Eigenvalue Spectrum of MOTS Stability in Weyl-Distorted Schwarzschild Blackholes

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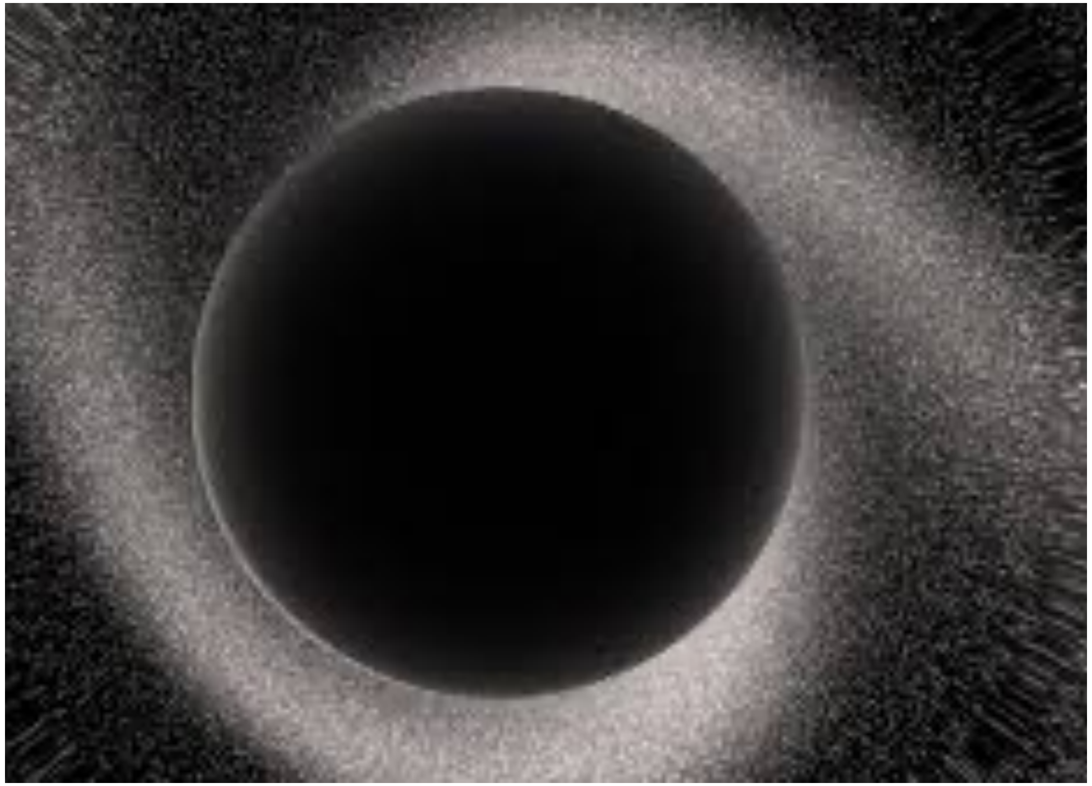
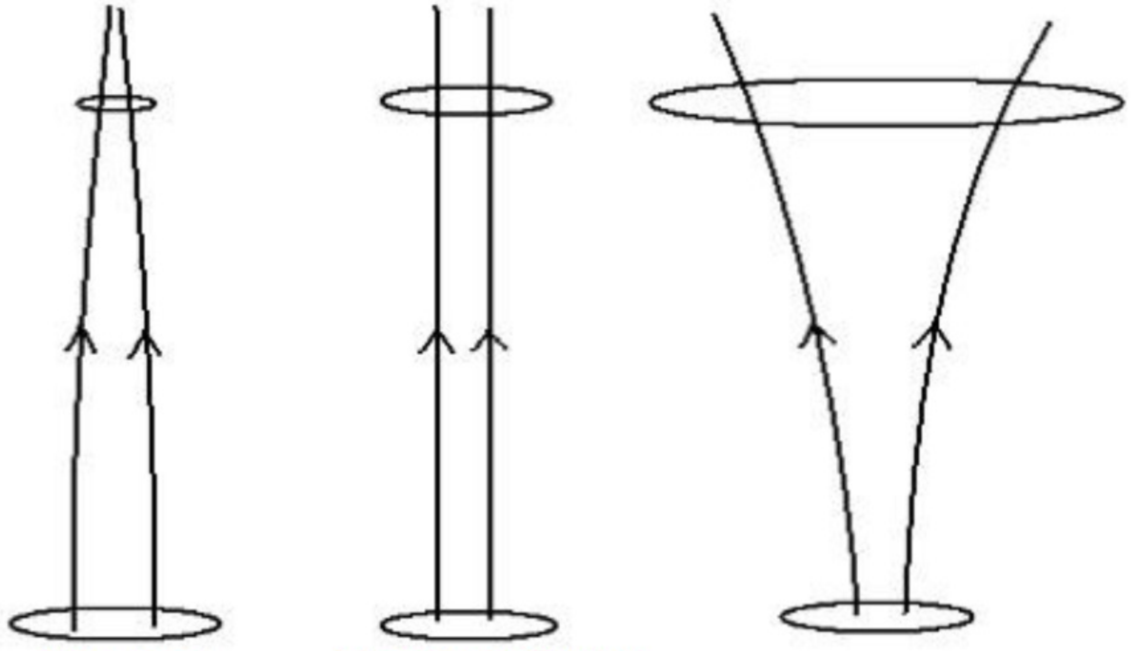
Background Of Study



- ❖ Black holes are celestial objects resulting from the gravitational collapse of massive stars.
- ❖ The challenge in defining a black hole revolves around understanding the boundary between the region occupied by the black hole and the surrounding Universe, a boundary known as the event horizon.
- ❖ A crucial concept in the study of black holes is that of Marginally Outer Trapped Surfaces (MOTS).



Congruence of null geodesics

Trapped Surfaces

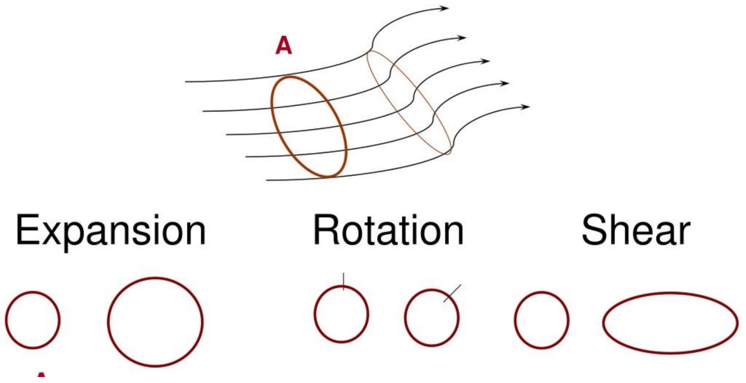
- A trapped surface is a two-dimensional surface in spacetime where the congruence of null geodesics emanating orthogonally from the surface converges (has negative expansion) in both the inward and outward directions.
- $\theta_{(n)} < 0$ and $\theta_{(\ell)} < 0$
- Trapped surfaces imply the existence of singularities.

Marginally Outer Trapped Surfaces (MOTS)

- A MOTS is a marginally trapped surface where the outward null expansion is zero $\theta_{(\ell)} = 0$.
- The Apparent Horizon is a MOTS that bounds the trapped region and is contained inside the event horizon of a black hole.

The Stability Operator

The cross-sectional area enclosing a congruence of geodesics.



The Stability Operator L for static axisymmetric solutions is defined by:

$$\delta_{\psi R}\theta_{(\ell)} = L_{\Sigma}\psi = -\Delta\psi + \left(\frac{1}{2}\mathcal{R} - 2 \parallel \sigma_{(\ell)} \parallel^{2}\right)\psi$$

The Eigenvalue Problem:

$$L\psi = \lambda\psi$$

The smallest_principal_eigenvalue_holds important information on MOTS stability.

- A positive principal eigenvalue $\lambda > 0$: indicates strict stability, suggesting the MOTS bounds a trapped region
- A negative principal eigenvalue $\lambda < 0$: indicates instability, suggesting the MOTS might not bound a trapped region.

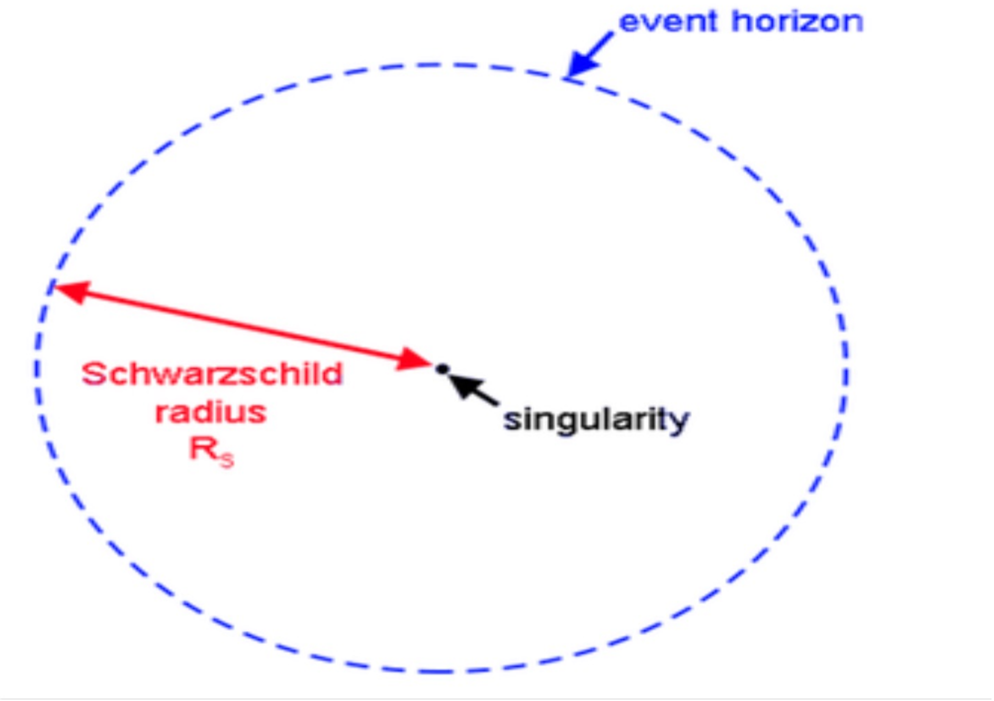
The Schwarzschild solution

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

The Schwarzschild solution characterizes the geometry of spacetime around a non-rotating, spherically symmetric mass.

Predicts the existence of a singularity at the center of the mass.

This solution depicts a spacetime that asymptotically approaches flat spacetime at large distances from the central mass



Weyl Solutions

- A static axisymmetric metric can be written in the general form:

$$ds^2 = -e^{U}\, dt^2 + e^{-2U+2V}(dz^2 + d\rho^2) + e^{-2U}\, \rho^2 d\phi^2$$

- For a metric to accurately represent the gravitational field in a vacuum, the function U(ρ, z) must satisfy the Laplace equation:

$$\nabla^2 U = \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial Z^2} = 0$$

and $\frac{\partial V}{\partial \rho} = \rho[(\frac{\partial U}{\partial \rho})^2 - (\frac{\partial U}{\partial Z})^2]$, $\frac{\partial V}{\partial Z} = 2\rho \frac{\partial U}{\partial \rho}\frac{\partial U}{\partial Z}$

- This allows a (Partial) superposition of solutions.

Weyl-Distorted Schwarzschild Metric

- The Metric is Given By:

$$ds^2 = -e^{2U}\left(1-\frac{2m}{r}\right) + e^{-2U+2V}\left(\frac{dr^2}{1-\frac{2m}{r}} + r^2 d\theta^2\right) + e^{-2U} r^2 \sin^2\theta d\phi^2$$

- If we demand a Schwarzschild singularity, then the metric at the event horizon r=2m:

$$ds^2 = 4m^2 e^{-2U}\left(e^{4U-4u_o} d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

- The potential U on the horizon is expressed as a series expansion in terms of Legendre polynomials $P_i(\cos\theta)$:

$$U(2m,\theta) = \sum_{i=0}^{\infty} \alpha_i P_i(\cos\theta) \text{ and } V(2m,\theta) = 2U(2m,\theta) - 2u_o$$

$U(2m, 0) = U(2m, \pi) = u_o,$

The coefficients α_i have specific values for the given problem:

$$\sum_{k=1}^{\infty} \alpha_{2k-1} = 0 \,,$$
$$\sum_{k=1}^{\infty} \alpha_{2k} = u_o$$

For the quadrupole distortion, we consider the α_2 term

Eigenvalue Analysis and Spectral Solution

Consider the differential operator

$$\mathcal{L}\psi(\theta) = -\frac{d^2\psi(\theta)}{d\theta^2} + 3\,\frac{d\psi(\theta)}{d\theta} + 2\psi(\theta),$$

with $\psi(x) = e^{-\frac{3x}{2}} \sin(n\pi x)$

which vanish at 0 and 1

$$\lambda_n = -\pi^2 n^2 - \frac{1}{4}.$$

Numerically we,

- ❖ Calculate the matrix elements $\mathcal{L}_{mn} = \langle \psi_m, \mathcal{L}\,\psi_n \rangle$ using integration techniques over the domain.
- ❖ Eigenvalue Problem: Solve the matrix equation $\mathcal{L}_{mn}\,\psi_n = \lambda_n\,\psi_n$

Breakdown of Spectral Method used:

Differential Operator Definition:

Consider the linear differential operator $\mathcal{L}$ defined as:

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$$\mathcal{L}\psi(\theta) = \frac{d^2\psi(\theta)}{d\theta^2} + 3\,\frac{d\psi(\theta)}{d\theta} + 2\,\psi(\theta)$$

where $\psi(\theta)$ is the function of interest.

We employ trial functions $\psi(\theta) = \sin(n\pi\theta)$, which are Fourier sine series basis functions. These functions form an orthogonal basis set over the interval [0,1].

These integrals are used to build the matrix $\mathcal{L}_{mn}$

Matrix Construction:

$$\mathcal{L}_{mn} = \int_0^1 (\sin m\pi\theta)\mathcal{L}[\sin(n\pi\theta)]d\theta$$

$$\mathcal{L}\,[\sin(n\pi\theta)] = (-n^2\pi^2 + 3in\pi + 2)\sin(n\pi\theta)\;\;\delta_{mn}$$

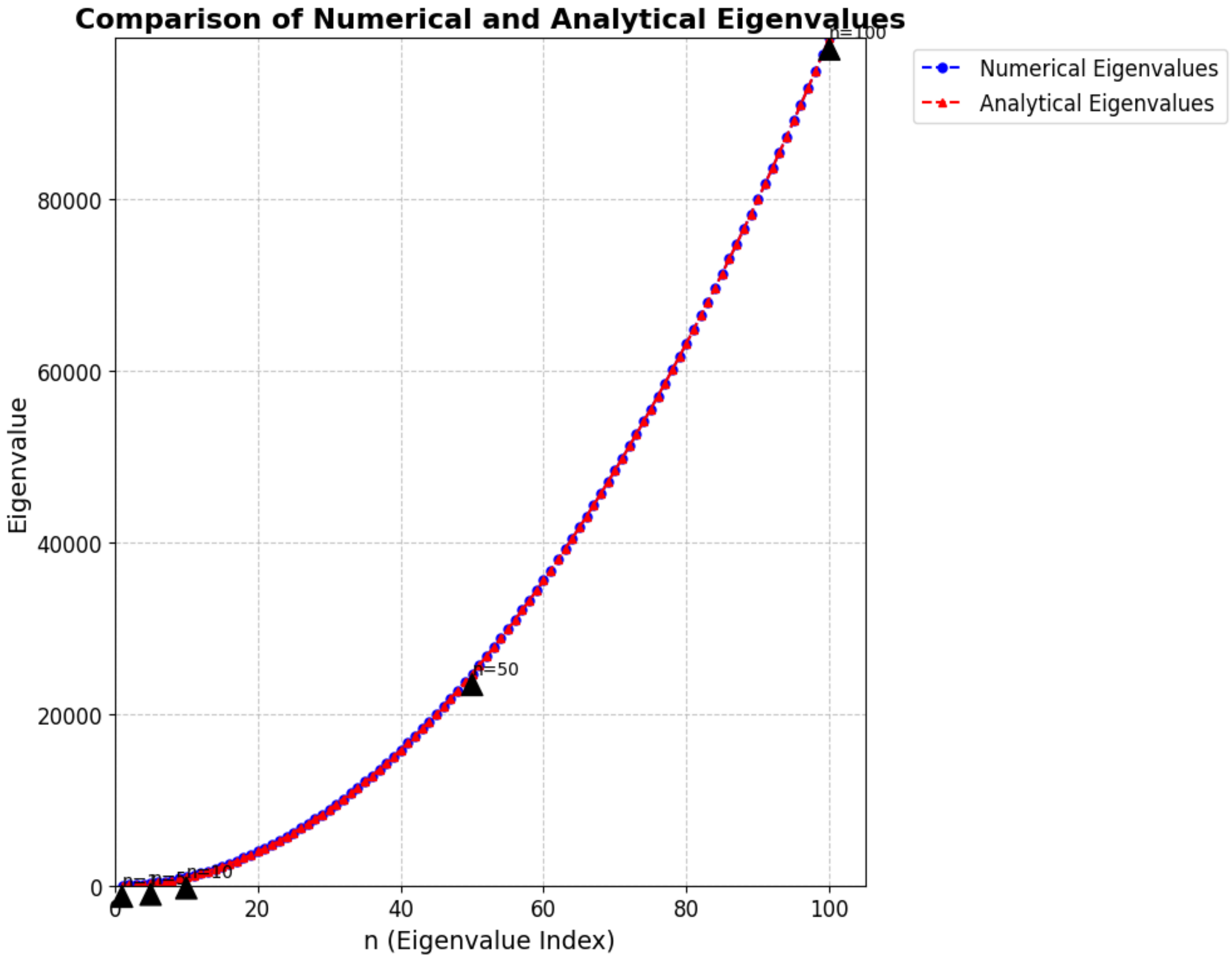
$$\mathcal{L}_{mn} = (-n^2\pi^2 + 3in\pi + 2)\;\delta_{mn}$$

Eigenvalue Computation: we compute

$$\mathcal{L}_{mn}\,\psi(\theta) = \lambda\psi(\theta)$$

The principal eigenvalue $\lambda_{principal}$ is identified, representing the dominant mode of the system.

Analytical vs. Numerical Solutions for Eigenvalues



Numerical Approach so far…

Spectral Methods: involves approximating functions by expansions in terms of basis functions (e.g., Fourier series, Chebyshev polynomials) and using these expansions to discretize the differential operator.

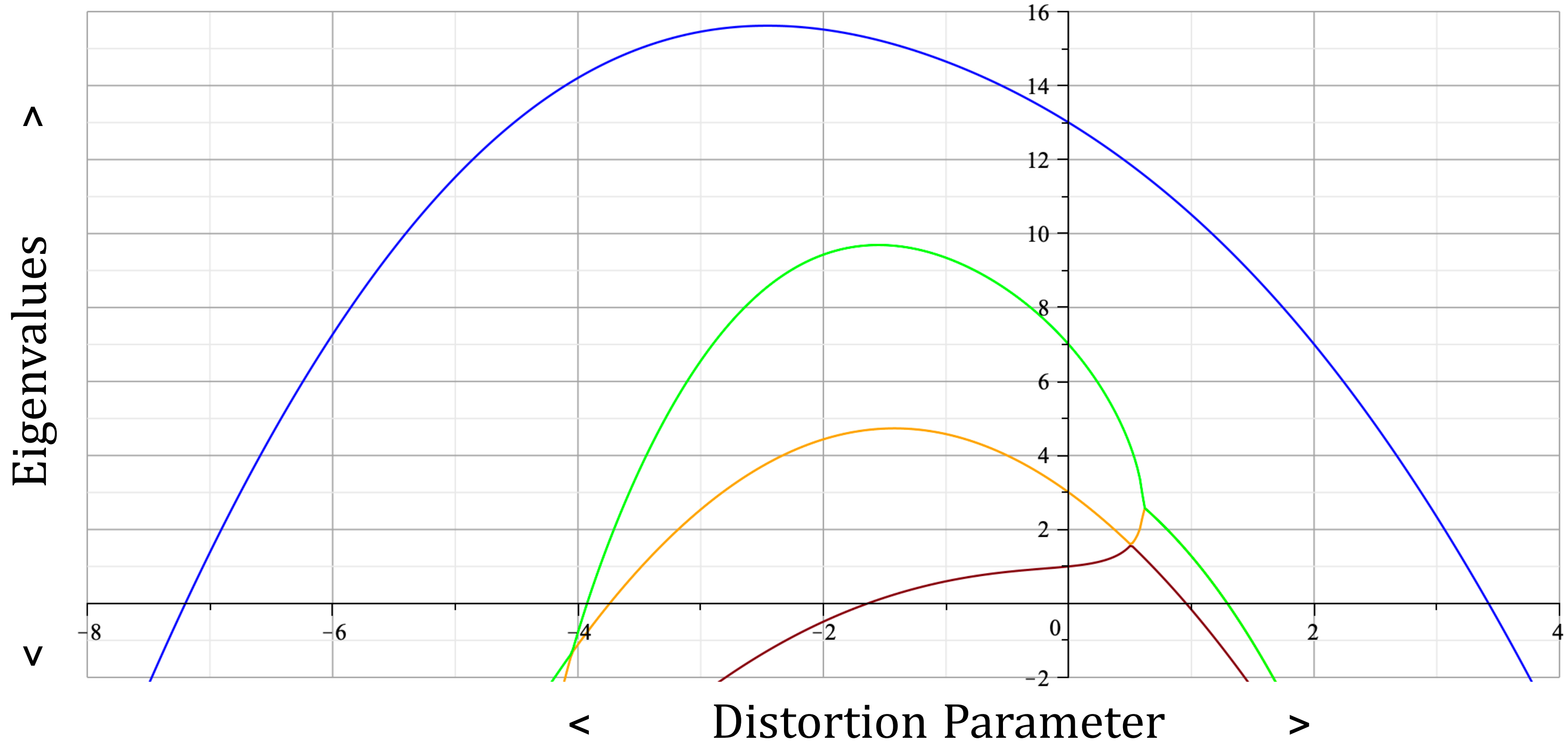
- Define the differential operator $\mathcal{L}$
- Specifies the basis functions (Fourier series approximation) for $\psi(\theta)$.
- Construct the matrix $\mathcal{L}_{mn}$ by substituting the Fourier series basis into $\mathcal{L}$.
- Computes the eigenvalues of $\mathcal{L}_{mn}$, which finds all eigenvalues of a given matrix.

Finite Difference Method: approximates derivatives using differences between function values at discrete points.

- Discretize the domain.
- Create the finite difference approximations for the first and second derivatives depending on the differential operator.
- Construct the differential operator matrix $\mathcal{L}$ using these approximations.
- Compute the eigenvalues of the matrix $\mathcal{L}_{mn}$.

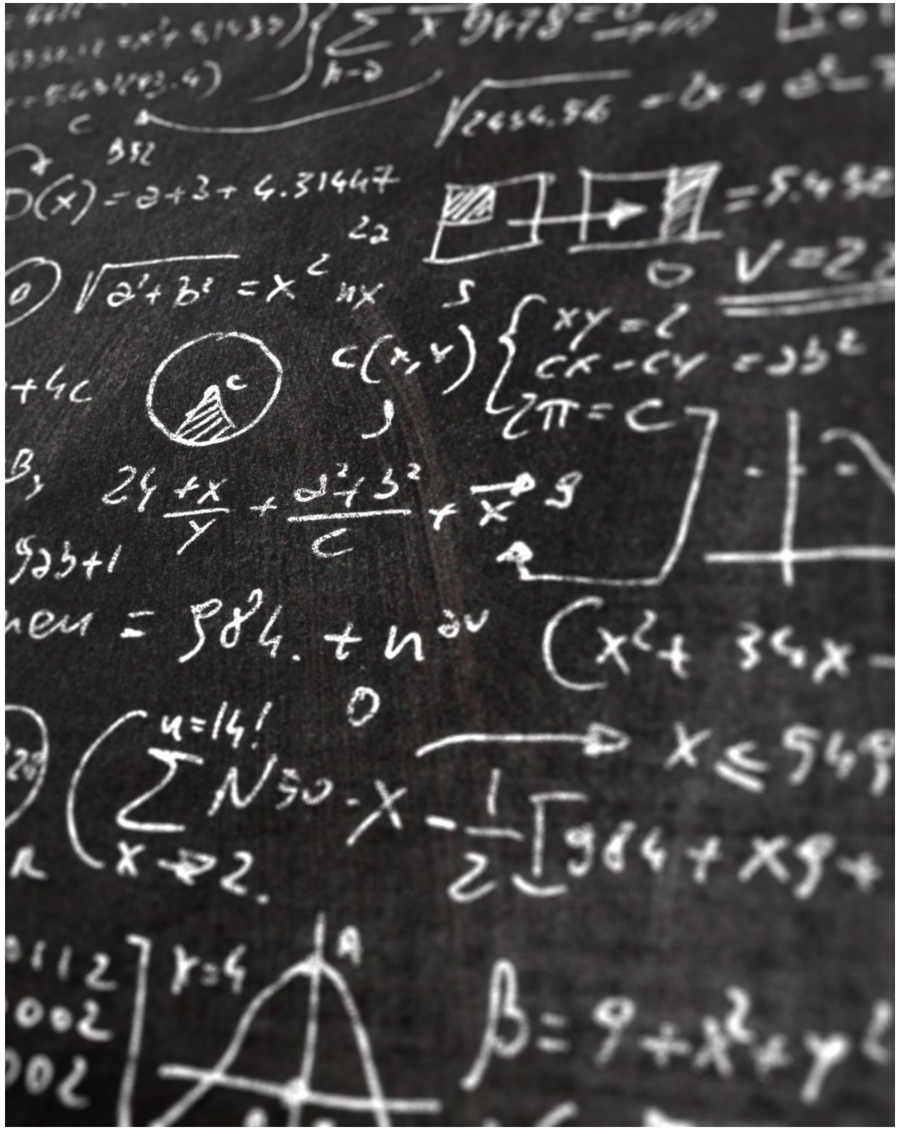
Calculating the eigenvalue spectrum

- A combined analytic/numerical calculation
- Not foliation dependent



IN PROGRESS (To Do…)

- Confirm our plot using the finite difference approximation
- Construct outsides MOTS
- Higher eigenvalues



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