Photon Surfaces and Shadows of Compact Objects

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Outline

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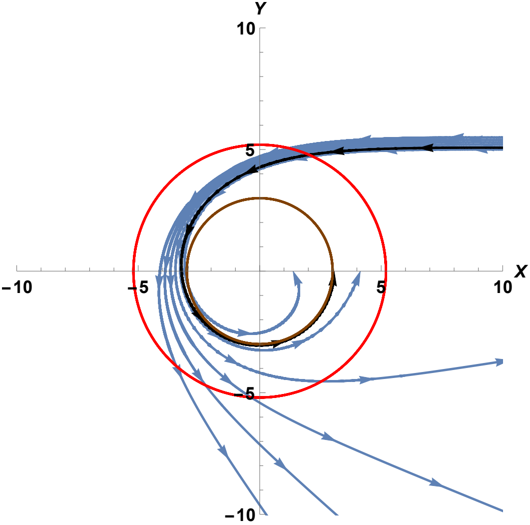
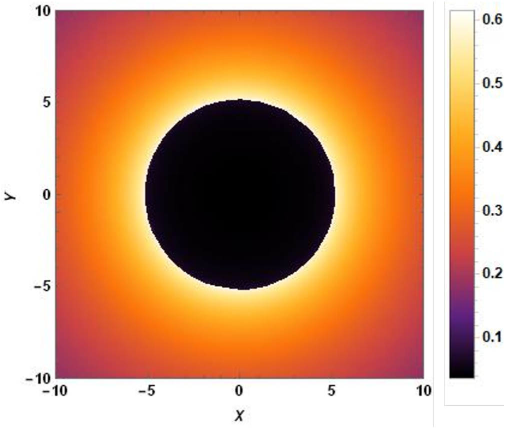
1. Shadow and Photon Rings
2. Standard procedure of determining photon sphere radius
3. Coordinate independent definition of photon surfaces
4. Invariant definition of photon surfaces in spin frame
5. Photon surfaces in Type-D solutions with $SO(3) \times R$ symmetry
6. Photon surface dynamics in collapsing spherically symmetric compact objects
7. Conclusion

Null geodesics in the Schwarzschild spacetime:



A.B. Joshi, D.Dey, et al., Phys. Rev. D **102**, no.2, 024022 (2020).
D.Dey, R.Shaikh, et al., Phys. Rev. D **103**, no.2, 024015 (2021).

SHADOW OF SCHWARZSCHILD BLACK HOLE:



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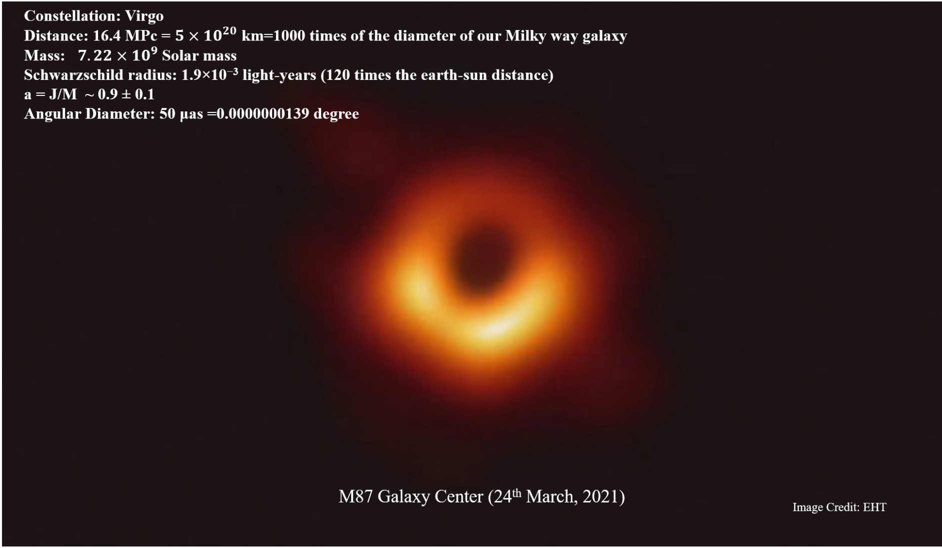
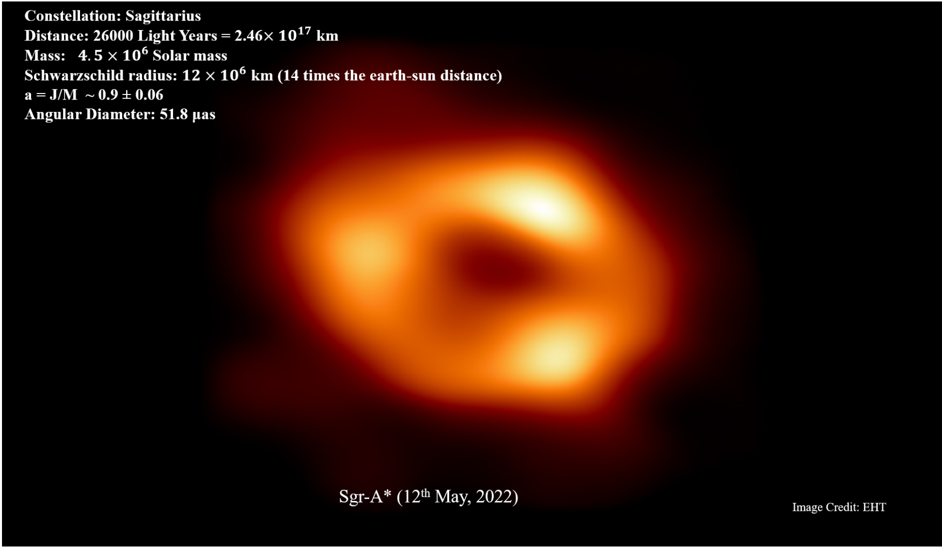
D. Dey, R. Shaikh, et al., Phys. Rev. D **102**, no.4, 044042 (2020).

SHADOW OF M87 GALAXY CENTER:



SHADOW OF SGR-A*:



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Photon sphere radius in a spherically symmetric static spacetime:

- The line element of a spherically symmetric, static spacetime can be written as,

$$dS^2 = -\mathcal{A}(r)dt^2 + \mathcal{B}(r)dr^2 + r^2 d\omega^2 \ , \tag{1}$$

where $d\omega^2 = d\theta^2 + \sin^2\theta \ d\phi^2$.

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- For null geodesics, we can write,

$$-\frac{e^2}{\mathcal{A}(r)} + \frac{l^2}{r^2} + \mathcal{B}(r)\dot{r}^2 = 0 \ , \tag{2}$$

  which implies,

$$V_{eff}(r) + \mathcal{A}(r)\mathcal{B}(r)\dot{r}^2 = e^2 \ , \tag{3}$$

  where $V_{eff}(r) = l^2 \frac{\mathcal{A}(r)}{r^2}$.

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- If there exists a photon sphere at a particular radius r_{ph} then

$$V_{eff}(r_{ph}) = e^2, V'_{eff}(r_{ph}) = 0, \ \text{and} \ V''_{eff}(r_{ph}) < 0,$$

which implies:

$$r_{ph} = \frac{2\mathcal{A}(r_{ph})}{\mathcal{A}'(r_{ph})} \ , \tag{4}$$

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If the spacetime mentioned in Eq. (1) does not allow any photon sphere then one would not find any real, positive solution of Eq. (4) for r_{ph}.

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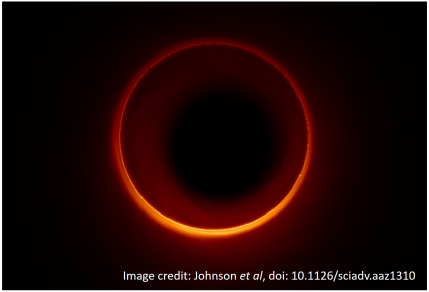
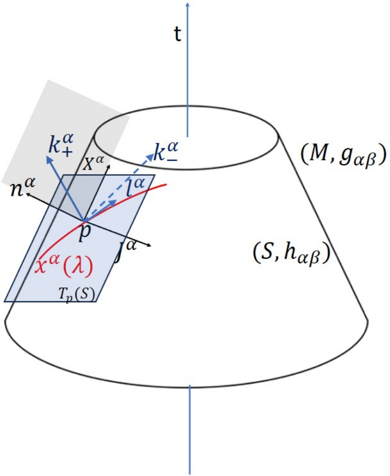


Definition of Photon surface: A photon surface S in a spacetime manifold $(\mathcal{M}, g^{\alpha\beta})$ is an immersed, non-spacelike hypersurface of $(\mathcal{M}, g^{\alpha\beta})$: for every point $p \in S$ and every null vector $l^{\mu} \in T_p S$, there exists a null geodesic $x^{\alpha}(\lambda) : (-\epsilon, \epsilon) \to \mathcal{M}$ of $(\mathcal{M}, g^{\alpha\beta})$ such that $\dot{x}^{\alpha}|_{\lambda=0} = l^{\alpha},\ x^{\alpha}(\lambda) \subset S$. (C. M. Claudel, K. S. Virbhadra and G. F. R. Ellis, J. Math. Phys. **42**, 818-838 (2001).)

A photon surface with $SO(3) \times R$ symmetry is known as the photon sphere.

CONDITIONS FOR PHOTON SURFACE:



D.Dey, A.A.Coley and N.T.Layden, Phys. Rev. D **109**, no.6, 064021 (2024).

Based on the definition of a photon surface given above, one can show the following conditions are equivalent (C. M. Claudel, K. S. Virbhadra and G. F. R. Ellis, J. Math. Phys. **42**, 818-838 (2001)):

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3. $\sigma_{\mu\nu} = 0$, where the $\sigma_{\mu\nu}$ is the traceless part of $K_{\mu\nu}$;
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The above conditions are true for all smooth generic spacetime manifolds without any symmetry.

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Invariant definition of a null frame:

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- CK algorithm employs the algebraic type and symmetry of a solution to fix a null frame invariantly.
- The initial step involves identifying a null frame in which the Riemann tensor and its derivatives take their canonical form. We refer to a null frame that fulfills this condition as a canonical frame. As an example, for a Petrov-type D solution, in a canonical null frame $(k_+^\alpha, k_-^\alpha, m^\alpha, \overline{m}^\alpha)$, the only non-zero component of Weyl tensor $(C_{\mu\nu\alpha\beta})$ is ψ_2, where

$$\psi_2 = C_{\mu\nu\alpha\beta} k_+^\mu m^\nu \overline{m}^\alpha k_-^\beta \neq 0,$$

and the other components

$$\begin{aligned}
\psi_0 &= C_{\mu\nu\alpha\beta} k_+^\mu m^\nu k_+^\alpha m^\beta = 0, \\
\psi_1 &= C_{\mu\nu\alpha\beta} k_+^\mu k_-^\nu k_+^\alpha m^\beta = 0, \\
\psi_3 &= C_{\mu\nu\alpha\beta} k_+^\mu k_-^\nu \overline{m}^\alpha k_-^\beta = 0, \\
\psi_4 &= C_{\mu\nu\alpha\beta} \overline{m}^\mu k_-^\nu \overline{m}^\alpha k_-^\beta = 0.
\end{aligned}$$

- After defining the canonical null frame, depending upon the symmetry of the given solution, the CK algorithm invariantly fixes the frame further preserving the canonical form of the Riemann tensor and its derivatives (for the above example: type-D canonical form of the Weyl tensor and its derivatives).

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- For a spherically symmetric Petrov type-D solution, it can be shown that a null frame can be invariantly defined up to the spatial spin. For type-D non-spherical solutions, one has to find an invariant way to fix the frame completely by fixing the spatial spin.

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- Cartan scalars are the projection of the Riemann tensor and the finite number of its derivatives on an invariantly defined null frame. Using the Cartan scalars, one can completely describe the local geometry of a spacetime manifold.
- Since the definition of photon surface given by C.M. Claudel, et al., is local, a photon surface must be invariantly defined in terms of the Cartan scalars.

PHOTON SURFACE DEFINITION IN TERMS OF CARTAN SCALARS:

- In the non-canonical frame $(l^{\alpha}, p^{\alpha}, M^{\alpha}, \bar{M}^{\alpha})$ where $l^{\alpha} \in T_p(S)$, the photon surface condition $K_{\mu\nu} l^{\mu} l^{\nu} = 0$ can be written in terms of the spin coefficients corresponding to that frame:

$$\begin{aligned} \tilde{\kappa} = \bar{\tilde{\kappa}} &= 0\,, \\ \tilde{\epsilon} + \bar{\tilde{\epsilon}} &= 0. \end{aligned} \tag{5}$$

- The above condition is true on a photon surface. However, since the null frame $(l^{\alpha}, p^{\alpha}, M^{\alpha}, \bar{M}^{\alpha})$ is not invariantly defined, the photon surface condition is also not defined invariantly.

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- Therefore, our primary task is to find a Lorentz transformation that transforms the null frame $(l^\alpha, p^\alpha, M^\alpha, \bar{M}^\alpha)$ to an invariant null frame $(k_+^\alpha, k_-^\alpha, m^\alpha, \overline{m}^\alpha)$ and investigate how the above condition changes under the Lorentz transformation.

Photon surface definition in terms of Cartan scalars:

- Using the general Lorentz transformation which relates the null frame $(l^\alpha, p^\alpha, M^\alpha, \bar{M}^\alpha)$ to the invariant frame $(k_+^\alpha, k_-^\alpha, m^\alpha, \overline{m}^\alpha)$, one can write down the photon surface condition in terms of spin coefficients corresponding to that invariant frame (D.Dey, A.A.Coley and N.T.Layden, Phys. Rev. D **109**, no.6, 064021 (2024).):

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&+\; E\left(DA + \frac{E\overline{E}}{A^2}\,\Delta A + \frac{\overline{E}}{A}\,\delta A + \frac{E}{A}\,\bar{\delta} A\right) + \left(\overline{E}E\;DE + \Delta E + \overline{E}\delta E + E\delta E\right) \;=\; 0\,,
\end{aligned}}
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where A is the real boost parameter, E and B are complex null rotation parameters, and the θ is the real spin parameter and all are functions of all of the spacetime coordinates in general.

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- It should be noted that the above invariant photon surface condition is true for all Petrov type-D solutions and independent of symmetry.

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- It should be noted that the above invariant photon surface condition is true for all Petrov type-D solutions and independent of symmetry.
- Now, we can use the above equation to redefine the photon surface condition using Cartan scalars. The above condition of the photon surface can be written concisely as $f(\mathcal{S}) = \overline{f}(\overline{\mathcal{S}}) = 0$, where f is the function of a set of spin coefficients (corresponding to the invariant frame) represented by $\mathcal{S}$.

Photon surface definition in terms of Cartan scalars:

- Since the condition $f(\mathcal{S}) = \bar{f}(\overline{\mathcal{S}}) = 0$ is always true on the photon surface, we can define the normal n^μ at any point on the surface as:

$$n_\alpha = \nabla_\alpha f(\mathcal{S})|_{f(\mathcal{S})=0} = \nabla_\alpha \bar{f}(\overline{\mathcal{S}})|_{\bar{f}(\overline{\mathcal{S}})=0}. \tag{6}$$

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- Using the above expression of normal and the fact that it is orthogonal to the null vector $l^{\alpha} \in T_p(S)$, i.e., $l^{\alpha} n_{\alpha} = 0$, we can write down the photon surface condition in the following way:

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- Since the above scalar equation consists of frame derivatives of spin coefficients, it can provide us with the photon surface condition in terms of Cartan scalars defined in the invariant frame $(k_+^\alpha, k_-^\alpha, m^\alpha, \bar{m}^\alpha)$.

Outline

Photon surface condition for spherically symmetric static Petrov Type-D solutions:

- For $SO(3) \times R$ symmetric Pertrov type-D solutions, the photon surface condition simplifies to:

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which for vacuum Schwarzschild spacetime becomes:

$$\psi_2 = -8\epsilon^2 \tag{7}$$

- For the following spherically symmetric and static spacetime:

$$dS^2 = -\mathcal{A}(r)dt^2 + \mathcal{B}(r)dr^2 + r^2 d\omega^2 \ ,$$

using the CK algorithm, it can be shown that the complete set of algebraically independent Cartan scalars is: $\psi_2, \phi_{00}, \phi_{11}, R, \epsilon, \rho,$ *and their frame derivatives*.

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using the CK algorithm, it can be shown that the complete set of algebraically independent Cartan scalars is: $\psi_2, \phi_{00}, \phi_{11}, R, \epsilon, \rho,$ *and their frame derivatives*.

- Hence, the condition for the photon surface i.e., $\phi_{00} - 8\epsilon^2 - \psi_2 - \frac{R}{12} = 0$ represents a specific relation among the Cartan scalars defined in the invariant null frame, and this relation holds exclusively on the photon surface.

- For the following spherically symmetric and static spacetime:

$$dS^2 = -\mathcal{A}(r)dt^2 + \mathcal{B}(r)dr^2 + r^2 d\omega^2 \ ,$$

  using the CK algorithm, it can be shown that the complete set of algebraically independent Cartan scalars is: $\psi_2, \phi_{00}, \phi_{11}, R, \epsilon, \rho,$ *and their frame derivatives*.

- Hence, the condition for the photon surface i.e., $\phi_{00} - 8\epsilon^2 - \psi_2 - \frac{R}{12} = 0$ represents a specific relation among the Cartan scalars defined in the invariant null frame, and this relation holds exclusively on the photon surface.
- Using the invariant condition of photon surfaces and the expressions of ψ_2, R, ϕ_{00}, we get:

$$\left.\frac{\mathcal{A}'(-r\mathcal{A}' + 2\mathcal{A}(r))}{4r\mathcal{A}(r)^2\mathcal{B}(r)}\right|_{r=r_{Ph}} = 0 \implies r_{Ph} = \left.\frac{2\mathcal{A}(r)}{\mathcal{A}'(r)}\right|_{r=r_{Ph}} , \tag{8}$$

  where r_{Ph} is the radius of photon sphere.

- We can determine the radius of the photon surface, if it exists in a given spacetime manifold, by solving the aforementioned algebraic equation for r_{Ph}. For Schwarzschild spacetime, the solution of the above equation is $r_{Ph} = 3M$, where M is the Schwarzschild mass.

Outline

- A collapsing spherically symmetric dust ball can be modeled by Limetre-Tolman-Bondi (LTB) spacetime:

$$dS^2 = -dt^2 + \frac{\mathcal{R}'^2(t,r)}{1+f(r)}\,dr^2 + \mathcal{R}^2(t,r)\,d\omega^2 \tag{9}$$

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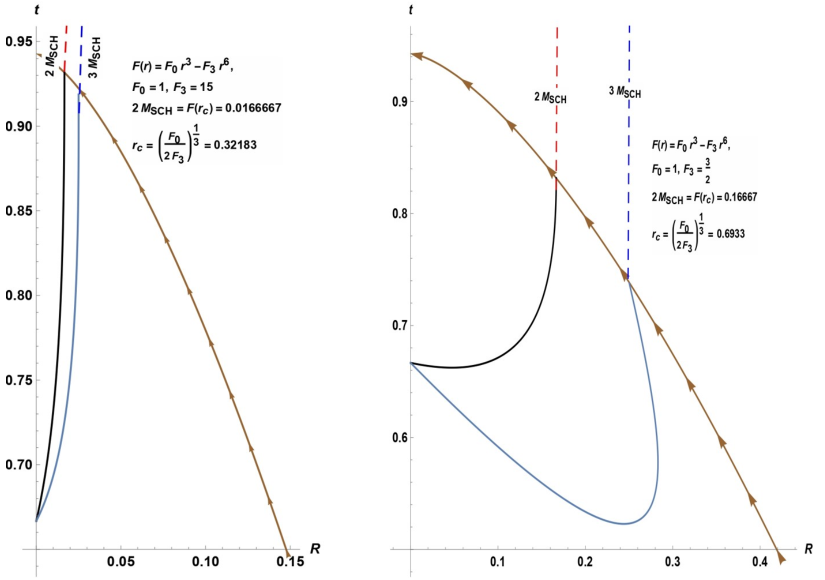
- For marginally bound scenario (i.e., $f(r) = 0$), the photon surface condition implies the following condition:

$$\boxed{\mathcal{R}\dot{\mathcal{R}}^3\dot{\mathcal{R}}' - \mathcal{R}\mathcal{R}'\ddot{\mathcal{R}} - \mathcal{R}'\left(\dot{\mathcal{R}}^2 - 1\right)^2 = 0.}$$


t
0.95
0.90
0.85
0.80
0.75
0.70
2 M_SCH
3 M_SCH
F(r) = F0 r^3 − F3 r^6,
F0 = 1, F3 = 15
2 M_SCH = F(r_c) = 0.0166667
r_c = (F0 / 2 F3)^(1/3) = 0.32183
0.05
0.10
0.15
R
t
0.9
0.8
0.7
0.6
2 M_SCH
3 M_SCH
F(r) = F0 r^3 − F3 r^6,
F0 = 1, F3 = 3/2
2 M_SCH = F(r_c) = 0.16667
r_c = (F0 / 2 F3)^(1/3) = 0.6933
0.1
0.2
0.3
0.4
R

The Vaidya spacetime can model collapsing null dust. The line element of the imploding Vaidya spacetime can be written as:

$$dS^2 = -\left(1 - \frac{2M(v)}{r}\right) dv^2 + 2dr\,dv + r^2 d\omega^2 \,, \tag{10}$$

where v is a null coordinate defined as: $v = t + r + 2M(v)\log\left(\frac{r}{2M(v)} - 1\right)$ and the ADM mass $M(v)$ is a positive definite function of v.

Photon surface dynamics in a spherically symmetric imploding null dust:

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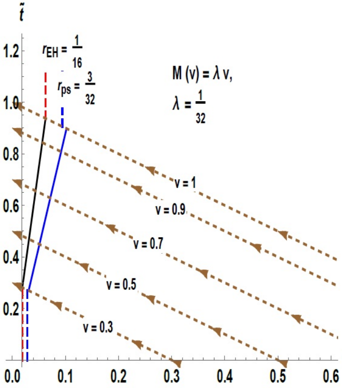

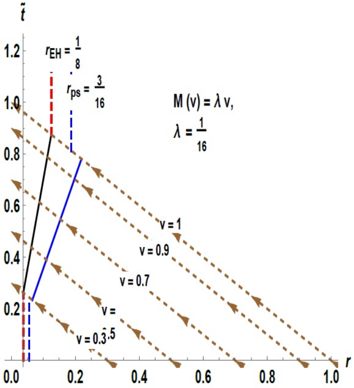

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- In the future, we aim to explore the photon surface dynamics in non-spherical type-D solutions, e.g., Kerr, Szekeres, etc.
- In addition to studying photon surfaces in type-D solutions, we are also interested in examining the properties of photon surfaces in colliding black hole scenarios. This research could have significant implications both theoretically and observationally.

Thank You