Group field theory: A view from below

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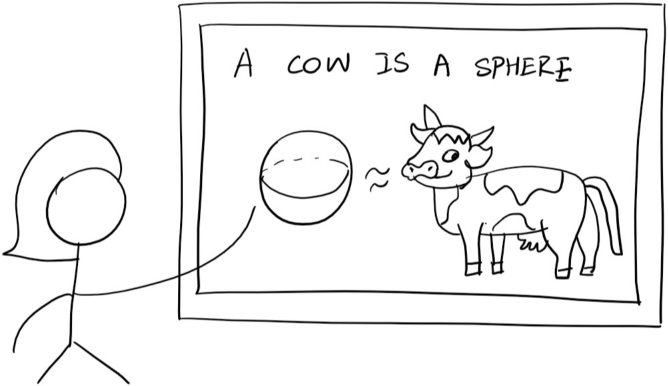
Outline

Why do we need toy models?

HK model

Group field theories: quantum gravity from the ground up

Why toy models?



Why toy models?

- Many of the issues in formulating a quantum theory of gravity are methodological.
- There is a plethora of approaches, e.g. string theory, loop quantum gravity, spinfoams, group field theories, etc.
- None of these is fully satisfactory *on its own* since each one misses one or another ideal that a quantum theory of general relativity (GR) in 4d is believed to have.
- Hence the need for *exactly soluble* toy models on which one can test their methods.
- Possibility to compare different approaches.

Toy models for gravity

Some notable examples: lower-dimensional gravity, Chern-Simons theory, topological quantum field theories, etc.

All these models are (more or less) exactly soluble.

But they are topological, i.e. do not possess local degrees of freedom. Hence not realistic.

We also have quantum cosmological models. But they lack the field-theoretic subtleties that make GR both interesting and hard to quantize.

A model bypassing many of these problems is the Husain-Kuchar (HK) model.

What is the HK model?

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Consider a 4d spacetime M. The action is

$$S = \int_M d^4x \, \epsilon_{ijk} e^i \wedge e^j \wedge F^k,$$

where e^i, $i \in \{1, 2, 3\}$, are $\mathfrak{su}(2)$-valued triads, and $F^i = dA^i + [A, A]^i$ for an $\mathfrak{su}(2)$-valued connection A^i.

How does this differ from general relativity?

If one replaces $\mathfrak{su}(2)$-valued *triads* e^i with $\mathfrak{so}(3,1)$-valued *tetrads* e^I, $I \in \{1, \cdots, 4\}$ (and the same for A), one recovers general relativity.

Features of HK

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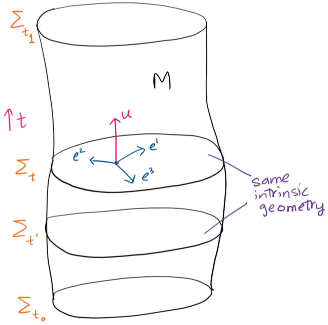
One can try defining a spacetime metric $g_{\alpha\beta} = \delta_{ij} e^i_\alpha e^j_\beta$.

But there exists a special direction $u^\alpha = (e \wedge e \wedge e)^\alpha$ such that $u^\alpha e^i_\alpha = 0$.

Therefore, $u^\alpha g_{\alpha\beta} = 0$, i.e. the metric is degenerate along one direction.



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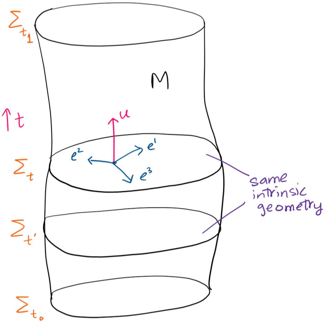

We can identify this direction with 'time'. In other words, there is **no Hamiltonian constraint** in the canonical formalism.

But there are **local degrees of freedom**: the equation of motion of e is $[e, F]^i = 0$, i.e. F is not flat.

Canonical quantization

What is the quantum theory of HK?

Loop quantum gravity redux

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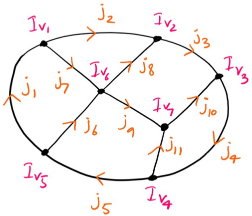
A spin network is a graph Γ with some edges $E_1, \cdots, E_N$ intersecting at vertices $v_1, \cdots, v_M$. The edges are labelled by unitary irreducible representations j_{E_i} of $SU(2)$ and the vertices by intertwiners I_{v_i}.



The intertwiners give a map between incoming and outgoing spins, like Clebsch-Gordon coefficients in angular momentum theory.

Loop quantum gravity redux

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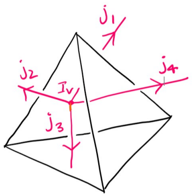
The answer lies in a quantum tetrahedron!

Every tetrahedron has a dual spin network.



Using angular momentum operators, one can define operators whose eigenvalues give the area and volume of the tetrahedron. These eigenvalues depend on $\{j_1, \cdots, j_4\}$ and I_v and are thus discrete.

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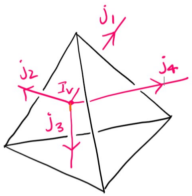

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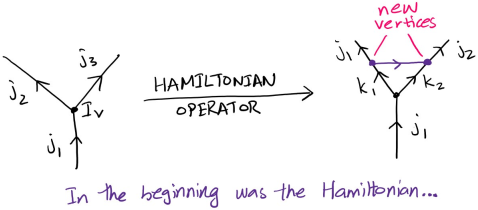
Multiple tetrahedra can be stacked together to triangulate 3-manifolds. Spin networks straddle these discretized manifolds. Higher number of tetrahedra yield finer resolutions of quantum geometry. Thus one can describe arbitrary 3-geometries in this framework.

Quantum dynamics

But in gravity, we are interested in how 3-geometries evolve. This is achieved by promoting the Hamiltonian constraint of general relativity to an operator acting on spin networks. Roughly, the Hamiltonian constraint adds vertices to a spin network.



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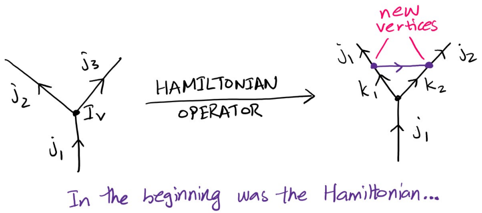

However, this step of the LQG program is plagued with serious ambiguities and difficulties. Fortunately, though, **HK does not have a Hamiltonian constraint. This is what makes it exactly soluble**.

It is as though LQG methods exist just to be applied to HK!

Our Lord, Our Saviour Mr. Feynman?

As per common physics folklore, path integral methods bypass the problem of dealing with nasty aspects of canonical quantization that stem from the underlying classical Hamitlonian formulation of a theory.

One might thus hope to develop LQG-inspired path integral frameworks as a way out of the nightmares induced by attempts to quantize the Hamiltonian constraint.

These methods are pursued in the so-called spinfoam approach to quantum gravity, and more generally, in something called group field theories.

However, the precise relations between canonical LQG and these approaches are rather obscure. We hope to shed light on this question by formulating a group field theory of the HK model.

But let's take two steps back

What should be the path integral for HK?

$$\langle(\Sigma_2, A_2, A_2)|(\Sigma_1, A_1, e_1)\rangle = \int_{\partial M = \Sigma_1 \cup \Sigma_2} De\, DA\, e^{-S_{HK}[e,A]}$$

$$\langle \Gamma_{\{j_{E_i}, I_{v_i}\}} | \Gamma_{\{\tilde{j}_{E_i}, \tilde{I}_{v_i}\}} \rangle = \delta_{\{j_{E_i}, I_{v_i}\}, \{\tilde{j}_{E_i}, \tilde{I}_{v_i}\}}$$

A group field theory primer

Recall that in canonical LQG, $SU(2)$ spin networks were used to describe the quantum geometry of space.

This geometry-spin-network correspondence is rather abstract. There is no a priori reference to a Riemannian 3-manifold. Rather, the latter seems to emerge from the underlying notion of abstract spin networks.

Group field theories carry this insight to its natural extreme and attempt to construct entire spacetimes from underlying group-theoretic data.

2d quantum gravity: spacetime as a Feynman diagram

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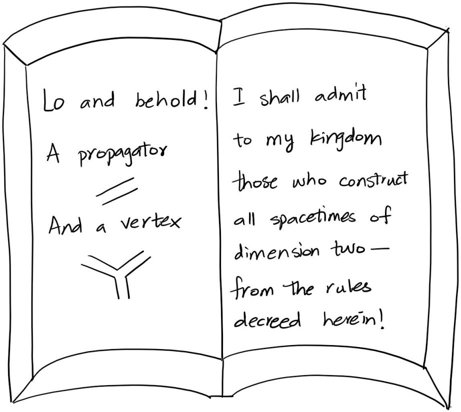
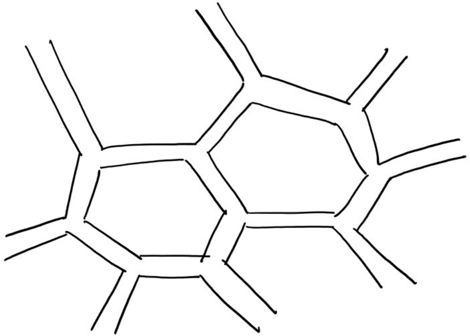
Suppose God sends an angel with a scripture containing guidance for theoretical physicists.

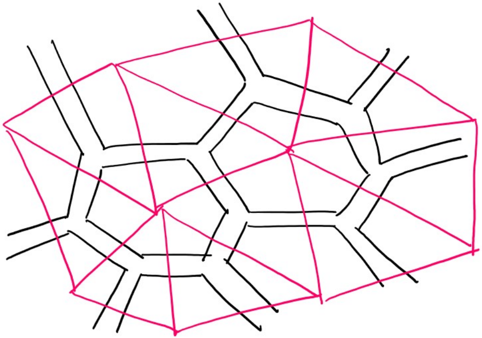
The scripture only contains one page with the following instructions.



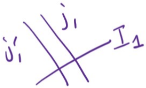
Recognizing a vertex and a propagator, particle physicists immediately proceed to construct possible Feynman diagrams.

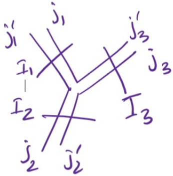


But clueless as to how to relate these diagrams to 2d manifolds, they turn to that brand of mathematicians known for their diabolic constructions, i.e. topologists, who reveal the fact that the propagators are strings which the vertices sew into triangles, which the Feynman diagrams in turn glue into triangulations of 2d manifolds.



To connect with LQG, one can label the propagators and vertices with group-theoretic data.





Then one can write the following action

$$S = \frac{1}{2} \sum_{\{j_i\},\{I_i\}} (M^{j_1 j_1' I_1})^2 - \frac{\lambda}{3!} \sum_{\{j_i\},\{I_i\}} M^{j_1 j_1' I_1} M^{j_2 j_2' I_2} M^{j_2 j_2' I_2}$$

whose kinetic term corresponds to the propagator and the interaction term represents the vertex.

If we then expand the partition function

$$Z = \int \prod_{j_i, j_i', l_i} dM^{j_i j_i' l_i} e^{-S[M]}$$

in powers of λ, we get a sum over possible Feynman diagrams, i.e. over triangulations of all 2d manifolds.

With appropriate modifications, one can generalize the entire construction to any number of dimensions. This, in essence, is what group field theories do.

Group field theory of HK

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Recall that classically, HK is a theory formulated on 4d manifolds that reduces upon analysis to a theory of non-dynamical 4-geometries. That is, a 3d initial data hypersurface remains 'frozen in time'.

Now, as we have seen, a d-dimensional group field theory gives a recipe to construct all d-dimensional manifolds by gluing together (d-1)-dimensional objects.

Thus, in a group field theory of HK, there should not be enough data to achieve this. That is, there shouldn't be enough information to glue 3d tetrahedra, say, to form 4d manifolds that are topologically distinct from 3d manifolds.

Group field theory of HK

Since it is the interaction term in a group field theory action that is responsible for gluing (d-1)-dim objects into nontrivial d-dim manifolds, we claim that HK should be described by a free group field theory, i.e. no interactions.

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We can confirm this suspicion by directly evaluating a group field theory transition amplitude for a 4-manifold whose boundary is a disjoint union of two 3-d manifolds, which can be thought of as initial and final boundary states. We find that the amplitude is nonzero only if the boundary states are identical.

Conclusions

- HK is a theory with local degrees of freedom that can be fully quantized using loop quantum gravity and group field theory methods. Since the results in both approaches exactly match, we have a clean bridge between 'canonical' and 'covariant' ways of concretely realizing the idea that geometry is discrete at a fundamental level.
- We learn that from a group field theory perspective, HK is to full quantum gravity as free scalar field theory is to $\lambda\phi^4$ theory.
- This further entails that the Hamiltonian constraint of general relativity is in some sense hidden in the interaction term of a group field theory. This is a partial answer to the quandary that spacetime diffeomorphisms are so essential in general relativity and yet totally elusive from the perspective of something as radical as a group field theory in which all inherent reference to spacetime goes away.