Invariants and the Characterization of Geometries

Exact and Numerical Applications

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The Cartan Equivalence Problem

Problem: How can we determine if two metrics describe the same spacetime, in different coordinate systems?

Cartan: Use the components of the curvature tensor and its derivatives in a particular frame, utilizing the Lorentz transformations to define a canonical frame invariantly, set the curvature tensor of each metric in this canonical form, and compare the components of them, and the sequences of transformations which lead to the canonical frames in each case.

Cartan-Karlhede Algorithm

- ▶ 1. Set differentiation order to q=0.
- ▶ 2. Calculate derivatives of the Riemann tensor up to qth order.
- ▶ 3. Fix the canonical frame of the Riemann tensor and derivatives.
- ▶ 4. Note the number of allowed frame transformations (dim H_q) – the dimension of the linear isotropy group.
- ▶ 5. Determine the number t_q of functionally independent components among the Riemann tensor and derivatives. (max is the dimension of the manifold, i.e. 4 in GR – number of coordinates your metric function has, essentially)
- ▶ 6. If the isotropy group and number of independent functions are the same as the previous step, we set $p + 1 = q$ and stop. Else repeat from step 2.

If any properties of the above are different (between solutions) at any order, the spacetimes are immediately distinguished from each other. If all properties are the same, then we can proceed to determining if coordinate transformations exist between the two metrics. (The hardest step)

Cartan-Karlhede Algorithm: The point

We use the components of the Riemann tensor and derivatives (the Cartan scalars) instead of coordinates to compare spacetimes. Thus we are directly comparing their geometric properties.

In principle, we should be able to use the lists of Cartan scalars of each metric to determine a coordinate transformation between the metrics. If two metrics are equivalent, the lists of Cartan scalars and the relationships amongst them will all be the same, and there will exist a coordinate transformation between the two sets.

The Newman Penrose Formalism

Karlhede's modification of Cartan's equivalence method relies on transforming the problem onto a frame basis in which the metric components are constants. One particular way of doing this is via the Newman Penrose formalism, where the metric has components:

$$\eta_{ab} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{1}$$

The frame consist of two real and two complex vectors, labelled $\{\ell^a, n^a, m^a, \bar{m}^a\}$, which are all null, and satisfy the following orthogonality conditions:

$$-\ell^a n_a = m^a \bar{m}_a = 1, \tag{2}$$

And all other inner products vanish.

The Newman Penrose Formalism

We can now project the curvature tensor, and its derivatives onto the frame, e.g.

$$
\begin{aligned}
R_{1212} &= R_{abcd}\ell^a n^b \ell^c n^d, \\
R_{1212;4} &= R_{abcd;e}\ell^a n^b \ell^c n^d \bar{m}^e,
\end{aligned}
\tag{3}
$$

We decompose the Riemann tensor into the Weyl tensor, Ricci tensor, and Ricci scalar, and define the algebraically independent components with the following labels:

$$
\begin{aligned}
\Psi_0 &= C_{1313}, \ \Psi_1 = C_{1213}, \\
\Psi_2 &= C_{1342}, \ \Psi_3 = C_{2124}, \\
\Psi_4 &= C_{2424}.
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
\Phi_{00} &= \frac{1}{2}S_{ab}\ell^a \ell^b, \ \Phi_{01} = \frac{1}{2}S_{ab}\ell^a m^b, \ldots \\
\Phi_{11} &= \frac{1}{4}S_{ab}(\ell^a n^b + m^a \bar{m}^b), \ldots \\
\Phi_{22} &= \frac{1}{2}S_{ab}n^a n^b, \ldots
\end{aligned}
\tag{5}
$$

The Newman Penrose Formalism

For convenience, we define the directional derivatives of the frame:

$$
\begin{aligned}
D &\equiv \ell^a \nabla_a, \\
\Delta &\equiv n^a \nabla_a, \\
\delta &\equiv m^a \nabla_a, \\
\bar{\delta} &\equiv \bar{m}^a \nabla_a.
\end{aligned}
\tag{6}
$$

The Newman Penrose Formalism

We define the connection coefficients of the frame, here referred to as *spin coefficients*:

$$-\kappa \equiv m^a \ell^b \ell_{a;b}, \quad (7a)$$

$$-\rho \equiv \bar{m}^a \ell^b m_{a;b}, \quad (7b)$$

$$-\sigma \equiv m^a m^b \ell_{a;b}, \quad (7c)$$

$$-\tau \equiv m^a \ell^b \ell_{a;b}, \quad (7d)$$

$$\nu \equiv \bar{m}^a n^b n_{a;b}, \quad (7e)$$

$$\mu \equiv \bar{m}^a m^b n_{a;b}, \quad (7f)$$

$$\lambda \equiv \bar{m}^a \bar{m}^b n_{a;b}, \quad (7g)$$

$$\pi \equiv \bar{m}^a \ell^b n_{a;b}, \quad (7h)$$

$$-\epsilon \equiv \frac{1}{2}(m^a \ell^b \bar{m}_{a;b} - n^a \ell^b \ell_{a;b}), \quad (7i)$$

$$-\gamma \equiv \frac{1}{2}(\ell^a n^b n_{a;b} - m^a n^b \bar{m}_{a;b}), \quad (7j)$$

$$-\beta \equiv \frac{1}{2}(n^a m^b \ell_{a;b} - \bar{m}^a m^b m_{a;b}), \quad (7k)$$

$$\alpha \equiv \frac{1}{2}(\ell^a \bar{m}^b n_{a;b} - m^a \bar{m}^b \bar{m}_{a;b}). \quad (7l)$$

The Newman Penrose Formalism: Lorentz Transformations

The usefulness of the Newman Penrose formalism for the CK algorithm is obvious after we specify the Lorentz transformations of the frame:

Boosts and spins: $\lambda_b, \theta \in \mathbb{R}$

$$
\begin{aligned}
\ell^a &\to \lambda_b \ell^a, \ n^a \to \frac{n^a}{\lambda_b}, \\
m^a &\to e^{i\theta} m^a, \ \bar{m}^a \to e^{-i\theta} \bar{m}^a.
\end{aligned}
\tag{8}
$$

Null rotation about ℓ: $E, B \in \mathbb{C}$

$$
\begin{aligned}
\ell^a &\to \ell^a, \\
n^a &\to n^a + E m^a + \bar{E} \bar{m}^a + E \bar{E} \ell^a, \\
m^a &\to m^a + E \ell^a, \\
\bar{m}^a &\to \bar{m}^a + \bar{E} \ell^a.
\end{aligned}
\tag{9}
$$

For null rotations about n, swap $E \to \bar{B}$ and $\ell \leftrightarrow n$.

A Simple Example

Given the Schwarzschild solution,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (10)$$

we wish to utilize the Cartan Karlhede algorithm to determine the Cartan scalars for a canonical frame for this solution.

Choosing an initial null frame to begin with, we can choose any basis, and orthogonalize it via the Gram-Schmidt process, or if the metric is in a nice enough form, we can determine and write down immediately the orthonormal frame. Then we construct the null vectors via simple combinations of the orthonormal frame.

Schwarzschild Initial Null Frame

$$\ell_\mu = \frac{1}{\sqrt{2}} \left\{ \sqrt{1 - \frac{2M}{r}}, \frac{1}{\sqrt{1 - \frac{2M}{r}}}, 0, 0 \right\}, \qquad \text{(11a)}$$

$$n_\mu = \frac{1}{\sqrt{2}} \left\{ \sqrt{1 - \frac{2M}{r}}, -\frac{1}{\sqrt{1 - \frac{2M}{r}}}, 0, 0 \right\}, \qquad \text{(11b)}$$

$$m_\mu = \frac{1}{\sqrt{2}} \left\{ 0, 0, r, ir\sin(\theta) \right\}, \qquad \text{(11c)}$$

$$\bar{m}_\mu = \frac{1}{\sqrt{2}} \left\{ 0, 0, r, -ir\sin(\theta) \right\}. \qquad \text{(11d)}$$

Schwarzschild Example

Given the initial frame, we can choose the canonical form ourselves. The only stipulation is that the frame of some other metric we are comparing to should also be transformed to this canonical frame via Lorentz transformations.

Since we know that the Schwarzschild solutions are Petrov Type D, we *know* there exists a frame such that the Weyl NP scalars have the following form:

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0,$$
$$\Psi_2 \neq 0. \tag{12}$$

So at zeroth order, we can demand that our frames must be put in a form with the above Weyl scalar properties. This choice uses up both of our null rotations.

Schwarzschild Example

For our typical choice of initial null frame for this Schwarzschild metric, the zeroth order tensor is already in this canonical form.

Now we need to determine the Lorentz transformations which keep the curvature tensor at zeroth order invariant. Since the scalar Ψ_2 is defined via the frame:

$$\Psi_2 = C_{abcd}\ell^a m^b \bar{m}^c n^d, \tag{13}$$

under a boost transformation, $\ell^a \to \lambda_b \ell^a, \ n^a \to n^a/\lambda_b$:

$$\Psi_2' = C_{abcd}\lambda_b \ell^a m^b \bar{m}^c n^d/\lambda_b = \Psi_2 = -\frac{M}{r^3} \tag{14}$$

Clearly the boost parameters cancel, thus Ψ_2 is invariant under a boost transformation. Similarly, it is also invariant under a spin transformation. Note: we can now replace the coordinate r with the Cartan scalar Ψ_2 in any coordinated expression!

Schwarzschild Example: Zeroth Order

At zeroth order, we can then conclude for the Schwarzschild metric, in our frame that we defined previously, we have fixed the null rotations by demanding that the frame is in its Petrov Type D form. The Algorithm at zeroth order thus concludes with the following properties:

- $t_0 = 1$
- $\dim H_0 = 2$ (spins and boosts)
- $C_0 = \{\Psi_2\}$

We now move to first order, taking a derivative of the curvature tensor...

Schwarzschild Example: First Order

We find that at first order, the curvature tensor only contains terms proportional to the following:

$$D\Psi_2, \Delta\Psi_2, \rho\Psi_2, \mu\Psi_2. \tag{15}$$

To fix the frame at this order, let us demand that the following relationship is satisfied among the curvature tensor components:

$$C_{1342;1} = D\Psi_2 = -\Delta\Psi_2 = -C_{1342;2}, \tag{16}$$

Since both are affected by a boost transformation, we can choose the boost that forces the frame to satisfy this relationship. Thus we have removed the boost transformation at first order. Due to the spherical symmetry, there is no way to remove the spin freedom, so we choose to arbitrarily set $\theta = 0$, as it makes no changes anywhere. All Lorentz freedom is now gone.

Schwarzschild Example: First Order

In coordinates, the Cartan scalar at first order can be written in terms of the Cartan scalar at zeroth order:

$$C_{1212;1} = \frac{6M\sqrt{r-2M}}{\sqrt{2}r^{9/2}} = \left(\frac{\Psi_2{}^4}{M}\right)^{1/3}\sqrt{\frac{1}{2} + \Psi_2{}^{1/3}M^{2/3}} \quad (17)$$

Notice how we have replaced the coordinate with the Cartan scalar. It is somewhat nice that we can do it in this case, its not always possible to solve the system of Cartan scalars for the coordinates.

The fact we still see the M inside the Cartan scalar after eliminating all coordinates means that M is an intrinsic feature of the Schwarzschild geometry, and two Schwarzschild solutions with different values of M are *not* equivalent!

Schwarzschild Example: First Order

Incidentally, our choice of boost also makes the spin coefficients $\rho = \mu$, so that now there is only a single algebraically independent component of the curvature tensor's first derivative. Additionally, the Ricci and Bianchi identities simplify to give us:

$$C_{1342;1} = 3\rho\Psi_2. \tag{18}$$

All other nonzero components of the curvature tensor are a constant multiple of this function.

Schwarzschild Example: Conclusion

We end at first order, as no new functionally independent components have arisen, thus the algorithm concludes with:

- $t_1 = 1$,
- $\dim H_1 = 0$,
- $C_1 = \{\Psi_2\}$.

The second order algorithm produces:

- $t_2 = 1$,
- $\dim H_2 = 0$,
- $C_2 = \{\Psi_2\}$.

Thus we can now stop, since the isotropy is spent, and we only have a single independent function.

Spherically Symmetric Perfect Fluids

We examined spherically symmetric metrics of the form:

$$ds^2 = -A(t,r)^2 dt^2 + B(t,r)^2 dr^2 + R(t,r)^2 d\Omega^2, \qquad (19)$$

with a perfect fluid source:

$$T_{ab} = (\tilde{\rho}(t,r) + p(t,r))u_a u_b + p g_{ab}, \qquad (20)$$

and the perfect fluid is barotropic, with linear equation of state:

$$p = \omega \rho, \qquad (21)$$

with $0 < \omega \leq 4/3$. We wish to model the collapse of a cloud of matter to a black hole.

Spherically Symmetric Perfect Fluids: Cartan Scalars

We applied the Cartan-Karlhede algorithm on the above spacetime to determine the Cartan scalars, which can be written as the NP functions, and in terms of fluid functions and metric functions:

$$\Phi_{00} = \Phi_{22} = 2\Phi_{11} = \kappa(\tilde{\rho} + p), \tag{22a}$$

$$\Psi_2 = \frac{\kappa\tilde{\rho}}{6} - \frac{M}{R^3}, \tag{22b}$$

$$\rho = \frac{-AR' + B\dot{R}}{\sqrt{2}ABR}, \tag{22c}$$

$$\mu = -\frac{AR' + B\dot{R}}{\sqrt{2}ABR}, \tag{22d}$$

$$\tag{22e}$$

The two spin coefficients ρ, μ are 'extended' Cartan scalars – i.e. scalars constructed out of Cartan scalars. Here, Φ_{22} and Ψ_2 are the functionally independent Cartan scalars. They can in principle replace the t and r coordinates.

Numerical Solutions

The field equations can be written as a set of first order coupled PDEs, for the functions A, B, R, M, subject to a constraint

$$\frac{\dot{R}^2}{A^2} = \frac{2M}{R} + \frac{R'^2}{B^2} - 1, \tag{23a}$$

$$M' = \frac{\kappa}{2}\rho R^2 R', \tag{23b}$$

$$\dot{M} = -\frac{\kappa}{2}p R^2 \dot{R}, \tag{23c}$$

$$B(t + \Delta t, r) = R' B_0 \exp\left(-\int_t^{t+\Delta t} \frac{A'}{A}\frac{\dot{R}}{R'} dt\right), \tag{23d}$$

$$\frac{A'}{A} = -\frac{p'}{\rho + p}. \tag{23e}$$

Numerical Solutions

The last equation is a conservation equation, and the only way to determine A. There is no dynamical equation for A. $B_0(r)$ is an arbitrary function, and the system is subject to boundary conditions:

$$R(t,0) = 0, \ M(t,0) = 0, \ R'(t,0) = 1, \ \dot{R}(t,0) = 0,$$
$$p(t,r_s) = 0, \ A(t,r_s) = 1, \tag{24}$$

the initial density profile and areal radius:

$$\tilde{\rho}(0,r) = 10\exp(-4r^2), \ R(0,r) = r, \tag{25}$$

and the arbitrary function:
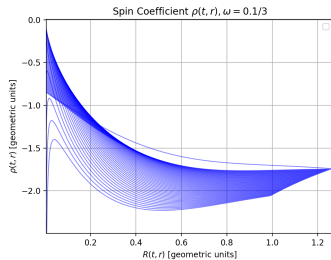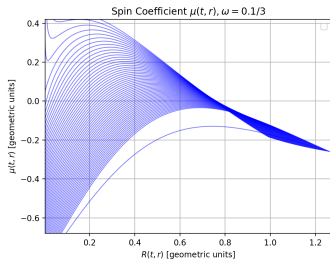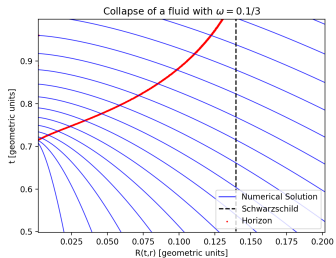
$$B_0(r) = 1. \tag{26}$$

The dynamical spacetime is matched to an exterior Schwarzschild solution at $r = r_s$.

The Basic Numerical Method

We wrote a C++ code to blast through the following numerical procedure:

1. In the integrated $B(t, r)$ equation, replace the integral with a trapezoidal integration over the time interval $[t, t + \Delta t]$.

2. From $\dot{R}_i$ and $\dot{M}_i$, integrate forward R_i and M_i to R_{i+1}^P, M_{i+1}^P using an Euler step, these are predicted values. Additionally, compute and store the numerical derivatives, $R_{i+1}'^P, M_{i+1}'^P$, using a fourth order finite difference method.

3. From $R_{i+1}'^P, M_{i+1}'^P$, compute ρ_{i+1}^P, p_{i+1}^P. Also compute and store p_{i+1}'.

4. From $p_{i+1}'^P, \rho_{i+1}^P, p_{i+1}^P$, solve the conservation equation for A_{i+1}^P, using a Runge-Kutta 4 step method, with boundary condition $A(r_s) = A_s$.

5. From the Friedmann constraint, solve for $\dot{R}_{i+1}^P$ using a bisection method.

6. If there are issues with bisection near $r = 0$, fill in the erratic data using a rational polynomial interpolation method, using the boundary condition $\dot{R}^P(t, 0) = 0$ (Turned off in all results below – experimental feature).

7. Compute $\dot{M}_{i+1}^P$.

8. Use $\dot{M}_{i+1}^P, \dot{R}_{i+1}^P$ and $\dot{R}_i, \dot{M}_i$ to compute a corrected value for R_{i+1}, M_{i+1} using the trapezoidal method.

9. With $\dot{R}, \dot{M}$ updated, repeat from step 2.

Numerical Results

Collapse of a fluid with $\omega = 0.1/3$

Spin Coefficient $\mu(t, r)$, $\omega = 0.1/3$

Spin Coefficient $\rho(t, r)$, $\omega = 0.1/3$

Applications of the Invariant Classification

In GR:

- ▶ Identifying geometric horizons (also apparent horizons, ergosurfaces).
- ▶ Locating/tracking shell crossing singularities and shell focussing singularities. These are necessary for cosmological models like in Szekeres, which generically form shell crossings.
- ▶ Numerical collapse and error tracking
- ▶ Photon surface definition and identification

In Brans Dicke:

- ▶ Properties of horizons in spherically symmetric vacuum spacetimes
- ▶ Locally naked singularities

Future Work

- Further development of the collapse problem to involve more sophisticated numerical methods.
- Photon Sphere Dynamics
- Applications of Cartan Scalars to Numerical Cosmology

References

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