Modularity of Calabi-Yau Manifolds and Connections to Flux Compactifications

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Number Theory and Physics?

Physics: Geometric realizations are usually defined on spaces over $\mathbb{R}$ or $\mathbb{C}$

Number Theory: Framework of finite fields $\mathbb{F}_p$ (for p prime)

Local-to-Global principle:
Geometries defined over finite fields contain information of those defined over $\mathbb{C}$

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Example: *Modularity of Calabi-Yau manifolds*

- relates certain Calabi-Yau Geometries (in a unique way) to modular forms
- based on tools from arithmetic geometry
- has strong implications on physics (flux compactifications, attractors,...)

[Weil, Deligne, Dwork, Serre, Wiles, Taylor, Moore, Bönisch, Candelas, de la Ossa, Elmi, Fischbach, Hulek, Kachru, Klemm, Kuusela, McGovern, Nally, Rodrigues-Villegas, van Straten, Verrill, Yang,...]

Flux Vacua, Attractors and Hodge Substructures

Setup: Type IIB string theory compactified on a Calabi-Yau threefold X_6

Flux superpotential for type IIB string compactifications: [Gukov,Vafa,Witten, 2000]

$$W = \int_{X_6} \Omega\left(z^i\right) \wedge (F - \tau H) \qquad \Omega \in H^{3,0}(X_6, \mathbb{C})$$

- z^i and τ scalar fields (the complex structure moduli and the axio-dilaton)
- Internal topological three-form fluxes $F, H \in H^3(X_6, \mathbb{Z})$

Supersymmetric vacuum constraints:

$$\partial_{z^i} W = 0 \ , \ \partial_\tau W = 0 \ , \ W = 0$$

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Supersymmetric vacuum constraints:

$$\int_{X_6} \Omega(z^i) \wedge F = 0\ ,\ \int_{X_6} \Omega(z^i) \wedge H = 0\ ,\ \int_{X_6} \partial_{z^i}\Omega\left(z^i\right) \wedge (F - \tau H) = 0$$

X_6 supports a non-trivial flux configuration with a supersymmetric vacuum only if

$$\langle F, H\rangle_{\mathbb{Z}} \subset \left[H^{2,1}(X_6, \mathbb{C}) \oplus H^{1,2}(X_6, \mathbb{C})\right] \cap H^3(X_6, \mathbb{Z})$$

defines a two-dimensional sublattice

Flux Vacua, Attractors and Hodge Substructures

Setup: M- (or F-)theory compactified on a Calabi-Yau fourfold X_8

Flux superpotential:

$$W = \int_{X_8} \Omega(z^i) \wedge G \qquad \Omega(z^i) \in H^4(X_8, \mathbb{C})$$

- z^i complex structure moduli
- $G \in H^4(X_8, \mathbb{Z})$ internal topological four-form flux

Supersymmetric vacuum constraints imply

X_8 supports a non-trivial flux configuration with a supersymmetric vacuum only if

$$\langle G \rangle_{\mathbb{Z}} \subset \left[H^{4,0}(X_8, \mathbb{C}) \oplus H^{2,2}(X_8, \mathbb{C}) \oplus H^{0,4}(X_8, \mathbb{C}) \right] \cap H^4(X_8, \mathbb{Z})$$

$\Rightarrow$ one-dimensional sublattice

Flux Vacua, Attractors and Hodge Substructures

Setup: Type IIB string theory compactified on a Calabi-Yau threefold X_6

BPS black hole solutions evolve towards a critical point of the central charge

$$Z(Q) = \frac{\int_{X_6} Q \wedge \Omega(z^i)}{\int_{X_6} \Omega(z^i) \wedge \Omega(z^i)} \qquad \Omega(z^i) \in H^{3,0}(X_6, \mathbb{C})$$

- z^i complex structure moduli
- $Q \in H^3(X, \mathbb{Z})$ charge of the black hole

Attractor points: Critical points of $Z(Q)$

- For $|Z(Q)| \neq 0$: $\qquad Q \in H^{3,0}(X_6, \mathbb{C}) \oplus H^{0,3}(X_6, \mathbb{C})$

If $|Z(Q)| \neq 0$, X_6 has a rank-two attractor point only if

$$\langle Q, Q' \rangle_{\mathbb{Z}} \subset \left[H^{3,0}(X_6, \mathbb{C}) \oplus H^{0,3}(X_6, \mathbb{C}) \right] \cap H^3(X_6, \mathbb{Z})$$

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Flux Vacua, Attractors and Hodge Substructures

Flux vacua for type IIB string compactifications:

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$\Rightarrow$ two-dimensional sublattice of $H^3(X_6,\mathbb{Z})$ with definite Hodge type

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Flux vacua for M- (or F-)theory compactifications:

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Arithmetic Geometry

- Assume that

$$X = \{f_i(x_k) = 0\} \quad f_i(x) \in \mathbb{Z}[x_1, \ldots, x_m]$$

  is some (affine or projective) complex variety
- Treat X to be defined over the finite field $\mathbb{F}_{p^r}$ with p^r elements (p prime, $r \in \mathbb{N}$)

$$X/\mathbb{F}_{p^r} := \{\bar{f}_i(x) = 0\} \subset (\mathbb{F}_{p^r})^m$$

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$$X/\mathbb{F}_{p^r} := \{\bar{f}_i(x) = 0\} \subset (\mathbb{F}_{p^r})^m$$
- (Finite) Number of points
$$N_{p^r}(X) := |X/\mathbb{F}_{p^r}|$$
collected in the generating **local zeta function**
$$\zeta_p(X, T) = \exp\left(\sum_{r=1}^{\infty} N_{p^r}(X) \frac{T^r}{r}\right)$$
- **"Local-to-global principle":**
$\zeta_p(X, T)$ contain information about the Hodge structure of $H^k(X, \mathbb{Z})$

Arithmetic Geometry

Weil conjectures: Constrain $\zeta_p(X, T)$ strongly [Weil, 1949]

Rationality:

$$\zeta_p(X, T) = \frac{R_1(X, T) \cdots R_{2n-1}(X, T)}{R_0(X, T) \cdots R_{2n}(X, T)} \quad , \; n = \dim_{\mathbb{C}}(X)$$

$R_k(X, T)$ are polynomials of degree $b^k = \dim(H^k(X, \mathbb{Q}))$

In particular: $R_k(X, T) = \det(\mathbb{1} - T\mathrm{Fr}_p^{-1})$ for linear maps

$$\mathrm{Fr}_p : H^k(X, \mathbb{Q}_p) \to H^k(X, \mathbb{Q}_p)$$

$H^k(X, \mathbb{Q}_p)$: p-adic cohomology groups

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$H^k(X,\mathbb{Q}_p)$: p-adic cohomology groups

Important fact:
If $H^k(X,\mathbb{Z})$ has a Hodge substructure, Fr_p becomes block-diagonal
$\Rightarrow R_k(X,T)$ factorizes for (almost) all primes p!

The Modularity Conjecture

Consider an elliptic curve $\mathcal{E}$:

$$R_1(\mathcal{E}, T) = 1 - a_p T + pT^2 \qquad \text{with } a_p = p + 1 - N_p(\mathcal{E})$$

Modularity: $f(\tau) := \sum_{p \text{ prime}} a_p q^p \ , \ q = e^{2\pi i \tau}$ is a modular form of weight two

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For a Calabi-Yau n-fold X: If $H^k(X, \mathbb{Z})$ has a two-dimensional Hodge substructure:

$$R_k(X, T) = R_\Lambda(X, T) \cdot R_\Sigma(X, T) \quad \text{with } R_\Lambda(X, T) = 1 - a_p p^\alpha T + p^\beta T^2$$

for some (fixed) $\alpha, \beta \in \mathbb{N}$

Serre's Modularity Conjecture: [Serre, 1975]

$$f(\tau) := \sum_{p \text{ prime}} a_p q^p \quad , \quad q = e^{2\pi i \tau} \text{ is a modular form}$$

Manifolds of this type are called **modular**

The Modularity Conjecture

Modularity Conjecture: If $H^k(X, \mathbb{Z})$ has a two-dimensional Hodge substructure, then X is modular and in particular $R_k(X, T)$ has a quadratic factor for (almost) all primes p

Remarks:

- Elliptic curves and rigid Calabi-Yau threefolds ($h^{2,1} = 0$) are proven to be modular
- Generic Calabi-Yau n-folds (with $n \geq 2$) are **not** modular

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Strategy to find supersymmetric flux vacua:

Use Modularity as necessary condition for Hodge substructures

$\Rightarrow$ Compute $R_k(X, T)$ for many primes

If $R_k(X, T)$ has a quadratic factor for (almost) all primes, then X is a candidate to have a non-trivial Hodge substructure of $H^k(X, \mathbb{Z})$

[Kachru, Nally, Yang, 2020], [Candelas, de la Ossa, van Straten, 2020],...

Arithmetic Search for Fluxes

Question: How can we find modular Calabi-Yau n-folds if these are non-generic?
$\Rightarrow$ Analyze not a single manifold but scan the full (complex structure) moduli space

Setup: X_z a family of Calabi-Yau n-folds, $z \in \mathbb{C}$ modulus

Algorithm: For $p \geq 7$ prime

- The moduli space reduces to the finite set $z_p \in \mathbb{F}_p$
- For each $z_p \in \mathbb{F}_p$ compute $R_k(X_{z_p}, T)$
- Count $|\{z_p \in \mathbb{F}_p \mid R_k(X_{z_p}, T) \text{ factorizes quadratically}\}|$

If there is at least one point of factorization per prime p:

- Find $z \in \bar{\mathbb{Q}} \subset \mathbb{C}$ s.t.

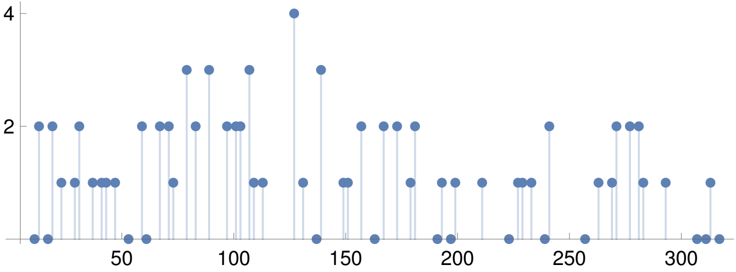
$$z_p \equiv z \mod p$$

  is a point of quadratic factorization for each prime p
- If such a $z \in \bar{\mathbb{Q}}$ exists, the (complex) variety X_z is a candidate to be modular

Example: Non-Modular Case

The mirror family of the complete intersection $\mathbb{P}^7[2,2,4]$: [Jockers, S.K., Kuusela, '23]

- Family of Calabi-Yau fourfolds X_z dependent on one modulus $z \in \mathbb{C}$
- Number of quadratic factorizations for each prime $7 \leq p \leq 317$:

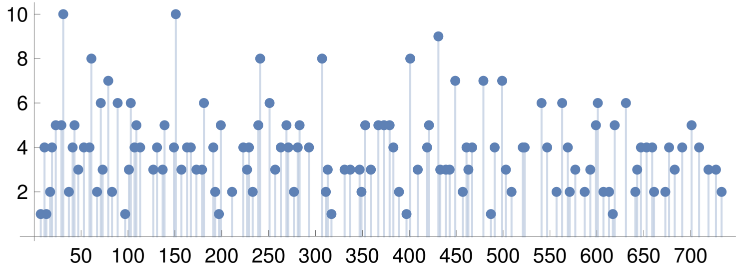


- Many primes p with no point $z_p \in \mathbb{F}_p$ s.t. $R_4(X_{z_p}, T)$ has a quadratic factorization
- The existence of an algebraic modulus $z \in \bar{\mathbb{Q}} \subset \mathbb{C}$ s.t. $H^4(X_z, \mathbb{Z})$ has a two-dimensional sublattice of definite Hodge type is highly unlikely

Example: Modular Case

A one-parameter family of Hulek-Verrill fourfolds HV_z^4: [Jockers, S.K., Kuusela, '23]

- Number of quadratic factorizations for each prime $7 \leq p \leq 733$



- At least one point $z_p \in \mathbb{F}_p$ for each prime s.t. $R_4(\mathrm{HV}_{z_p}^4, T)$ has a quadratic factorization
- There is potentially a modulus $z \in \bar{\mathbb{Q}}$ s.t. HV_z^4 is modular

Example: Modular Case

Reconstruction of possible modular points $z \in \bar{\mathbb{Q}} \subset \mathbb{C}$ from p-adic data:

- Collection of points $z_p \in \mathbb{F}_p$ with quadratic factorization

prime p	$z_p \in \mathbb{F}_p$			
$p = 11$	1	6	8	10
$p = 13$	1			
$p = 17$	1	15		

prime p	$z_p \in \mathbb{F}_p$			
$p = 19$	1	2	7	17
$p = 23$	1	4	5	12
$p = 29$	1	6	11	24

- (Rational) solution $z \in \mathbb{Q}$ s.t. $z_p \equiv z \mod p$ appears in this table:

$$z = 1$$

- $\mathrm{HV}^4_{z=1}$ is a candidate for a modular Calabi-Yau fourfold!

A Modular Calabi-Yau Fourfold

Consistency checks:

- Coefficients a_p of quadratic factor

$$R_\Lambda(\mathrm{HV}_1^4, T) = 1 - a_p p T + p^2 T^2$$

give the q-expansion of a unique modular form

- Identified generators of the two-dimensional Hodge substructure

$$\Lambda = \left[H^{3,1}(\mathrm{HV}_1^4, \mathbb{C}) \oplus H^{1,3}(\mathrm{HV}_1^4, \mathbb{C})\right] \cap H^4(\mathrm{HV}_1^4, \mathbb{Z})$$

by $\mathrm{Re}(\nabla_z \Omega(1)), \mathrm{Im}(\nabla_z \Omega(1))$

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- Remainder

$$\Sigma = \left[H^{4,0}(\mathrm{HV}_1^4, \mathbb{C}) \oplus H^{2,2}(\mathrm{HV}_1^4, \mathbb{C}) \oplus H^{0,4}(\mathrm{HV}_1^4, \mathbb{C})\right] \cap H^4(\mathrm{HV}_1^4, \mathbb{Z})$$

  defines suitable four-form fluxes

- In particular:

$$G := C \cdot \mathrm{Re}(\Omega(z))\big|_{z=1} \in \Sigma \, , \quad C \in \mathbb{R}$$

Physical Implications of Modularity

So far: Used modularity as a tool to obtain geometric information on X

For flux vacua of type IIB compactified on a Calabi-Yau threefold X:

- X is modular with modular form f_X
- Corresponding to f_X, there exists an (unique) elliptic curve $\mathcal{E}(f_X)$, s.t. $f_X = f_{\mathcal{E}(f_X)}$
- Axio-dilaton constraint: $\int_X \partial_z \Omega(z) \wedge (F - \tau H) = 0$
- The axio-dilaton is fixed by the complex structure of $\mathcal{E}(f_X)$ as

$$\tau = \tau(\mathcal{E}(f_X))$$

[Candelas, de la Ossa, Kuusela, McGovern, '23]

Similar for rank-two attractor points:

- f_X determines the area of the event horizon of the attractive BH

[Candelas, de la Ossa, Elmi, van Straten, '19]

Conclusions: Number theory and Physics!

Arithmetic geometry can be used as a tool to investigate varieties which are defined over $\mathbb{C}$

Modularity serves as a necessary condition for (two-dimensional) Hodge substructures, i.e. for

- supersymmetric flux vacua
- rank-two attractor points
- topology changing transition loci?

The corresponding modular form f_X contains physical information

- For type IIB flux vacua: The axio-dilaton τ
- For rank-two attractor points: The BH entropy S_{BH}