Analysis of Causality and Stability in relativistic hydrodynamics

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Study of hydrodynamics

Motivation behind ⋯

- Hydrodynamics : A long-wavelength effective theory of fluids that describes evolution of the system in terms of it's state variables and their derivatives.
- Riddle : There can be many possible hydrodynamic theories; how can we filter out unphysical ones?

Key features for theory to be physically acceptable ⋯

- Causality: Signal propagation should not exit the light cone; no superluminal velocities.
- Stability: Perturbations around global equilibrium must decay down with time.
- Pathology free: Both stable as well as causal.

Established theories ⋯

- Muller-Israel-Stewart (MIS): Higher order stable-causal theory. Ref: Israel, Annals Phys. 100 (1976), 310-331, Israel and Stewart, Annals Phys. 118, 341-372, Muller, Z. Phys. 198 (1967), 329-344.

Recent attempt and current field of study ⋯

- BDNK: First order stable-causal theory. Ref: Bemfica, Disconzi and Noronha, PRD 98, no.10, 104064 (2018), PRD 100, no.10, 104020 (2019), P. Kovtun, JHEP 10, 034 (2019), JHEP 06, 067 (2020).

Causality and stability - a choice to be endured

Desirable features preferred by a hydrodynamic theory

1. They are defined only in terms of the dynamical variables, such as temperature (T), fluid velocity (u^μ), and chemical potential (μ), that are related to some conserved quantities $\Longrightarrow$ characterizing the equilibrium.
2. Even out of equilibrium, the fluid fields are well defined that can attach some physical meaning to it $\Longrightarrow$ defined in terms of some field theory operators.
3. The final differential equations for the fluid variables must contain a finite number of derivatives $\Longrightarrow$ the theory must be truncated.

The features (i) and (ii) are needed to build an unambiguous connection between the fluid variables and the experimentally measurable quantities.

The third feature (iii) is a must to have computationally tractable evolution equations.

Is the holy trinity ever compatible with causality ?

Causality and stability - a choice to be endured

In reality, the journey of relativistic dissipative hydro is far from this desirable holy trinity

1. Theory defined in terms of fundamental fluid variables (T, u^μ, μ) + Landau or Eckart frame (fluid variables are defined in a physically measurable way) + truncated at finite order $\Longrightarrow$ causality violation.

   $\Downarrow$

   Relativistic first-order (Navier-Stokes) theory : Superluminal signal propagation and thermodynamic instability.

2. Causal theory + second or higher (but finite) order derivative corrections + Landau or Eckart frame (fluid variables have first principle definition)

   $\Downarrow$

   Muller-Israel-Stewart theory : New degrees of freedom (with no equilibrium counterparts) are needed.

3. Causal theory + defined only in terms of fundamental fluid variables (T, u^μ, μ) (no extra 'non-fluid' variables) + truncated at finite order of derivative corrections

   $\Downarrow$

   First order Bemfica-Disconzi-Noronha-Kovtun (BDNK) theory : definition of the fluid variables away from equilibrium are not fixed (theory is pathology-free only in frames other than Landau or Eckart).

4. Causal theory + defined only in terms of fundamental fluid variables (T, u^μ, μ) + Landau frame $\Longrightarrow$ infinite number of derivatives.

   $\Downarrow$

   Practical limitations for simulation purpose.

Causality and stability - a choice to be endured

The current work indicates :

- There is a tension between the three desirable features of hydrodynamics.
- To maintain causality and stability we have to give up at least one of them.

A connection between different hydro formalisms (MIS and BDNK) in terms of these features

BDNK stress tensor Ref : Kovtun, JHEP 10 (2019), 034 , Bemfica et al., PRD 100 (2019), 104020

$$T^{\mu\nu} = (\varepsilon + \mathcal{A})\left[u^\mu u^\nu + \frac{\Delta^{\mu\nu}}{3}\right] + [u^\mu Q^\nu + u^\nu Q^\mu] - 2\eta\sigma^{\mu\nu}$$

$$\mathcal{A} = \chi\left[3\frac{DT}{T} + \nabla_\mu u^\mu\right], \quad Q^\mu = \theta\left[\frac{\nabla^\mu T}{T} + Du^\mu\right]$$

Fluid frame transformation of BDNK theory : $u^\mu = \hat{u}^\mu + \delta u^\mu, \qquad T = \hat{T} + \delta T$

Frame transformation order by order :

$$\delta u^\mu = \sum_{n=1}^{\infty} \delta u_n^\mu, \quad \delta T = \sum_{n=1}^{\infty} \delta T_n$$

$$\delta T_1 = -\tilde{\chi}\left[\frac{\hat{D}\hat{T}}{\hat{T}} + \frac{1}{3}\hat{\nabla}_\mu \hat{u}^\mu\right], \qquad \delta u_1^\mu = -\tilde{\theta}\left[\frac{\hat{\nabla}^\mu \hat{T}}{\hat{T}} + \hat{D}\hat{u}^\mu\right],$$

$$\delta T_n = -\tilde{\chi}\left[\frac{1}{\hat{T}}\hat{D}\delta T_{n-1} + \frac{1}{3}\hat{\nabla}_\mu \delta u_{n-1}^\mu\right] \quad , \quad \delta u_n^\mu = -\tilde{\theta}\left[\frac{1}{\hat{T}}\hat{\nabla}^\mu \delta T_{n-1} + \hat{D}\delta u_{n-1}^\mu\right] \quad \text{for } n \geq 2$$

Causality and stability - a choice to be endured

BDNK theory in Landau frame : temperature and velocity have first principle definition

$$T^{\mu\nu} = \hat{\varepsilon}\left[\hat{u}^\mu \hat{u}^\nu + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right] + \hat{\pi}^{\mu\nu}$$

The price is paid : (i) either by an infinite number of derivatives

$\hat{\pi}^{\mu\nu} = -2\eta\left[\hat{\sigma}^{\mu\nu} + \sum_{n=1}^{\infty} \partial^{\langle\mu}\delta u_n^{\nu\rangle}\right]$: summed over temporal derivatives under the all order frame transformation

$$\hat{\pi}^{\mu\nu} = -2\eta\Bigg[\frac{\hat{\nabla}^{\langle\mu}\hat{u}^{\nu\rangle}}{(1+\tilde{\theta}\hat{D})} + \frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})}\frac{\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{(1+\tilde{\chi}\hat{D})} + \frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})^2}\frac{(-\frac{1}{3}\tilde{\chi})}{\left(1+\tilde{\chi}\hat{D}\right)}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u}$$

$$+\frac{(-\tilde{\theta})^2}{(1+\tilde{\theta}\hat{D})^2}\frac{\left(-\frac{1}{3}\tilde{\chi}\right)}{(1+\tilde{\chi}\hat{D})^2}\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}^2\hat{T} + \frac{(-\tilde{\theta})^2}{(1+\tilde{\theta}\hat{D})^3}\frac{\left(-\frac{1}{3}\tilde{\chi}\right)^2}{(1+\tilde{\chi}\hat{D})^2}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}^2\hat{\nabla}\cdot\hat{u} + \cdots\Bigg]$$

- In each increasing spatial gradient, the temporal gradient resulting from the infinite sum also increases in the denominator, such that they exactly balance each other $\Longrightarrow$ a necessary condition of causality.
- $\pi^{\mu\nu}$ has time derivatives in the denominator $\Longrightarrow$ nonlocality in time.
- Such nonlocalities can be recast into a local set of equations by introducing new 'non-fluid' variables just like the MIS theory.

Causality and stability - a choice to be endured

BDNK theory in Landau frame : temperature and velocity have first principle definition

$$T^{\mu\nu} = \hat{\varepsilon}\left[\hat{u}^\mu\hat{u}^\nu + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right] + \hat{\pi}^{\mu\nu}$$

The price is paid : (ii) or by introducing new 'non-fluid' variables as in MIS

$$\begin{aligned}
(1+\tilde{\theta}\hat{D})\hat{\pi}^{\mu\nu} &= -2\eta\hat{\sigma}^{\mu\nu} + \rho_1^{\mu\nu}\\
(1+\tilde{\chi}\hat{D})\rho_1^{\mu\nu} &= (-2\eta)(-\tilde{\theta})\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{T} + \rho_2^{\mu\nu}\\
(1+\tilde{\theta}\hat{D})\rho_2^{\mu\nu} &= (-2\eta)(-\tilde{\theta})\left(-\frac{\tilde{\chi}}{3}\right)\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}\cdot\hat{u} + \rho_3^{\mu\nu}\\
(1+\tilde{\chi}\hat{D})\rho_3^{\mu\nu} &= (-2\eta)(-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}^2\hat{T} + \rho_4^{\mu\nu}\\
(1+\tilde{\theta}\hat{D})\rho_4^{\mu\nu} &= (-2\eta)(-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)^2\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}^2\hat{\nabla}\cdot\hat{u} + \cdots\\
&\vdots
\end{aligned}$$

Relaxation time scales are provided by the poles of the infinite sum of temporal derivatives

To maintain causality and stability we have to give up at least one of (i), (ii), (iii)

S Bhattacharyya, S Mitra and S Roy, Phys. Lett. B 856 (2024), 138918

Causality and stability - a choice to be endured

The method of 'integrating in' new 'non-fluid' variables is not unique

Frame transformation in one go :

$$\frac{\delta T}{\hat{T}} + \tilde{\chi}\left[\frac{\hat{D}\hat{T}}{\hat{T}} + \frac{\hat{\nabla}_\mu \hat{u}^\mu}{3}\right] + \tilde{\chi}\left[\frac{\hat{D}\delta T}{\hat{T}} + \frac{\hat{\nabla}_\mu \delta u^\mu}{3}\right] = 0$$

$$\delta u^\mu + \tilde{\theta}\left[\hat{D}\hat{u}^\mu + \frac{\hat{\nabla}^\mu \hat{T}}{\hat{T}}\right] + \tilde{\theta}\left[\hat{D}\delta u^\mu + \frac{\hat{\nabla}^\mu \delta T}{\hat{T}}\right] = 0$$

Alternate way of introducing new 'nonfluid' degree of freedom :

$$T^{\mu\nu} = \hat{\varepsilon}\left[\hat{u}^\mu \hat{u}^\nu + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right] + \hat{\pi}^{\mu\nu}$$

$$\left[(1+\tilde{\theta}\hat{D})(1+\tilde{\chi}\hat{D}) - \tilde{\theta}\frac{\tilde{\chi}}{3}\,\hat{\nabla}^2\right]\left\{(1+\tilde{\theta}\hat{D})\hat{\pi}^{\mu\nu} + 2\eta\hat{\sigma}^{\mu\nu}\right\} = 2\eta\tilde{\theta}\left\{1+(\tilde{\theta}+\tilde{\chi})\hat{D}\right\}\frac{\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{\hat{T}}$$

Similar to MIS theory, only one 'non-fluid' tensorial degree of freedom, but it follows a complicated inhomogeneous PDE, second order in spatial but third order in temporal derivatives.

S Bhattacharyya, S Mitra and S Roy, Phys. Lett. B 856 (2024), 138918

Acausality in truncated 'MIS' theory

The all order information (over temporal derivative) is required for causality

MIS theory - an all order gradient correction theory

MIS theory :

$$\partial_\mu T^{\mu\nu} = 0 \ , \quad T^{\mu\nu} = \varepsilon\left(u^\mu u^\nu + \frac{1}{3}\Delta^{\mu\nu}\right) + \pi^{\mu\nu}$$

$$\pi^{\mu\nu} + \tau_\pi D\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$$

What if we demand the theory to be finitely truncated

$$\pi^{\mu\nu} = \sum_n \pi_n^{\mu\nu} \ , \qquad \pi_1^{\mu\nu} = -2\eta\sigma^{\mu\nu} \ , \qquad \pi_n^{\mu\nu} = -\tau_\pi D\pi_{n-1}^{\mu\nu} \ , n \geq 2$$

$$\pi^{\mu\nu} = -2\eta\left\{\sum_{n=0}^{N}(-\tau_\pi D)^n\right\}\sigma^{\mu\nu}$$

only with $N \to \infty$ and $|\tau_\pi D| < 1$

$$\pi^{\mu\nu} = -2\eta\,(1 + \tau_\pi D)^{-1}\,\sigma^{\mu\nu}$$

Acausality in truncated 'MIS' theory

Linear stability and causality analysis : $\psi(t,x) = \psi_0 + \delta\psi(t,x)$

Fluctuations are expressed in plane wave solutions via a Fourier transformation : $\delta\psi(t,x) \rightarrow e^{i(kx-\omega t)}\delta\psi(\omega,k)$

shear channel dispersion relation : $(i\omega) + \tilde{\eta}(ik)^2\left[\sum_{n=0}^{N}(\tau_\pi i\omega)^n\right] = 0$

sound channel dispersion relation : $(i\omega)^2 + \frac{4}{3}\tilde{\eta}(i\omega)(ik)^2\left[\sum_{n=0}^{N}(\tau_\pi i\omega)^n\right] - \frac{1}{3}(ik)^2 = 0$

Violates relativistic quantum theory causality condition $\mathrm{Im}(\omega(k)) \leq |\mathrm{Im}(k)|$ for any finite N

S Mitra, Phys. Rev. D 109 (2024) no.12, L121501

$N \rightarrow \infty$

shear channel : $\tau_\pi\omega^2 + i\omega - \tilde{\eta}k^2 = 0$

sound channel : $\tau_\pi\omega^3 + i\omega^2 - \left(\frac{4}{3}\tilde{\eta} + \frac{1}{3}\tau_\pi\right)\omega k^2 - \frac{1}{3}ik^2 = 0$

Respects relativistic quantum theory causality condition $\mathrm{Im}(\omega(k)) \leq |\mathrm{Im}(k)|$

Abide by the conservation rule of fluid modes : $\mathcal{O}_\omega[F(\omega, \boldsymbol{k} \neq 0)] = \mathcal{O}_{|\boldsymbol{k}|}[F(\omega = a|\boldsymbol{k}|, \boldsymbol{k} = \mathbf{b}|\boldsymbol{k}|]$

$\downarrow$

Condition for causality

Ref: Hoult and Kovtun, PRD 109 (2024) no.4, 046018

Probing physical origin of the newly promoted 'non-fluid' degrees of freedom - a kinetic theory perspective

- The underlying microscopic theories are always known to be free from pathologies concerning subluminal signal propagation.
- Then the process of coarse-graining must have to do something with the rising causality related issues.
- The truncation scheme could be the answer ?

Boltzmann transport equation : $\tilde{p}^\mu \partial_\mu f = -\frac{\tilde{E}_p}{\tau_R}\delta f = -\frac{\tilde{E}_p}{\tau_R}(f - f_0)$

Momentum distribution correction : $$\delta f = -\frac{\left[\frac{\tau_R}{\tilde{E}_p}\tilde{p}^\mu\partial_\mu\right] f_0}{\left[1 + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^\mu\partial_\mu\right]} = \frac{\frac{\tau_R}{\tilde{E}_p} f_0 \left[\tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}\sigma_{\mu\nu} - \frac{1}{(\varepsilon+P)}\tilde{E}_p\tilde{p}_{\langle\nu\rangle}\nabla_\rho\pi^{\nu\rho}\right]}{\left[1 + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^\lambda\partial_\lambda\right]}$$

Exact expression of shear viscous flux (no truncation so far)

$$\frac{\pi^{\alpha\beta}}{T^2} = \int dF_p \frac{\frac{\tau_R}{\tilde{E}_p}\tilde{p}^{\langle\alpha}\tilde{p}^{\beta\rangle}\tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}\sigma_{\mu\nu}}{\left[1 + \tau_R D + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^{\langle\lambda\rangle}\nabla_\lambda\right]} - \frac{\tau_R}{(\varepsilon+P)}\int dF_p \frac{\tilde{p}^{\langle\alpha}\tilde{p}^{\beta\rangle}\tilde{p}_{\langle\mu\rangle}\nabla_\nu\pi^{\mu\nu}}{\left[1 + \tau_R D + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^{\langle\lambda\rangle}\nabla_\lambda\right]}$$

Newer degrees of freedom - a microscopic derivation

Hydrodynamics - systematic build up of gradients of the fluid variables

$$\tau_R D + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^{\langle\lambda\rangle}\nabla_\lambda \quad \sim \quad \frac{\lambda_{\mathrm{mfp}}}{L} = K_n$$

K_n = Knudsen number : decides the region of validity of the resulting coarse-grained (hydrodynamic) theory

$K_n < 1 \quad \sim \quad |\tau_R D + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^{\langle\lambda\rangle}\nabla_\lambda| < 1$: limit for hydro to be valid in terms of systematic build up of gradients

- $|\tau_R D + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^{\langle\lambda\rangle}\nabla_\lambda|$ is expanded as an infinite derivative sum series operating on thermodynamic forces.
- The infinite sum over the temporal derivatives forms a closed structure that creates relaxation operator like forms $(1+\tau_R D)$ in the denominator.

$$\frac{\pi^{\alpha\beta}}{T^2} = \sum_{m=0}^{\infty}\left\{\frac{\tau_R}{1+\tau_R D}\right\}^{2m+1}\left[\int \frac{dF_p}{\left(\tilde{E}_p\right)^{2m+1}}\tilde{p}^{\langle\alpha}\tilde{p}^{\beta\rangle}\tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}\tilde{p}^{\langle\lambda_1\rangle}\cdots\tilde{p}^{\langle\lambda_{2m}\rangle}\right]\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2m}}\sigma_{\mu\nu}$$
$$+\frac{1}{(\varepsilon+P)}\sum_{n=0}^{\infty}\left\{\frac{\tau_R}{1+\tau_R D}\right\}^{2n+2}\left[\int \frac{dF_p}{\left(\tilde{E}_p\right)^{2n+1}}\tilde{p}^{\langle\alpha}\tilde{p}^{\beta\rangle}\tilde{p}_{\langle\mu\rangle}\tilde{p}^{\langle\lambda_1\rangle}\cdots\tilde{p}^{\langle\lambda_{2n+1}\rangle}\right]\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n+1}}\nabla_\nu\pi^{\mu\nu}$$

In each term the order of spatial gradient in the numerator exactly agrees with the order of temporal derivative in the denominator → **causality**

Newer degrees of freedom - a microscopic derivation

Exact expression of shear viscous flux (no truncation done)

$$\frac{\pi^{\alpha\beta}}{4!P}=\sum_{m=0}^{\infty}(-1)^m\frac{1}{(2m+5)}\frac{1}{(2m+3)}\frac{1}{(2m+1)}\left\{\frac{\tau_R}{1+\tau_R D}\right\}^{2m+1}\Big[\left(\nabla^2\right)^m\sigma^{\alpha\beta}+(4m)\left(\nabla^2\right)^{m-1}\nabla^{\langle\alpha}\nabla^{\nu}\sigma^{\beta\rangle}_{\nu}$$
$$+(2m)(m-1)\left(\nabla^2\right)^{m-2}\nabla^{\langle\alpha}\nabla^{\beta\rangle}\nabla_{\langle\mu}\nabla_{\nu\rangle}\sigma^{\mu\nu}\Big]$$
$$+\frac{1}{(\varepsilon+P)}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+5)}\frac{1}{(2n+3)}\left\{\frac{\tau_R}{1+\tau_R D}\right\}^{2n+2}\Big[\left(\nabla^2\right)^n\nabla^{\langle\alpha}\nabla^{\nu}\pi^{\beta\rangle}_{\nu}+n\left(\nabla^2\right)^{n-1}\nabla^{\langle\alpha}\nabla^{\beta\rangle}\nabla_{\langle\mu}\nabla_{\nu\rangle}\pi^{\mu\nu}\Big]$$

The nonlocal set of equations can be recast into a local set of equations by ‘integrating in’ new ‘non-fluid’ variables $\Longrightarrow$ **new degrees of freedom**

$$\left(1+\tau_R D\right)\pi^{\alpha\beta}=2\eta\sigma^{\alpha\beta}+\rho_1^{\alpha\beta}$$
$$\left(1+\tau_R D\right)^2\rho_1^{\alpha\beta}=-\eta\tau_R^2\left[\frac{2}{7}\nabla^2\sigma^{\alpha\beta}+\frac{12}{35}\nabla^{\langle\alpha}\nabla^{\nu}\sigma^{\beta\rangle}_{\nu}\right]+\rho_2^{\alpha\beta}$$
$$\left(1+\tau_R D\right)^2\rho_2^{\alpha\beta}=\eta\tau_R^4\left[\frac{2}{21}\nabla^4\sigma^{\alpha\beta}+\frac{32}{105}\nabla^2\nabla^{\langle\alpha}\nabla^{\nu}\sigma^{\beta\rangle}_{\nu}+\frac{4}{105}\nabla^{\langle\alpha}\nabla^{\beta\rangle}\nabla_{\langle\mu}\nabla_{\nu\rangle}\sigma^{\mu\nu}-\frac{16}{35}\Delta^{\alpha\beta}_{ab}\Delta^{b\nu}_{cd}\nabla^a\nabla^c\nabla_\nu\nabla_\rho\sigma^{\rho d}\right]+\rho_3^{\alpha\beta}$$
$$\vdots$$

S Mitra, Phys. Lett. B 860 (2025), 139174

For each higher order of spatial gradient truncation a new degree of freedom needs to be introduced
Otherwise, the causality of the theory is bound to be compromised

Field redefinition - impact on non-hydrodynamic spectra

The question at hand $\cdots$

- Outside equilibrium the fluid variables have ambiguity in their definition.
- They can be defined with arbitrary corrections that become nonzero with spacetime variations.
- These ambiguities are typically resolved by imposing constraints on the conserved currents, akin to gauge fixing - **frame choice**.
- The linearized spectrum changes with field redefinition - may pick up artefacts from arbitrary definitions.
- Crucial to ensure that the physical constraints derived from linearized perturbation analysis represent the true underlying physics.
- To probe the causality analysis one must incorporate **non-hydrodynamic modes**.
- Field redefinitions may change the structure of non-hydrodynamic modes - fluid that is causal in one frame might be acausal after field redefinition.

Which of the fluid models should be considered physical ?

Field redefinition - impact on non-hydrodynamic spectra

Set of variables $\{\Phi_i\}$ $\longrightarrow$ velocity, temperature, and conserved charges

$\downarrow$

Governed by a system of nonlinear coupled PDEs : $\mathcal{E}(\{\Phi_i\}) = 0 \longrightarrow$ stress tensor and charge current conservation

$\downarrow$

$\{\Phi_i\} = \{\bar{\Phi}_i\}$: equilibrium value invariant under spacetime translations and spatial rotations

$\downarrow$

Linearizing $\mathcal{E}$ around $\{\bar{\Phi}_i\}$: $\Phi_i^s = \bar{\Phi}_i + \delta\Phi_i(\omega, k)e^{-i\omega t + i\vec{k}\cdot\vec{x}}$

$\downarrow$

Linearization matrix equation : $\mathcal{E}(\{\Phi_i = \Phi_i^s\}) = 0 \Rightarrow \sum_j M_{ij}(\bar{\Phi}, \omega, k)\delta\Phi_j = 0$

Field redefinition : $\Phi_i \to \Psi_i = \Phi_i + \Delta\Phi_i$

$\downarrow$

At equilibrium, Φ_i and Ψ_i coincide $\longrightarrow$ $\bar{\Psi}_i = \bar{\Phi}_i$

$\downarrow$

Under field redefinition, the equations of motion transform as : $\mathcal{E}(\Phi) \to \tilde{\mathcal{E}}(\Psi)$ $\to$ distinctly different structure

$\downarrow$

Linearizing $\tilde{\mathcal{E}}$ around $\{\bar{\Phi}_i\}$: $\Psi_i^s = \bar{\Psi}_i + \delta\Psi_i(\omega, k)e^{-i\omega t + i\vec{k}\cdot\vec{x}}$

$\downarrow$

Linearization matrix equation : $\tilde{\mathcal{E}}(\{\Psi_i = \Psi_i^s\}) = 0 \Rightarrow \sum_j \tilde{M}_{ij}(\bar{\Phi}, \omega, k)\delta\Psi_j = 0$

Our objective is to establish the relationship between M_{ij} and $\tilde{M}_{ij}$

Field redefinition - impact on non-hydrodynamic spectra

Field redefinition impacts linearized Fourier spectra

Fourier representation of field redefinition : $\Delta\Phi_i(\Phi^s) = \sum_j S_{ij}(\bar{\Phi}, \omega, k)\delta\Phi_j(\omega, k)e^{-i\omega t + i\vec{k}\cdot\vec{x}}$

Modification in linearized spectra : $\mathrm{Det}[M] = \mathrm{Det}[\tilde{M}]\,\mathrm{Det}[\mathbf{1} + S]$

$\downarrow$

The zeros of $\mathrm{Det}[\mathbf{1} + S]$ represent new modes in the Φ frame that are absent in the Ψ frame

The problem $\cdots$

- These new modes are artefacts of the frame transformation with no physical significance.
- These artificial modes may appear unstable or acausal, even if the theory in the ψ frame is entirely valid.
- However, in the ϕ frame, there is no straightforward way to distinguish these artefacts from genuine physical modes.

Field redefinition - impact on non-hydrodynamic spectra

The solution $\cdots$

Field redefinition :

Frame-1 : $\{\hat{u}^\mu, \hat{T}\}$ Frame-2,: $\{u^\mu, T\}$ $u^\mu = \hat{u}^\mu + \Delta u^\mu(\hat{u}, \hat{T})$ $T = \hat{T} + \Delta T(\hat{u}, \hat{T})$

The most general expressions for Δu^μ and ΔT

$$\Delta u^\mu = F_u(\hat{u}\cdot\partial)\hat{u}^\mu + F_T(\frac{\hat{P}^{\mu\alpha}\partial_\alpha \hat{T}}{\hat{T}}) + R_u(\hat{P}^{\mu\theta}\hat{P}^{\alpha\beta}\partial_\alpha\partial_\beta\hat{u}_\theta) \qquad \frac{\Delta T}{\hat{T}} = G_u(\partial\cdot\hat{u}) + G_T(\frac{\hat{u}^\alpha\partial_\alpha\hat{T}}{\hat{T}}) + R_T(\frac{\hat{P}^{\alpha\beta}\partial_\alpha\partial_\beta\hat{T}}{\hat{T}})$$

$F_{u(T)}$, $G_{u(T)}$ and $R_{u(T)}$ are are linear differential operators

$$F_{u(T)} \equiv \sum_{m,n} f^{u(T)}_{m,n}\,[\hat{u}\cdot\partial]^m \left[\hat{P}^{\alpha\beta}\partial_\alpha\partial_\beta\right]^n, \quad G_{u(T)} \equiv \sum_{m,n} g^{u(T)}_{m,n}\,[\hat{u}\cdot\partial]^m \left[\hat{P}^{\alpha\beta}\partial_\alpha\partial_\beta\right]^n, \quad R_{u(T)} \equiv \sum_{m} r^{u(T)}_{m}[\hat{P}^{\alpha\beta}\partial_\alpha\partial_\beta]^m$$

General field redefinition of the fluid variables introduces an additional factor to the dispersion polynomial

$$\mathcal{F}(\omega,k^2) \equiv \mathrm{Det}[\mathbf{1}+S] = (1-i\omega F_u - k^2 R_u)\left[(1-i\omega G_T - k^2 R_T)(1-i\omega F_u - k^2 R_u) + k^2 F_T\, G_u\right]$$

- None of the zeros of $\mathcal{F}$ will have the form $\lim_{k\to 0}\omega(k) = 0 \longrightarrow$ redefining fluid variables will not affect the spectrum of hydrodynamic modes.
- At least one new mode will appear with a finite, non-zero frequency as $k \to 0$, resembling a genuine "non-hydrodynamic" mode.

Field redefinition - impact on non-hydrodynamic spectra

'Frame-2' dispersion polynomial has more non-hydro modes than in 'Frame-1' due to frame transformation

$$P(\omega, k^2) = \hat{P}(\omega, k^2)\mathcal{F}(\omega, k^2)$$

Our aim$\cdots$

- Identify a frame transformation that eliminates non-hydro modes while retaining precisely the original hydrodynamic sector - the "Hydro frame".
- This redefinition cannot fully erase the information about any physical non-hydrodynamic mode.
- The resulting stress tensor becomes an infinite series of all orders.
- The infinite series of the final stress-energy tensor has a radius of convergence located at the lowest non-hydro mode in the complex frequency space.
- The non-hydro mode acts as a cutoff in the hydrodynamic expansion.

If the transformed stress-energy tensor, after eliminating the non-hydrodynamic modes, include up to infinite orders derivative, then the original fluid theory is physical.

If the stress-energy tensor turns out to have a finite number of terms, then the non-hydro modes of the original theory must emerge from field redefinition artifacts solely and should not be considered physical.

S Bhattacharyya, S Mitra, S Roy, and R Singh, arXiv:2410.19015 [nucl-th]

Conclusion and outlook

Summary···

- Causality and stability require a consensus between the defining fluid variables, their gauge fixing and the theory to be truncated - so far at least one has to be compromised.
- The causality related issue arise because of the gradient truncation scheme of the hydro theory from the microscopic derivation.
- No truncated theory can be causal - the all order information retains either via new degrees of freedom or field redefinition.

Future remarks···

- Nonlinear causality analysis?
- Generalization to other stable-causal hydrodynamic models?