with probability $P = \min(1, \exp(-\Delta F/T))$



The Metropolis scheme

- The Metropolis update reproduces the flux on average, and also its variance

$$\langle \vec{q} \rangle = -\Delta t \, \Gamma \, \overrightarrow{\nabla} \frac{\delta F}{\delta \phi} + \mathcal{O}(\Delta t^2)$$

$$\langle \vec{q}^2 \rangle = 2\Gamma \, T \, \Delta t + \mathcal{O}(\Delta t^2)$$

- Probability of a new configuration,

$$P\left(\phi(t,\vec{x}) \to \phi^{new}(t,\vec{x})\right) \sim \exp\left[-\left(F[\phi^{new}] - F[\phi]\right)\right]$$

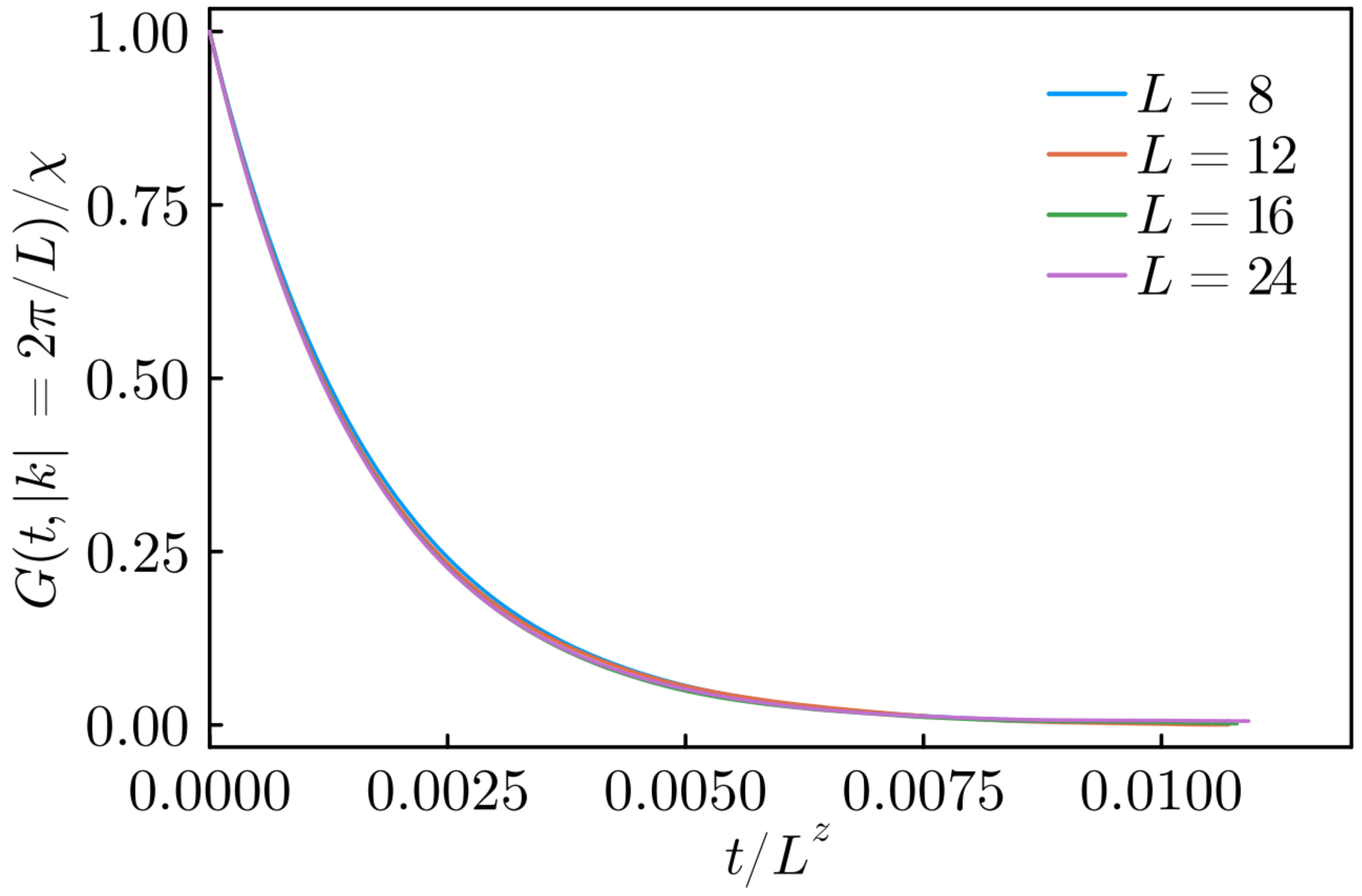
irrespective of order of updates.

- The equilibrium distribution $\exp(-F[\phi]/T)$ is sampled even if Δt is not small.

- If Δt is not small, the diffusion eq. is approximately realized.

Results: Dynamic scaling

C.C., J. Ott, T. Schaefer, V. Skokov (PRD 108 (2023) 074004)



Data collapse occurs for $z \approx 3.97$. Theoretical expectation $z = 4 - \eta, \eta \approx 0.03$

- Scaling Hypothesis: Near a critical point the dynamic correlator, $\langle \phi(0, k)\, \phi(t, -k) \rangle$

$$G(t, k) = \tilde{G}(t/\xi^z, k\xi)$$

$\tilde{G}$ is a universal function.

- At the critical point $\xi \sim L$, thus $G(t, k)$ obtained in different volumes should collapse

$$G(t, k = 2\pi/L) \to \tilde{G}\left(\frac{t}{L^z}, 2\pi\right)$$

if time is scaled by L^z.

- z is the dynamic scaling exponent

Critical dynamics in Model H

- Couple the order parameter ϕ to a fluid's momentum density $\vec{\pi}$

$$\frac{\partial \phi}{\partial t} = \Gamma \nabla^2 \frac{\delta H}{\delta \phi} - \left(\nabla \phi \cdot \frac{\delta H}{\delta \vec{\pi}_T} \right) + \zeta$$

diffusion advection noise

- Stochastic evolution equation of the momentum density

$$\frac{\partial \vec{\pi}_T}{\partial t} = \eta \nabla^2 \frac{\delta H}{\delta \vec{\pi}_T} + \left(\vec{\nabla} \phi \right) \cdot \frac{\delta H}{\delta \phi} - \left(\frac{\delta H}{\delta \vec{\pi}_T} \cdot \vec{\nabla} \right) \vec{\pi}_T + \vec{\xi}$$

diffusion Stress energy of ϕ advection noise

- The Hamiltonian $H = \int d^3x \left[\frac{\vec{\pi}_T^2}{2\rho} + \frac{1}{2} \left(\vec{\nabla} \phi \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right]$

Coupling to a fluid (Model H)

- Couple the order parameter to a fluid's momentum density $\vec{\pi}$

$$\frac{\partial \phi}{\partial t} = \Gamma \nabla^2 \frac{\delta H}{\delta \phi} - \left(\nabla \phi \cdot \frac{\delta H}{\delta \pi_T} \right) + \zeta$$

- Evolution equation of the momentum density

$$\frac{\partial \vec{\pi}_T}{\partial t} = \eta \nabla^2 \frac{\delta H}{\delta \vec{\pi}_T} + \left(\vec{\nabla} \phi \right) \cdot \frac{\delta H}{\delta \phi} - \left(\frac{\delta H}{\delta \vec{\pi}_T} \cdot \vec{\nabla} \right) \vec{\pi}_T + \vec{\xi}$$

- The Hamiltonian

$$H = \int d^3x \left[\frac{\vec{\pi}_T^2}{2\rho} + \frac{1}{2} \left(\vec{\nabla} \phi \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right]$$

Assumptions:

- Non-relativistic fluid
- The momentum density is transverse $\vec{\nabla} \cdot \vec{\pi} = 0$

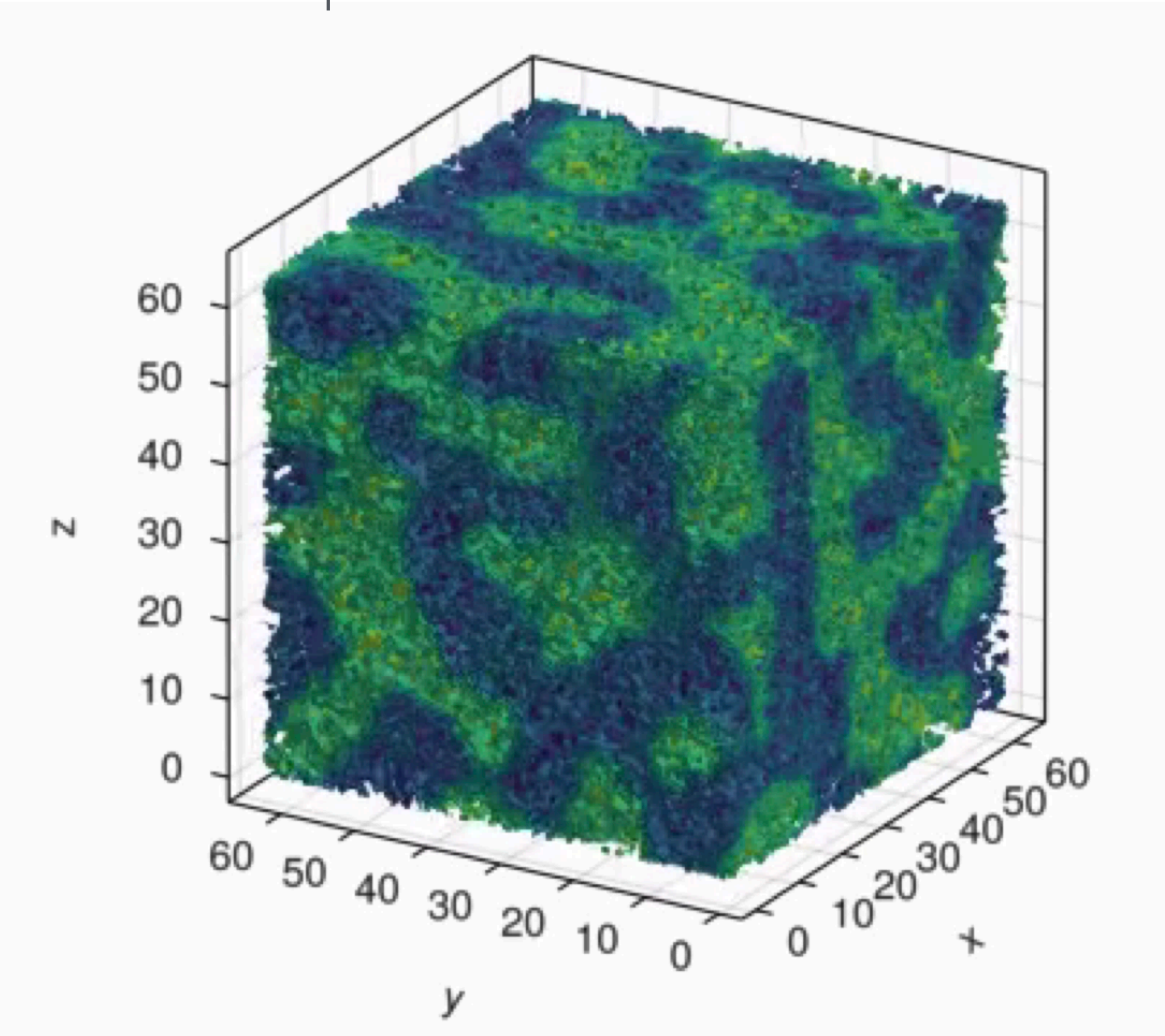
There are shear waves but no sound. No coupling to energy density or pressure.

Model H simulations

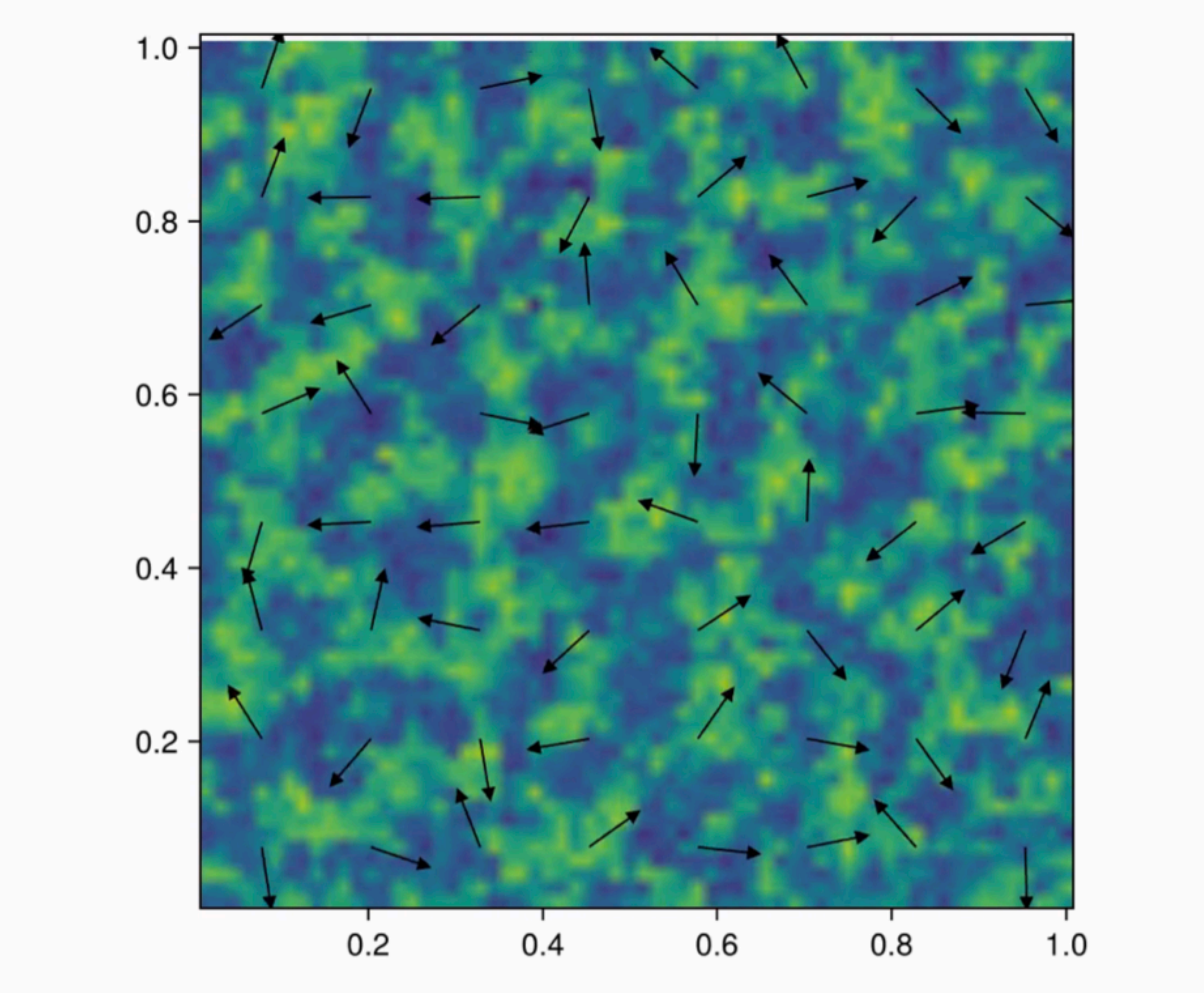
- Evolution consists of both stochastic/dissipative and conservative parts.
- Use Metropolis for the stochastic/dissipative update.

C.C., J. Ott, T. Schaefer, V. Skokov, arXiv:2411.15994

Order parameter field in 3d



Order parameter + velocity field in 2d



Simulations by Josh Ott

Effective viscosity

- Consider the time-dependent correlation function of the momentum density

C.C., J. Ott, T. Schaefer, V. Skokov PRL 133 (2024) 032301

$$\langle \pi_i^T(0,\vec{k})\,\pi_j^T(0,-\vec{k})\rangle \equiv C_{ij}(t,\vec{k}),$$

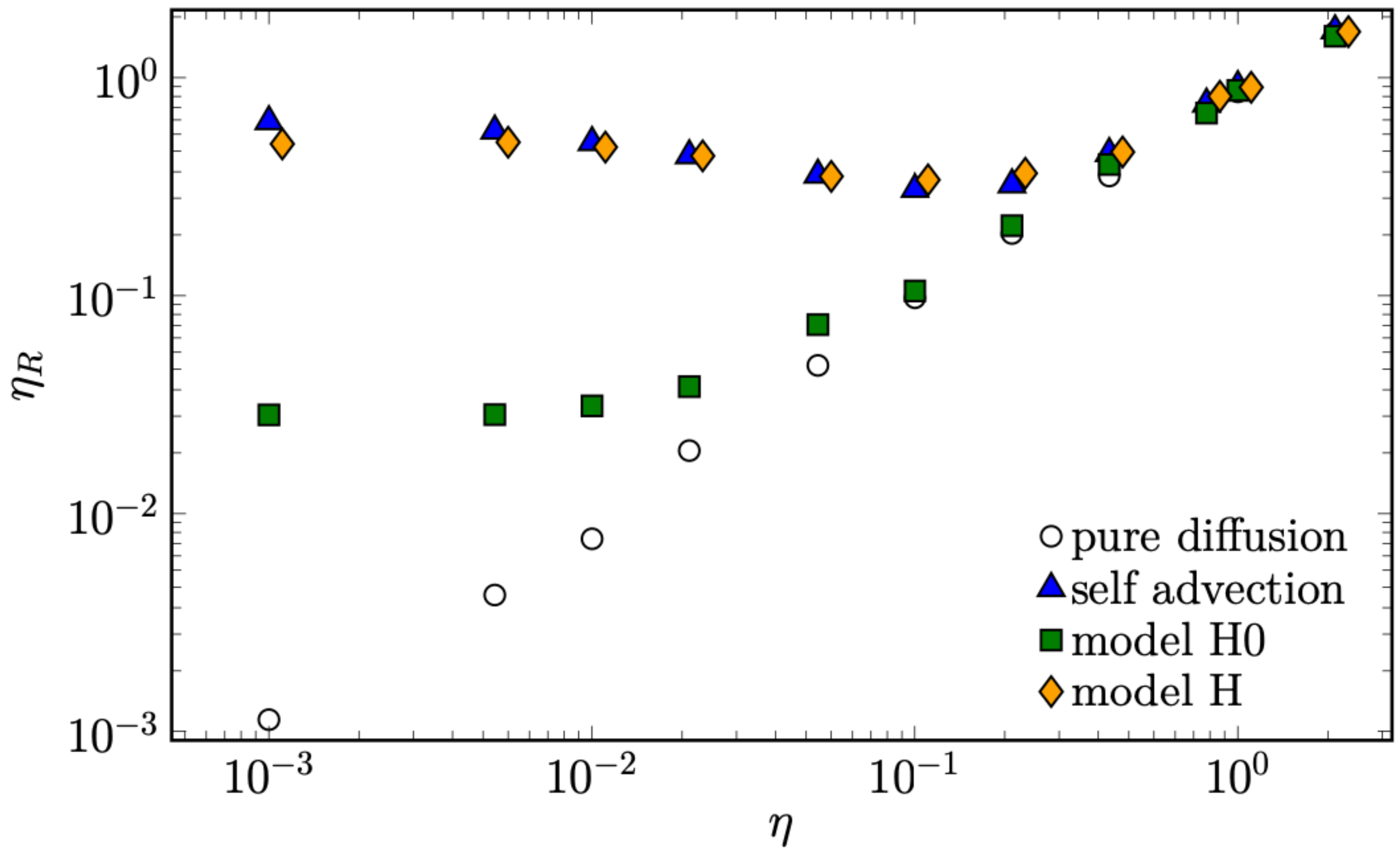
here $C_{ij}(t,\vec{k}) = \left(\delta_{ij} - \hat{k}_i\hat{k}_j\right) C_\pi(t,k)$

- In linearised hydro: $C_\pi(t,k) = \rho\, T \exp\left(-\frac{\eta}{\rho}k^2\, t\right)$

The "stickiness of shear"

$$\eta_R = \eta + \frac{7}{60\pi^2}\,\frac{\rho T\Lambda}{\eta}$$

Schaefer & Chafin



Thermal fluctuations + Non-linearity of hydro

$\Longrightarrow$ shear viscosity has a minimum

$$\frac{\partial \vec{\pi}_T}{\partial t} + \frac{\vec{\pi}_T}{\rho}\cdot\vec{\nabla}\,\vec{\pi}_T = \frac{\eta}{\rho}\nabla^2\vec{\pi}_T + \vec{\nabla}\,\phi\,\nabla^2\phi + \vec{\xi}$$

Self-advection dominates

In analogy to "stickiness of sound" Kovtun, Moore & Romatschke

Extraction of dynamic critical exponent

- Compute time dependent correlator of the order parameter

$$C(t,\vec{k}) = \langle \phi(0,\vec{k})\,\phi(t,-\vec{k})\rangle$$

at the critical point.

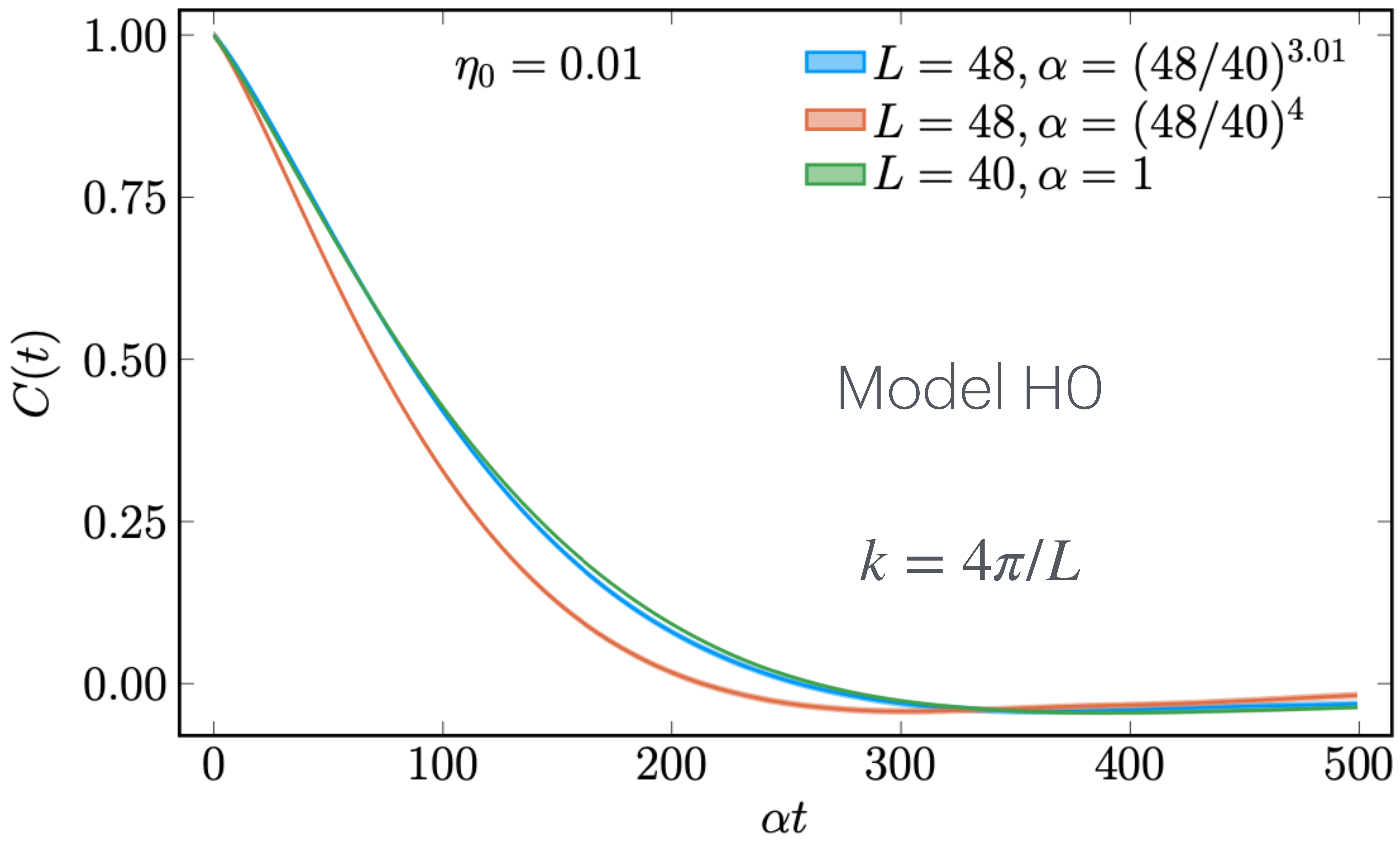
- a wave-number dependent relaxation rate is defined:

$$C(t,\vec{k}) \sim \exp(-\Gamma_k t)$$

- Dynamic scaling at critical point :

$$C(t,k) = \tilde{C}\left(t/L^z, k\,L\right)$$

- Hold kL fixed, vary lattice size. Extract z by looking for data collapse.

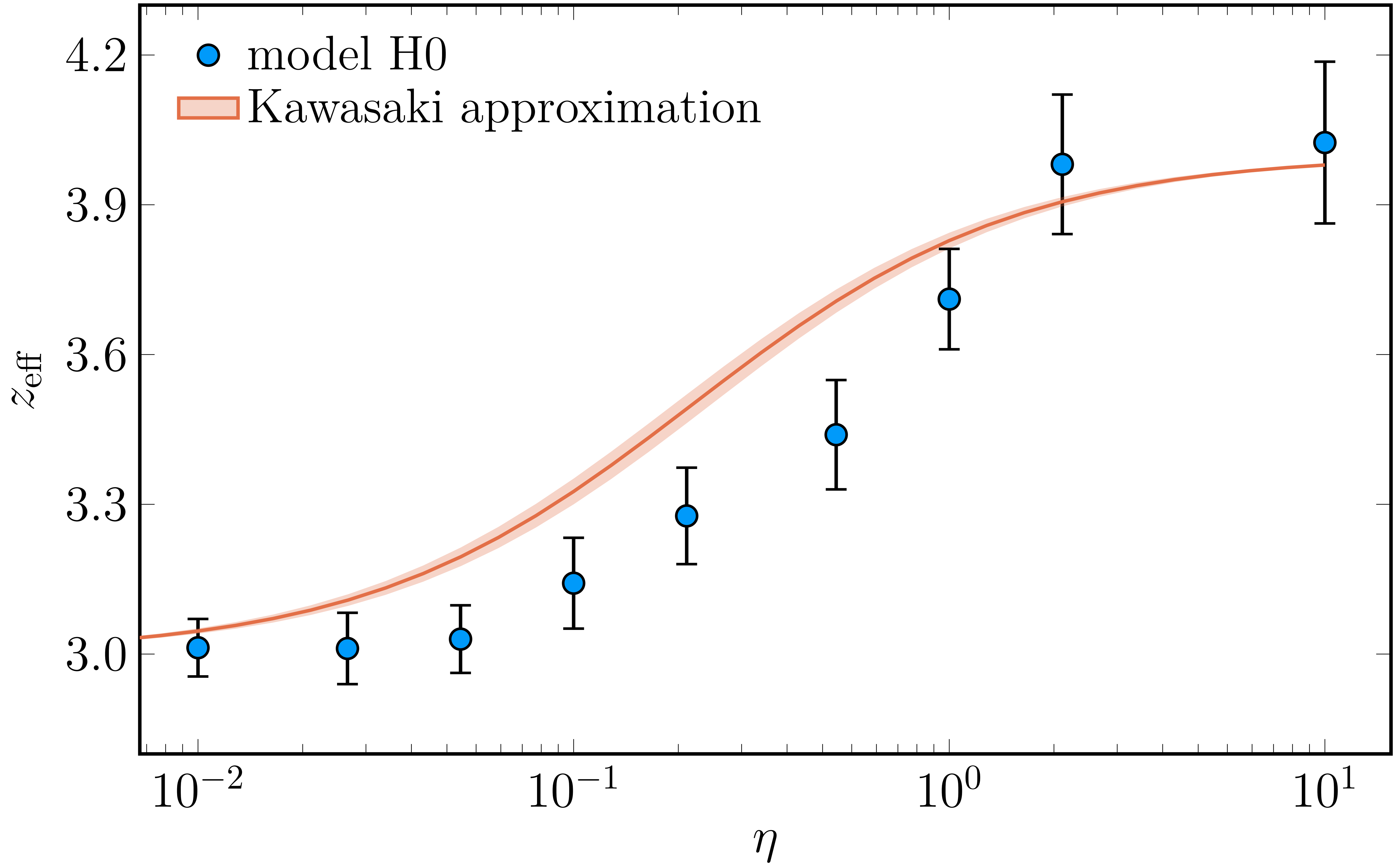


$$z\,(\eta = 0.01) = 3.01$$

Variation of z with η

- Extract z for various η

- In Model H0, η_R can become quite small.
    - Dynamic exponent crosses over from $z = 4$ (pure diffusion) to $z = 3$ (Model H expectation)



The Kawasaki approximation:

$$\Gamma_k = \frac{\Gamma}{\xi^4}\left(k\xi\right)^2\left(1 + (k\xi)^2\right) + \frac{T}{6\pi\eta_R\xi^3}K(k\xi)$$

Summary & Outlook

- Performed numerical simulations of stochastic fluid dynamics near a critical point. Observed renormalization of shear viscosity and dynamical scaling.
  - Self-coupling of momentum density is important in limiting the smallness of effective viscosity.
  - Dynamic scaling exponent depends sensitively on value of correlation length and effective shear viscosity.
  - Pure Model H behavior $z \approx 3$ requires both large ξ and small η_R .

To generalize this to relativistic fluids with non-trivial expansions and cooling, inclusion of sound modes and critical equation of state.

Thank you!

Extra Slides

The Maximum-Entropy framework

- To re-construct δf solely using quantities appearing in $T^{\mu\nu}$, i.e., $(e, u^\mu, \pi^{\mu\nu}, \Pi)$

- What is the most probable distribution? Let there be several micro states i with probabilities P_i. The Shannon entropy is given by

$$S = -\sum_i P_i \log(P_i)$$

For the kinetic distribution function f(x,p) the non-equilibrium entropy density is given by

$$s = -\int dP\ (u \cdot p)\ (f\ \log f - f)$$

De Groot, van Leeuwen, van Weert, Relativistic Kinetic theory

Holds for Boltzmann particles. Can be generalized for Fermi-Dirac or Bose particles

Model B in mean-field approximation

- In the free-energy functional set $\lambda = 0$

$$F[\phi] = \int d^3x \left[\frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4\right]$$

- Evolution of ϕ becomes linear. The equal-time correlator $N_k(t) = \langle\phi(t,\vec{k})\,\phi(t,-\vec{k})\rangle$ satisfies

$$\frac{\partial N_k}{\partial t} = -2\Gamma_k(N_k - N_k^{eq})$$

Equilibrium correlator $N_k^{eq} = \dfrac{T}{k^2+m^2}$ and relaxation-rate $\Gamma_k = \Gamma\,k^2(k^2+m^2)$

- Near $m^2 = 0$, mean-field predicts $\Gamma_k \sim k^z$ with a dynamic exponent $z = 4$.

- Later: interactions, coupling of ϕ to hydro modes lead to modifications from $z = 4$.

Model B: the non-linear case

- Interactions renormalize m^2. For chosen values of (T, λ) it is possible to tune m^2 to hit the critical point.

$$F[\phi] = \int d^3x \left[\frac{1}{2} \left(\nabla \phi \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right]$$

- To determine m_c^2 for an infinite system from finite volume calculations. Quantities like $\langle M^2 \rangle$, $\langle M^4 \rangle$ show peaks whose location depends on L.

- At the true critical point, leading order finite volume effects on the Binder cumulant U cancel

$$U \equiv 1 - \frac{\langle M^4 \rangle}{3(\langle M^2 \rangle)^2}$$

- Model B configs have long thermalization time $\tau_R \sim L^z$ with $z \approx 4$.

- Determine m_c^2 using Model A (purely relaxational dynamics), lies in same static universality class, easier to thermalize $\tau_R \sim L^2$.

T. Schaefer and V. Skokov PRD 014006 (2022)

Metropolis step for Model B

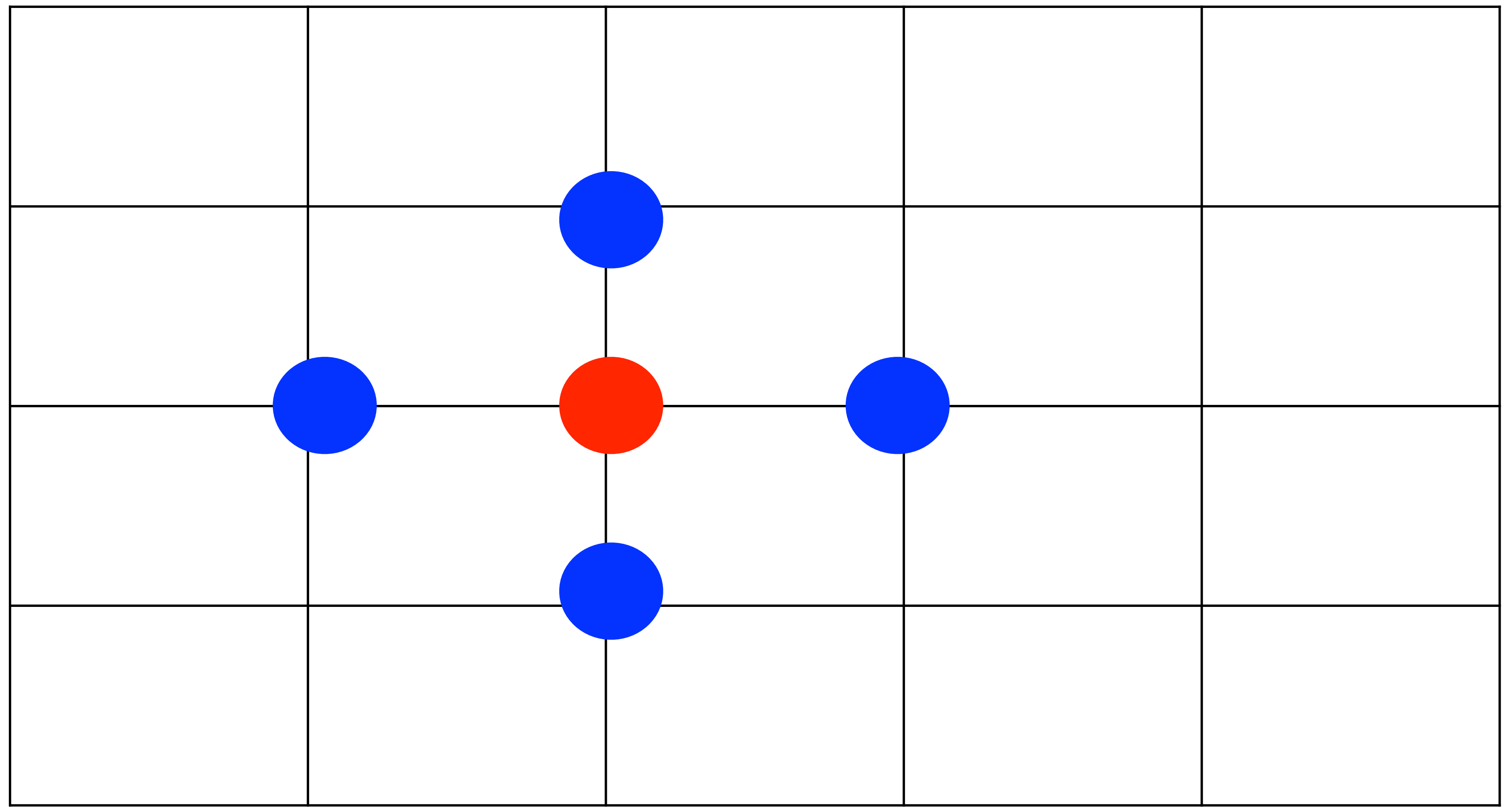
- Choose a trial update at $\vec{x}$ and $\vec{x} + \hat{\mu}$

$$\phi^{\text{trial}}(t + \Delta t, \vec{x}) = \phi(t, \vec{x}) - q_\mu, \qquad \phi^{\text{trial}}(t + \Delta t, \vec{x} + \hat{\mu}) = \phi(t, \vec{x} + \hat{\mu}) + q_\mu$$

$$q_\mu = \sqrt{2\,\Gamma\,T\,\Delta t}\;\xi_\mu$$

- The change in free energy $\Delta F(\vec{x}, \vec{x} + \hat{\mu}) = \Delta F(\vec{x}) + \Delta F(\vec{x} + \hat{\mu}) + q_\mu^2$

$$\Delta F(x) = \left(d + \frac{m^2}{2}\right)\left(\phi^2_{\text{trial}}(x) - \phi^2(x)\right) + \frac{\lambda}{4}\left(\phi^4_{\text{trial}}(x) - \phi^4(x)\right)$$
$$- \left(\phi_{\text{trial}}(x) - \phi(x)\right)\sum_{\hat{\mu}=1}^{d}\left(\phi(x + \hat{\mu}) - \phi(x - \hat{\mu})\right)$$



Metropolis step for Model B

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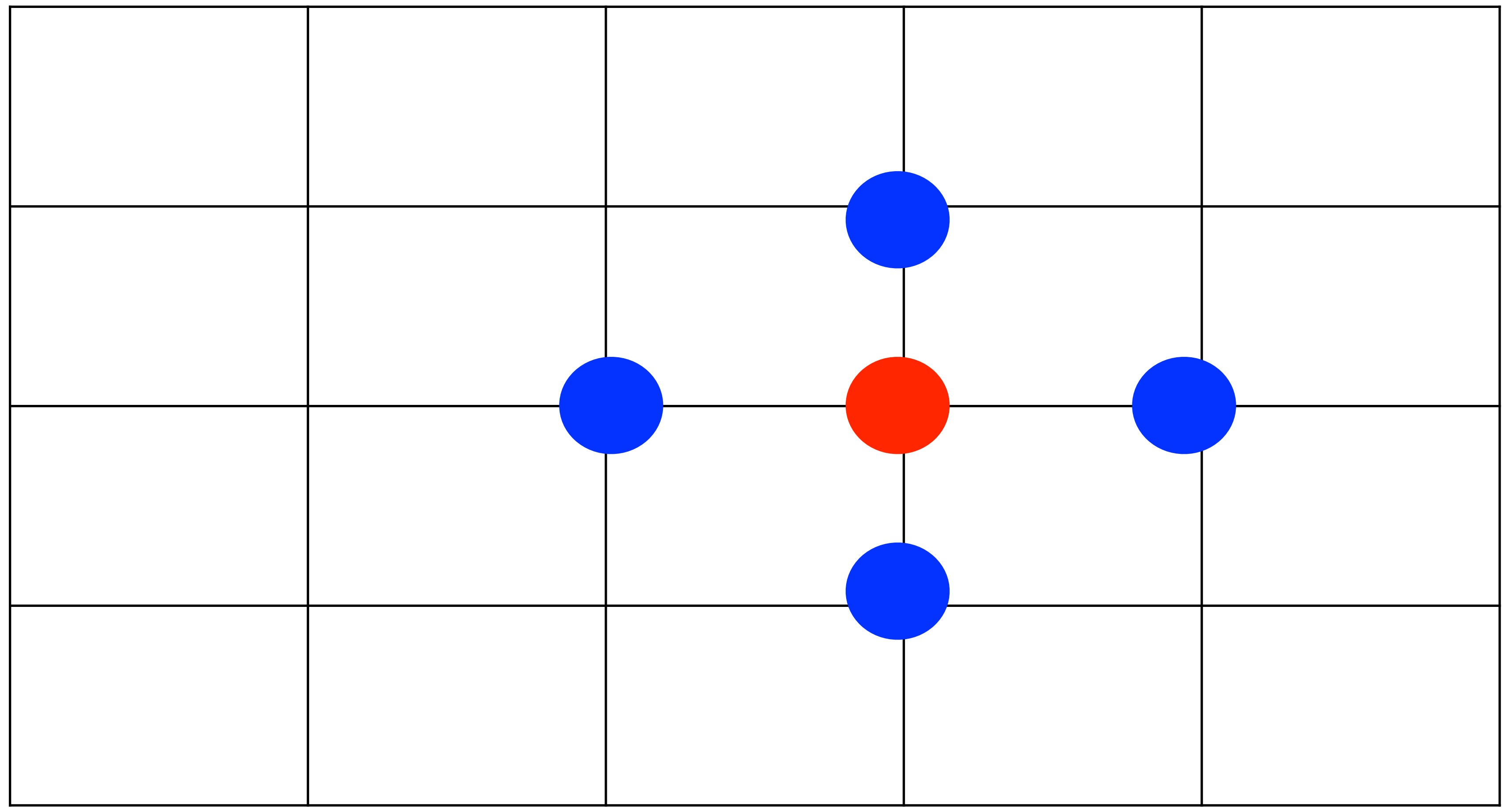
$$q_\mu = \sqrt{2\,\Gamma\,T\,\Delta t}\,\xi_\mu$$

- The change in free energy $\Delta F(\vec{x}, \vec{x} + \hat{\mu}) = \Delta F(\vec{x}) + \Delta F(\vec{x} + \hat{\mu}) + q_\mu^2$



Metropolis step for Model B

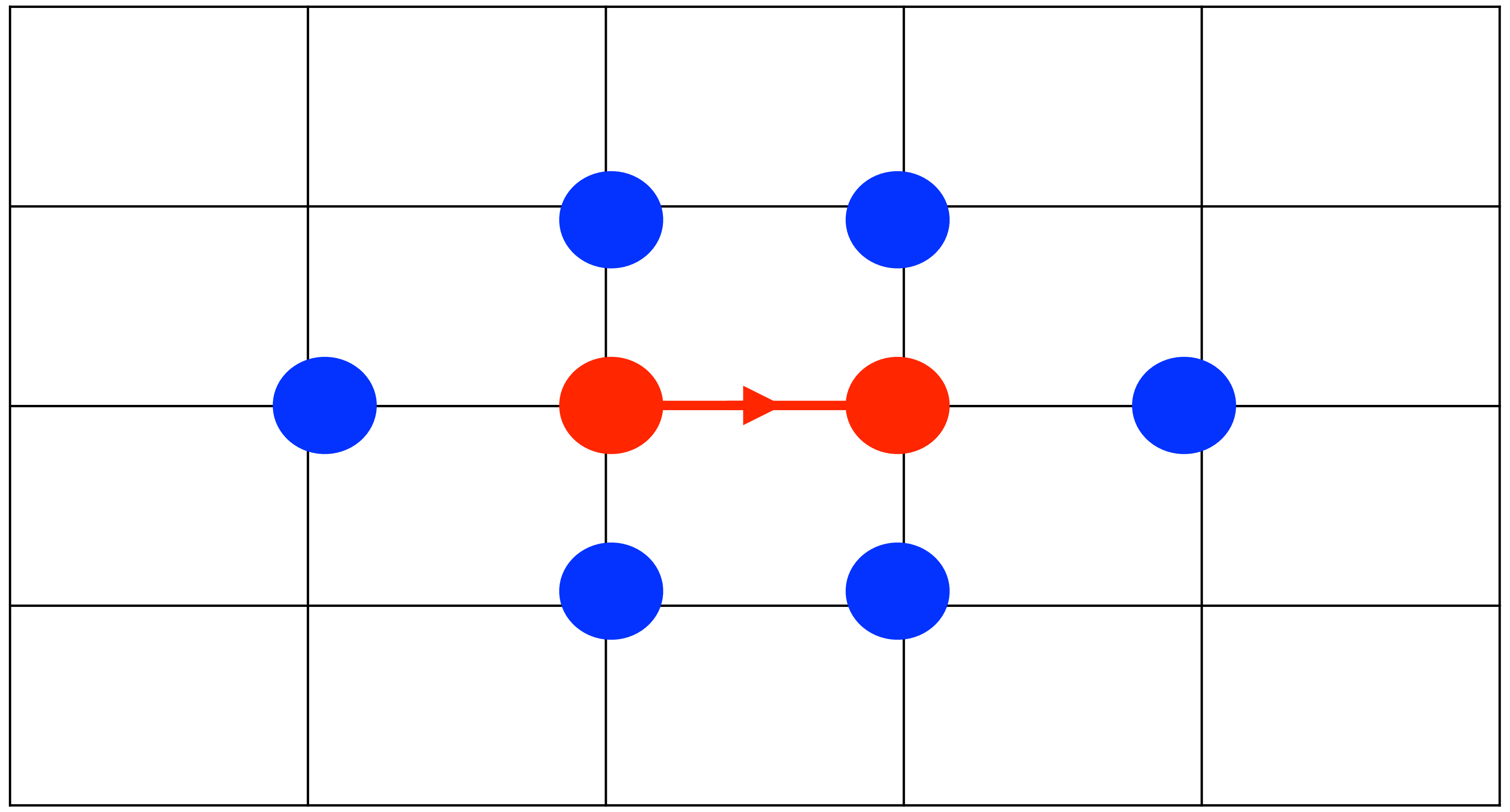
- Choose trial updates at $\vec{x}$ and $\vec{x} + \hat{\mu}$

$$\phi^{\text{trial}}(t + \Delta t, \vec{x}) = \phi(t, \vec{x}) - q_\mu, \quad \phi^{\text{trial}}(t + \Delta t, \vec{x} + \hat{\mu}) = \phi(t, \vec{x} + \hat{\mu}) + q_\mu$$

$$q_\mu = \sqrt{2 \Gamma T \Delta t}\, \xi_\mu$$

- The change in free energy $\Delta F(\vec{x}, \vec{x} + \hat{\mu}) = \Delta F(\vec{x}) + \Delta F(\vec{x} + \hat{\mu}) + q_\mu^2$

- Accept with probability $P = \min(1, \exp(-\Delta F/T))$



Model H (deterministic part)

- Let's consider only the non-dissipative part of the equations

$$\frac{\partial\phi}{\partial t} + \frac{\vec{\pi}_T}{\rho}\cdot\vec{\nabla}\phi = 0, \qquad \frac{\partial\vec{\pi}_T}{\partial t} + \frac{\vec{\pi}_T}{\rho}\cdot\vec{\nabla}\vec{\pi}_T = \vec{\nabla}\phi\,\vec{\nabla}^2\phi \longleftarrow$$

Third-order term, goes beyond usual Navier-Stokes

The third-order term is necessary for conserving energy

$$\frac{dH}{dt} = \int d^3x\left[\dot{\vec{\pi}}_T\cdot\frac{\vec{\pi}_T}{\rho} - \dot{\phi}\,\nabla^2\phi + V'(\phi)\,\dot{\phi}\right] = 0$$

where the equations of motion have been used along with standard continuum manipulations

$$\int_x V'(\phi)\frac{\vec{\pi}_T}{\rho}\cdot\nabla\phi = \int_x\vec{\nabla}\cdot\left(\frac{\vec{\pi}_T}{\rho}V(\phi)\right) = 0 \qquad \frac{\pi_i^T}{\rho}\left(\frac{\pi_i^T}{\rho}\nabla_j\right)\pi_i^T = \nabla_i\left(\frac{\pi_i^T}{\rho}\frac{\pi_T^2}{2\rho}\right)$$

- These continuum manipulations are not necessarily allowed in the discretized theory.

Model H numerics (deterministic part)

- The equations in manifestly conserving form $\dot{\phi} = \vec{\nabla} \cdot \left(\frac{\vec{\pi}_T}{\rho} \phi \right)$ $\quad \dot{\pi}_i^T = -P_{ij}^T \nabla_k \left(\frac{1}{\rho} \pi_T^k \pi_T^j + \nabla_k \nabla_j \phi \right)$

- Use a skew symmetric derivative for the non-linear term

$$\nabla_\mu \left(\frac{1}{\rho} \pi_\mu^T \pi_\nu^T \right) \bigg|_{skew} \equiv \frac{1}{2} \nabla_\mu \left(\frac{1}{\rho} \pi_\mu^T \pi_\nu^T \right) + \frac{1}{2} \frac{\pi_\mu^T}{\rho} \nabla_\mu \pi_\nu^T$$

Morinishi, Lund, Vasilyev, Moin, Journal of computational physics (143, 90 (1998)

along with a centred difference $\nabla_\mu^c \psi = (\psi(x + \hat{\mu}) - \psi(x - \hat{\mu})/2$

- The discretized evolution equations:

$$\dot{\phi} = -\frac{1}{\rho} \pi_T^\mu \nabla_\mu^c \phi, \qquad \dot{\pi}_T^\mu = -\left[\nabla_\mu \left(\frac{1}{\rho} \pi_\mu^T \pi_\nu^T \right) \bigg|_{skew} + \left(\nabla_\mu^c \phi \right) \left(\nabla_\nu^c \nabla_\nu^c \phi \right) \right]$$

Model H numerics (deterministic part)

- The discretized eqs.

$$\dot{\phi} = -\frac{1}{\rho}\pi_T^{\mu}\nabla_{\mu}^{c}\phi \qquad \dot{\pi}_T^{\mu} = -\left[\nabla_{\mu}\left(\frac{1}{\rho}\pi_{\mu}^{T}\pi_{\nu}^{T}\right)\Big|_{skew} + \left(\nabla_{\mu}^{c}\phi\right)\left(\nabla_{\nu}^{c}\nabla_{\nu}^{c}\phi\right)\right]$$

conserves the kinetic energy of the system exactly:

$$\frac{dT}{dt} = \frac{d}{dt}\int d^3x\left[\frac{\pi_T^2}{2\rho} + \frac{(\nabla\phi)^2}{2}\right] = 0$$

- The equations are integrated in time using a Runge-Kutta scheme. After each step, project onto transverse part in Fourier space

$$\pi_{\mu}^{T} = P_{\mu\nu}^{T}\pi_{\nu} \qquad P_{\mu\nu}^{T} = \delta_{\mu\nu} + \frac{\tilde{k}_{\mu}\tilde{k}_{\nu}}{\tilde{k}^2}$$

- Total energy conservation in the deterministic step is found to hold to very good accuracy.

Results: Dynamics of momentum density

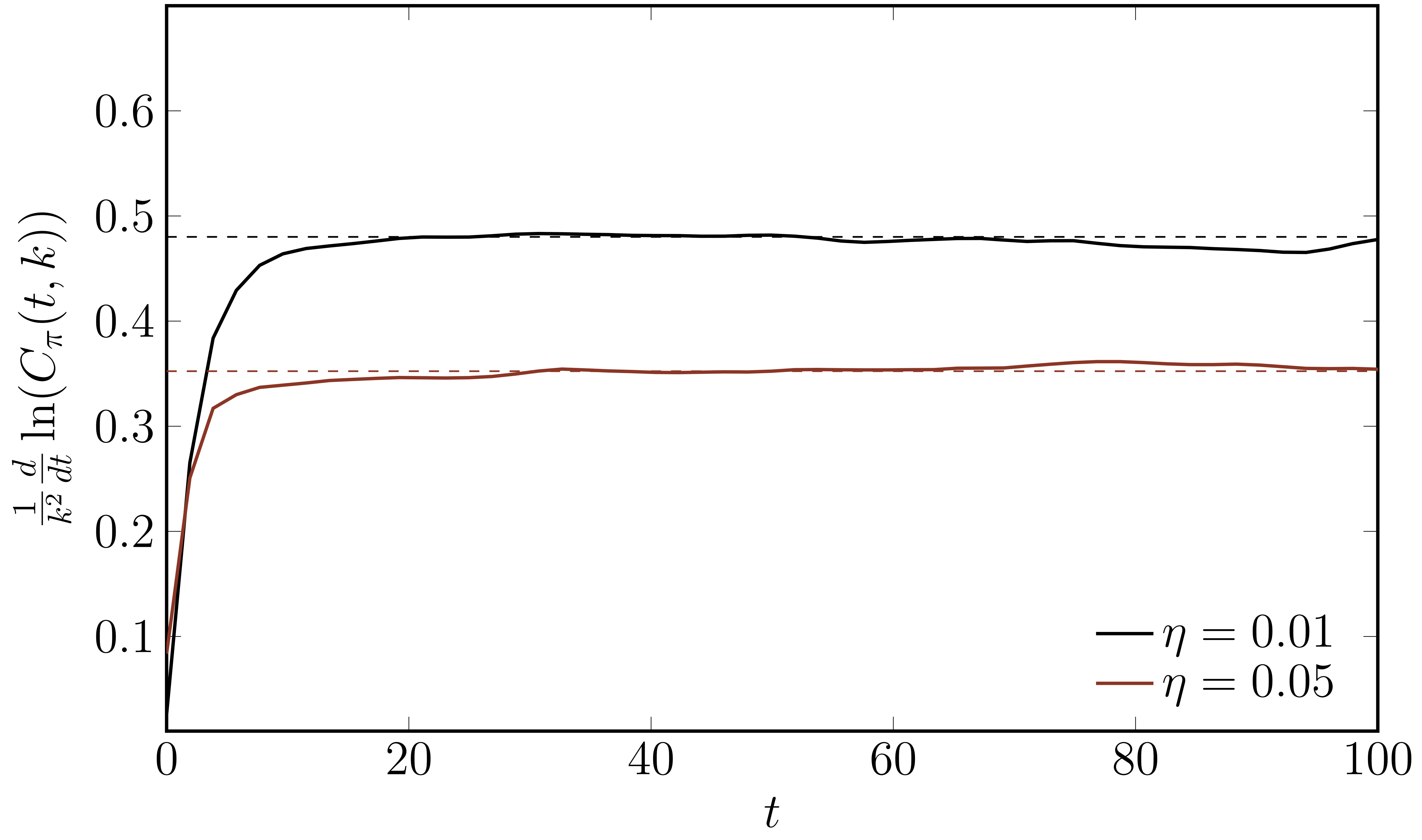
- Consider the time-dependent correlation function of the momentum density

$$\langle \pi_i^T(0,\vec{k})\, \pi_j^T(0,-\vec{k})\rangle \equiv C_{ij}(t,\vec{k}), \quad \text{where} \quad C_{ij}(t,\vec{k}) = \left(\delta_{ij} - \hat{k}_i\hat{k}_j\right) C_\pi(t,k)$$

- In linearized hydrodynamics $C_\pi(t,k) = \rho\, T \exp\left(-\frac{\eta}{\rho}k^2\, t\right)$

- Compute $C_\pi(t,k)$ in Model H to extract effective η

- Thermal fluctuations and non-linear effects modify linear hydro result (even away from T_c)



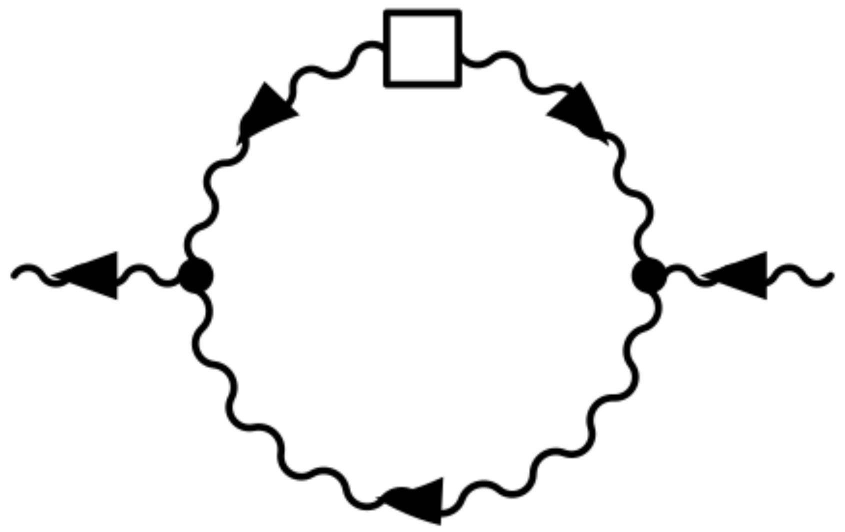
Dynamics: Loop corrections

Non-linear interactions between modes $\vec{\pi}_T, \phi$ can be represented diagrammatically

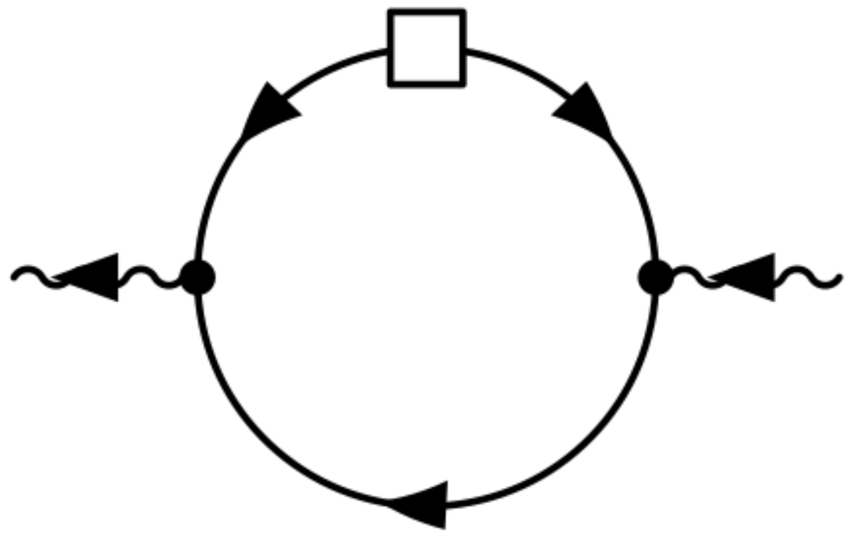
Green's functions for π_T 

Green's functions for ϕ

Corrections to momentum corr. function

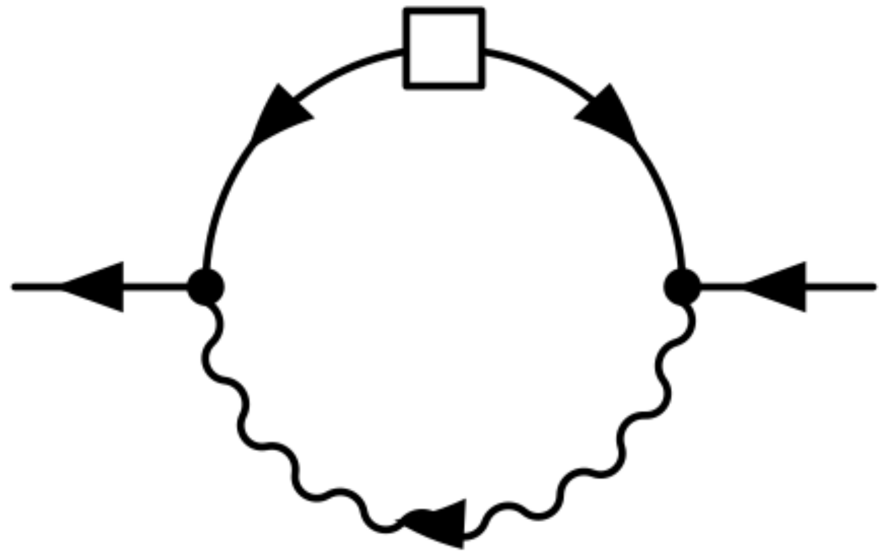


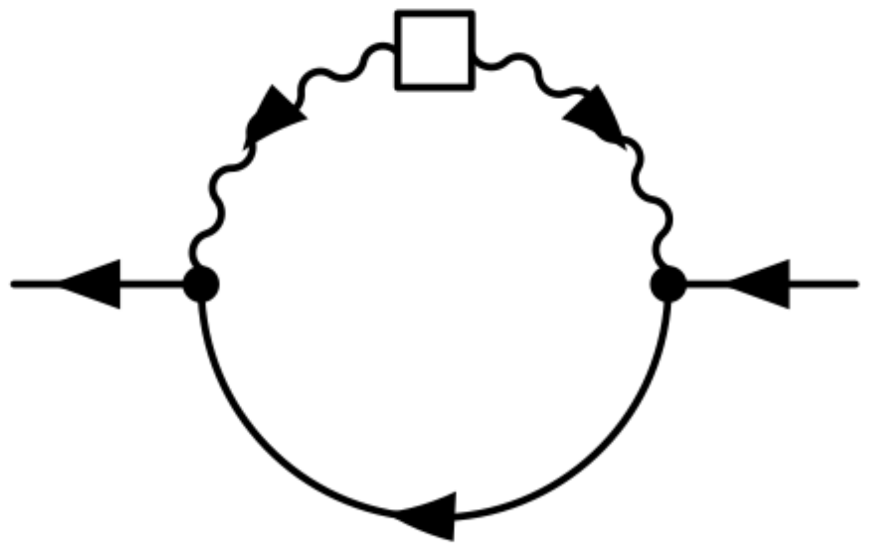
Self-advection of π_T



Coupling of π_T to ϕ

Corrections to corr. function of ϕ





Advection of ϕ by π_T

Dynamics: Order parameter

- Using the time dependent correlation function of the order parameter

$$C(t, \vec{k}) = \langle \phi(0, \vec{k})\, \phi(t, -\vec{k}) \rangle$$

a wave-number dependent relaxation rate is defined $C(t, \vec{k}) \sim \exp(-\Gamma_k t)$

- A model for Γ_k was proposed by Kawasaki:

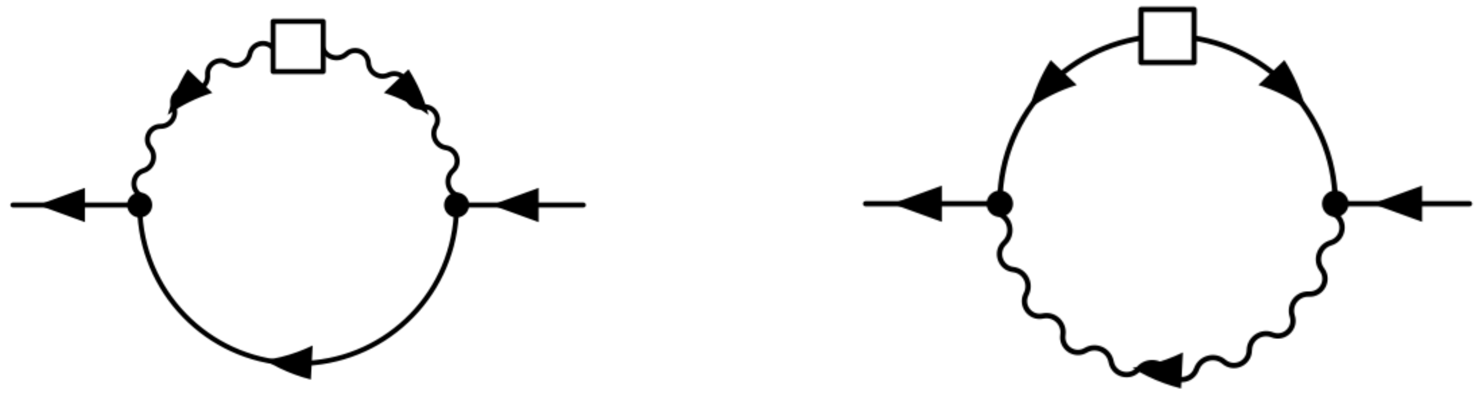
Kawasaki function

$$\Gamma_k = \frac{\Gamma}{\xi^4} (k\xi)^2 \left(1 + (k\xi)^2\right) + \frac{T}{6\pi\eta_R \xi^3} K(k\xi)$$

Pure Model B prediction using mean field approx.

Arises from coupling between ϕ and π_T

Diagrams computed with certain approximations

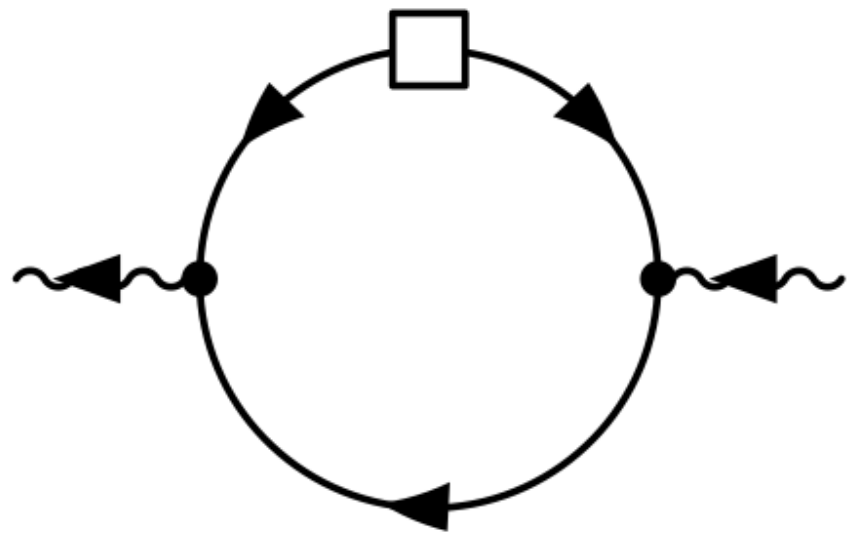


Dynamics: Kawasaki approximation

- The Kawasaki approximation: $$\Gamma_k = \frac{\Gamma}{\xi^4}\,(k\xi)^2\left(1+(k\xi)^2\right) + \frac{T}{6\pi\eta_R\xi^3}K(k\xi)$$

- Near critical point, relaxation-rate for wavenumbers $k = k_* \sim 1/\xi$ should cross over from $z = 4$ (pure diffusive dynamics) to $z = 3$ (pure Model H behavior).

- Digression: Using Γ_k one can re-recompute the renormalization of η due to coupling of π_T to ϕ:



$$\eta_R = \eta\left[1 + \frac{8}{15\pi^2}\log\left(\frac{\xi}{\xi_0}\right)\right]$$

Near critical point, viscosity diverges, but only weakly

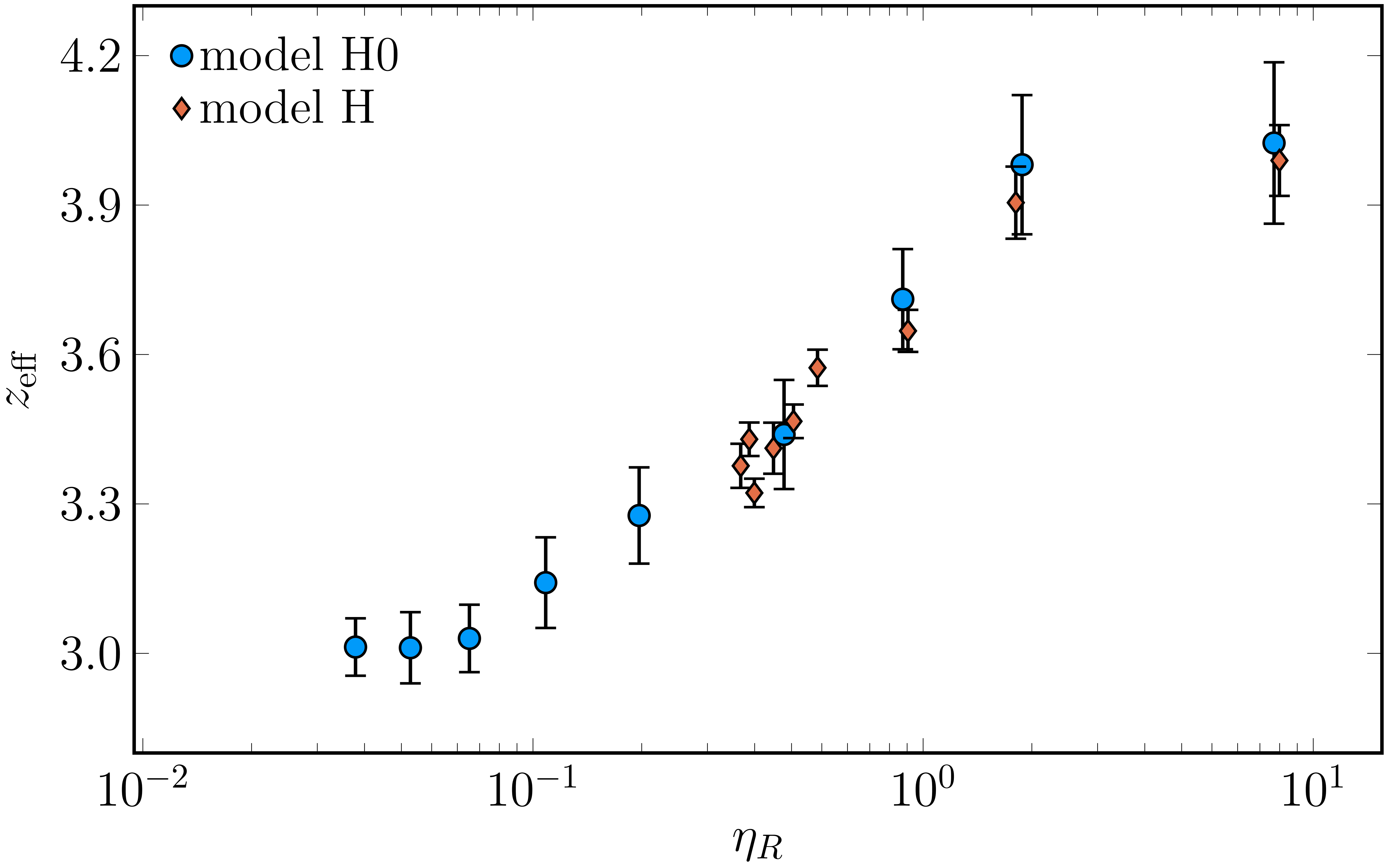
$$\eta_R \sim \xi^{x_\eta} \quad \text{with } x_\eta \approx 0.05$$

Cross-over of z

- Dynamic scaling exponent as a function of renormalized viscosity.

- z for full Model H coincides with Model H0

- In full Model H, η_R cannot become too small $\Longrightarrow$ min(z) ≈ 3.3



Model H $\quad \dfrac{\partial \vec{\pi}_T}{\partial t} + \dfrac{\vec{\pi}_T}{\rho} \cdot \vec{\nabla}\,\vec{\pi}_T = \dfrac{\eta}{\rho}\,\nabla^2 \vec{\pi}_T + \cdots$

Model H0 $\quad \dfrac{\partial \vec{\pi}_T}{\partial t} + \cancel{\dfrac{\vec{\pi}_T}{\rho} \cdot \vec{\nabla}\,\vec{\pi}_T} = \dfrac{\eta}{\rho}\,\nabla^2 \vec{\pi}_T + \cdots$

Evolution of higher moments

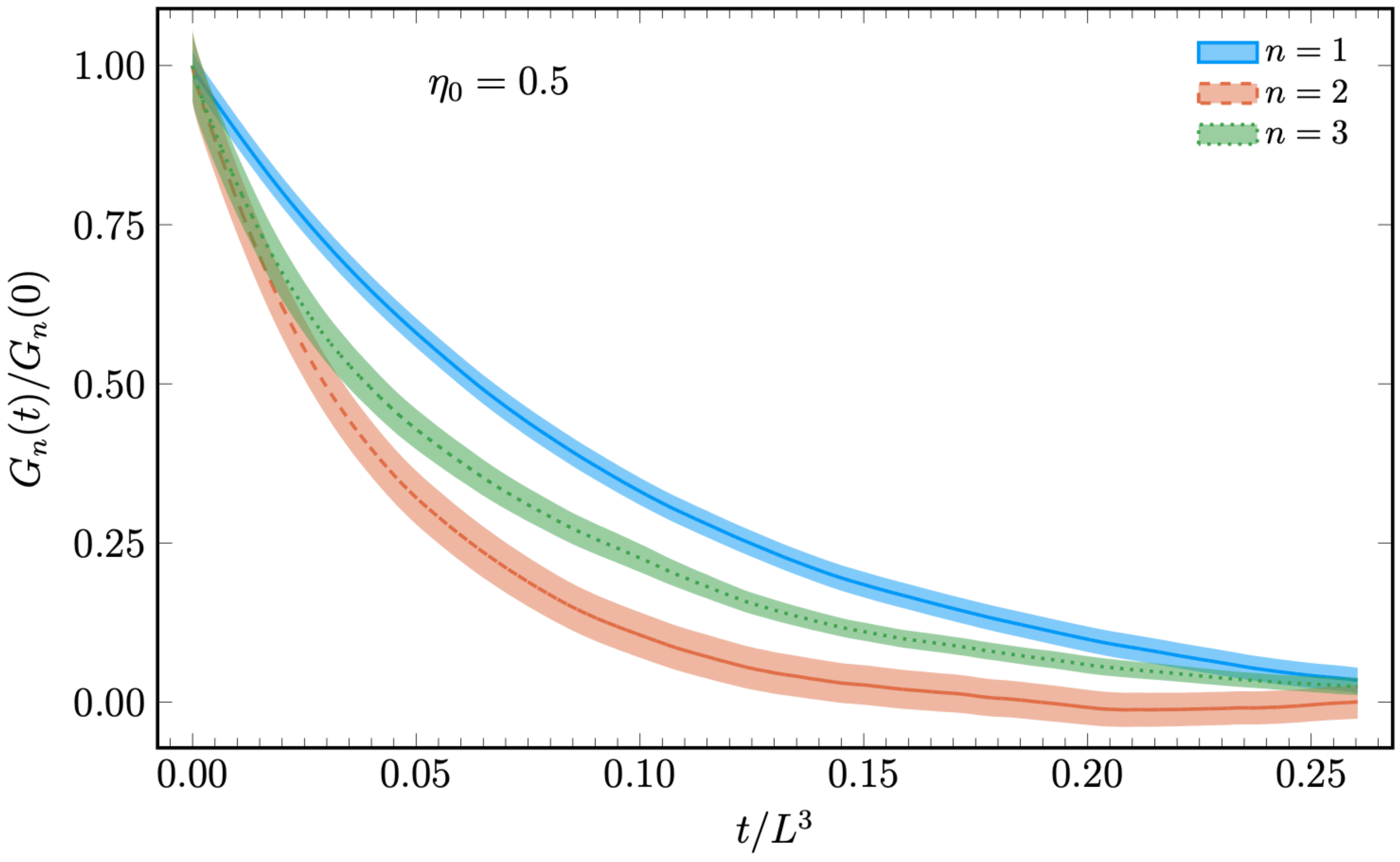
- Consider higher-point correlations

$$G_n(t) = \langle M^n(t) M^n(0) \rangle$$

$$M(t) = \int_V d^3x\, \phi(t, \vec{x})$$

- Correlation functions satisfy dynamical scaling

- Relaxation rate depends on 'n'. Not compatible with mean field expectations



Backup: determination of m_c^2 in Model A

- At a critical point, susceptibilities $\langle M^2 \rangle$ diverge (infinite vol). In finite volume there are peaks. Possible strategy: Thermalize Model B configurations, compute $\langle M^2 \rangle$ at different m^2 and look for peaks.

- Mean-field estimates that Model B configurations take $\tau_{\text{therm}} \sim L^z$ with z ~ 4 to thermalize. Computationally demanding.

- Use a model in the same static universality class but with smaller $z \Longrightarrow$ Model A, relaxational dynamics of an order-parameter (z = 2).

$$\frac{\partial \phi}{\partial t} = -\Gamma \frac{\delta F}{\delta \phi} + \zeta \qquad F[\phi] = \int d^3x \left[\frac{1}{2}\left(\nabla \phi\right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4\right]$$

$$\langle \zeta(t, \vec{x})\, \zeta(t', \vec{x}') \rangle = 2\Gamma\, T\, \delta(\vec{x} - \vec{x}')\, \delta(t - t')$$

Backup: The stickiness of sound

Kovtun, Moore & Romatschke

Linearized energy-momentum tensor in presence of noise

$$T_{00,\xi} = \delta e \qquad T_{0i,\xi} = -\left(e_0 + P_0\right)\delta u_i \qquad T_{ij,\xi} = \delta_{ij}\, c_s^2\, \delta e - \eta\left(\partial_i \delta u_j + \partial_j \delta u_i - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\delta\vec{u}\right) + \xi_{ij}$$

Noise is Gaussian: $\langle \xi_{ij}(x)\xi_{kl}(y)\rangle = 4\,\eta\, T\Delta_{ijkl}\,\delta^4(x-y)$

Averages of any quantity is obtained by using a functional integral $\langle \mathcal{O}\rangle \equiv \int D\xi_{ij}\, e^{-S_\xi}\,\mathcal{O}$

$$S_\xi = \int d^3x\, \xi_{ij}\left(\frac{1}{8T\eta}\,\Delta^{ijkl}\right)\xi_{kl}$$

Can compute any correlation functions, for eg., $\langle T^{12}(x)\, T^{12}(y)\,\rangle \equiv G^{12,12}(x,y)$

Backup: The stickiness of sound

Beyond linearized regime, consider terms up to 2nd order in perturbation (also take low momentum limit) $T_{\xi}^{12} = \left(e_0 + P_0\right)\, \delta u^1\, \delta u^2\, + \xi^{12}$

The symmetric correlator $G_{\rm sym}^{12,12}(x,y) = \langle \xi^{12}(x)\xi^{12})(y)\rangle_\xi + (\epsilon_0 + P_0)^2 \langle \delta u^1(x)\delta u^2(x)\delta u^1(y)\delta u^2(y)\rangle_\xi.$

In Fourier space,

$$G_{\rm sym}^{12,12}(\omega, k\to 0) = 2T\eta + \int \frac{d\omega'}{2\pi}\frac{d^{d-1}k'}{(2\pi)^{d-1}} \left[G_{\rm sym}^{01,01}(\omega',\mathbf{k}')G_{\rm sym}^{02,02}(\omega-\omega',-\mathbf{k}') + G_{\rm sym}^{01,02}(\omega',\mathbf{k}')G_{\rm sym}^{02,01}(\omega-\omega',-\mathbf{k}')\right]$$

For example, $G_{\rm sym}^{01,01} = -\dfrac{2T}{\omega}\left(e_0 + \dfrac{k^2\eta}{i\omega - \gamma_\eta k^2}\right)$ $\qquad \gamma_\eta = \eta/(e_0 + P_0)$

Finally, one obtains

$$G^{12,12}(\omega, k\to 0) = -i\omega\left(\eta + \frac{17T\Lambda_{UV}}{120\pi^2\gamma_\eta}\right) + (1+i)\omega^{3/2}\frac{\left(7 + \left(\frac{3}{2}\right)^{3/2}\right)T}{240\pi\gamma_\eta^{3/2}}$$

Renormalization of shear

Kovtun, Moore & Romatschke