Motivating semiclassical gravity

An approximation for bipartite quantum systems

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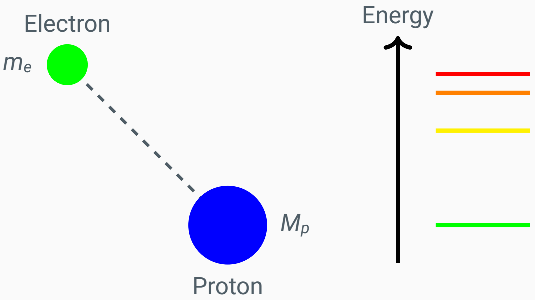
Introduction A simple example

The ***spectrum*** of Hydrogen with proton classical and electron quantum is ***correct***.



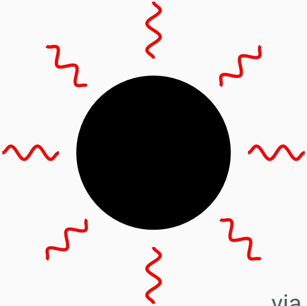
This treatment is ***asymmetric*** though—the ***electron*** cannot be classical instead.

It works as it is the famous ***Born-Oppenheimer*** approximation, and its error parameter m_e/M_p is small.

There is no need to account for ***backreaction***, for m_e/M_p stays small.

There is ***no consensus*** on a quantum theory of gravity to date.

. . . so we rely on approximations with ***classical*** gravity and ***quantum*** matter.

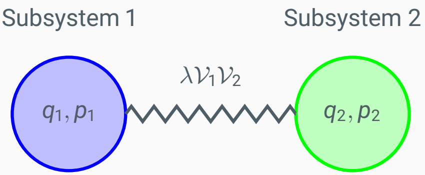


Quantized matter
carrying energy to infinity
from a classical black hole

Black hole mass is corrected
via energy conservation ($dM/dt \sim -1/M^2$)

We need to account for backreaction but do so in an ***ad hoc*** manner.

1. Are such ***classical-quantum*** approximations derivable from the fundamentals?
2. If so, how good are they, or what are their ***regimes of validity***?



A bipartite system (subsystem 1 + subsystem 2)

$$\mathcal{H}(q_1, p_1, q_2, p_2) = \mathcal{H}(q_1, p_1) + \mathcal{H}(q_2, p_2) + \lambda \mathcal{V}_1(q_1, p_1)\mathcal{V}_2(q_2, p_2) \text{ (\textbf{\textit{Classical-classical}})}$$

$$\hat{H} = \hat{H}_1 \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{H}_2 + \lambda \hat{V}_1 \otimes \hat{V}_2 \text{ (\textbf{\textit{Quantum-quantum}})}$$

A classical-quantum approximation for this system

Classical eqs. with ***quantum*** expectations $\begin{cases} \partial_t q_1 = \partial_{p_1}\left(\mathcal{H}_1 + \lambda \langle\psi|\,\hat{V}_2\,|\psi\rangle\,\mathcal{V}_1\right) \\ \partial_t p_1 = -\partial_{q_1}\left(\mathcal{H}_1 + \lambda \langle\psi|\,\hat{V}_2\,|\psi\rangle\,\mathcal{V}_1\right) \end{cases}$

Quantum eq. with ***classical*** trajectory $\left\{\iota\partial_t\,|\psi\rangle = \left(\hat{H}_2 + \lambda\mathcal{V}_1(q_1,p_1)\hat{V}_2\right)|\psi\rangle\right.$ [1]

This is reminiscent of the ***semiclassical Einstein equation***: $G_{\alpha\beta} = M_{Pl}^{-2}\,\langle\psi|\,\hat{T}_{\alpha\beta}\,|\psi\rangle$.

[1] V. Husain, I. Javed, and S. Singh, Phys. Rev. Lett. 129, 111302 (2022).

Could the said classical-quantum (CQ) approximation be derived from the known correct quantum-quantum (QQ) dynamics, which is given by the following?

$$\iota\partial_t |\psi\rangle = \hat{H} |\psi\rangle$$
$$\hat{H} = \hat{H}_1 \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{H}_2 + \lambda \hat{V}_1 \otimes \hat{V}_2$$

Others too have attempted to derive this CQ approximation but ***without much success***.[2]

Our approach relies on ***somewhat different assumptions*** from theirs.

[2] T. Singh and T. Padmanabhan, Ann. Phys. (N.Y.) 196, 296 (1989)
C. Kiefer and T. P. Singh, Phys. Rev. D 44, 1067 (1991).

$$\imath\partial_t |\psi\rangle = \hat{H}|\psi\rangle \xRightarrow{\text{Approximation}} \begin{cases} \partial_t q_1 = \partial_{p_1}\left(\mathcal{H}_1 + \lambda \langle\psi|\hat{V}_2|\psi\rangle \mathcal{V}_1\right) \\ \partial_t p_1 = -\partial_{q_1}\left(\mathcal{H}_1 + \lambda \langle\psi|\hat{V}_2|\psi\rangle \mathcal{V}_1\right) \\ \imath\partial_t |\psi\rangle = \left(\hat{H}_2 + \lambda\mathcal{V}_1(q_1,p_1)\hat{V}_2\right)|\psi\rangle \end{cases}$$

Assumptions allowing the approximation

We find that the CQ approximation is valid if the following hold.

1. Coupling parameter λ is small.
2. Entanglement between subsystems is small.
3. Quantum state of subsystem 1 is a ***semiclassical*** state.

We start by assuming that the system is in a nearly product state (entanglement is small):

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle + \mathcal{O}(\lambda).$$

$|\psi\rangle$ may be written as a matrix, Z, which, in turn, defines ***reduced density matrices*** $\hat{\rho}_1 = ZZ^\dagger$ and $\hat{\rho}_2 = Z^\mathsf{T} Z^*$ for the two subsystems.

Schrodinger equation then leads us to

$$\iota \partial_t \hat{\rho}_1 = \left[\hat{H}_1, \hat{\rho}_1\right] + \lambda \left[\hat{V}_1, Z \hat{V}_2^\mathsf{T} Z^\dagger\right]$$

and

$$\iota \partial_t \hat{\rho}_2 = \left[\hat{H}_2, \hat{\rho}_2\right] + \lambda \left[\hat{V}_2, Z^\mathsf{T} \hat{V}_1^\mathsf{T} Z^*\right].$$

If we use the expansion for Z, we could read off ***effective Hamiltonians*** from the equations of motion for $\hat{\rho}_1$ and $\hat{\rho}_2$:

$$\iota\partial_t\hat{\rho}_{1,2} = \left[\hat{H}^{\text{eff}}_{1,2}, \hat{\rho}_{1,2}\right] + \mathcal{O}\left(\lambda^2\right),$$

where $\hat{H}^{\text{eff}}_{1,2} = \hat{H}_{1,2} + \lambda \langle\psi_{2,1}| \hat{V}_{2,1} |\psi_{2,1}\rangle \hat{V}_{1,2}$.

Finally, we assume a sharply peaked semiclassical state for subsystem 1 such that

$$\partial_t q_1 \approx \partial_{p_1}\mathcal{H}_1 + \lambda \langle\psi_2|\,\hat{V}_2\,|\psi_2\rangle\,\partial_{p_1}\mathcal{V}_1$$

and

$$\partial_t p_1 \approx -\partial_{q_1}\mathcal{H}_1 - \lambda \langle\psi_2|\,\hat{V}_2\,|\psi_2\rangle\,\partial_{q_1}\mathcal{V}_1,$$

where $q_1 = \langle\psi_1|\,\hat{q}_1\,|\psi_1\rangle$, $p_1 = \langle\psi_1|\,\hat{p}_1\,|\psi_1\rangle$, and $\langle\psi_1|\,\hat{V}_1\,|\psi_1\rangle \approx \mathcal{V}_1(q_1, p_1)$.

Being an approximation after all, the CQ scheme holds for a ***finite*** amount of time.

Failure could be determined by a time scale called ***scrambling time***.

Scrambling time

It is the characteristic time for the growth of entanglement between subsystem 1 and subsystem 2 from 0 to $\mathcal{O}(\lambda)$.

Scrambling time is defined as above, for it is calculated through linear perturbation theory.

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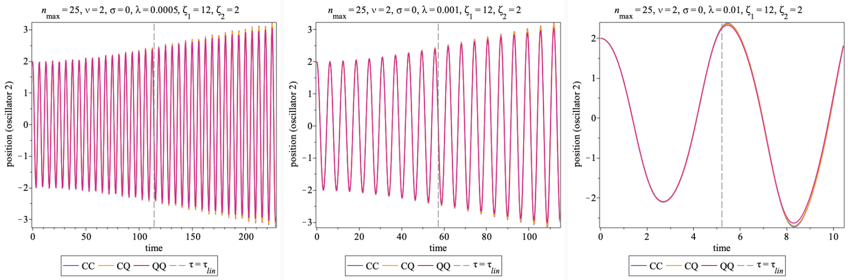
For von Neumann entanglement entropy $S_{\mathrm{VN}}(t)$ to be $\mathcal{O}(\lambda) \ll \ln(d)$, where d is the minimum of the dimensions of the two Hilbert spaces involved,

$$t \ll t_{\mathrm{VN}} = \frac{\ln(d)}{\mathcal{E}(t_{\mathrm{VN}})},$$

where t_{VN} is the srambling time and $\mathcal{E}(t) = (\lambda/t) \int_0^t dt' \sqrt{\left\langle \hat{V}_1(t')^2 \right\rangle_{(0)} \left\langle \hat{V}_2(t')^2 \right\rangle_{(0)}}$.

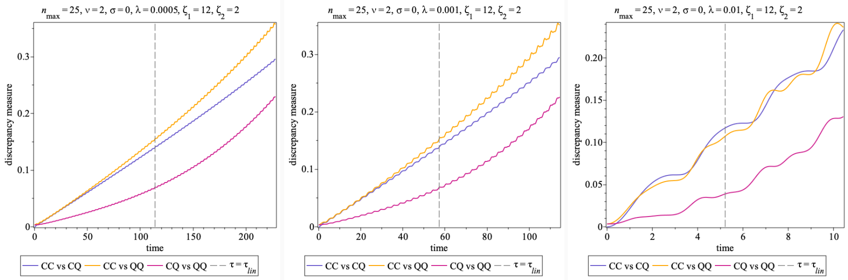
A similar calculation could be done with ***linear entropy*** as well.

This whole formalism, in fact, could be redone through ***alternative Hamiltonians*** as shown in detail in our paper (Phys. Rev. D 108, 086033).

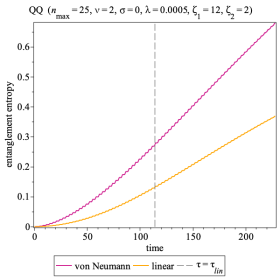


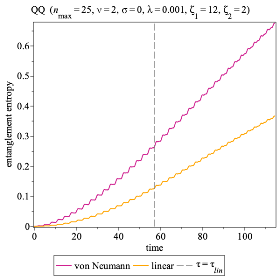
$n_{max} = 25, \nu = 2, \sigma = 0, \lambda = 0.0005, \zeta_1 = 12, \zeta_2 = 2$
position (oscillator 2)
time
CC
CQ
QQ
$\tau = \tau_{lin}$
$n_{max} = 25, \nu = 2, \sigma = 0, \lambda = 0.001, \zeta_1 = 12, \zeta_2 = 2$
position (oscillator 2)
time
CC
CQ
QQ
$\tau = \tau_{lin}$
$n_{max} = 25, \nu = 2, \sigma = 0, \lambda = 0.01, \zeta_1 = 12, \zeta_2 = 2$
position (oscillator 2)
time
CC
CQ
QQ
$\tau = \tau_{lin}$

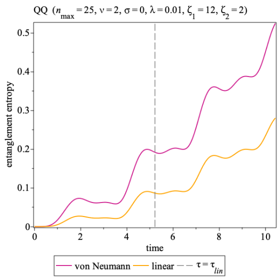
Results and discussion CQ vs. QQ and CC



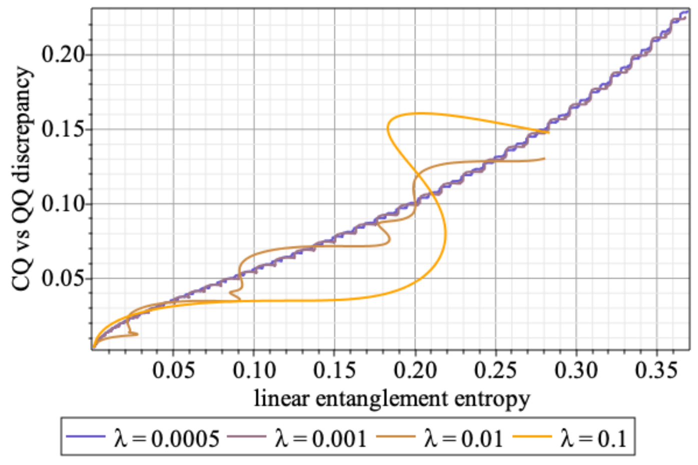
Results and discussion CQ vs. QQ







Results and discussion Error and entropy



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Key messages

1. ***Derivation*** of a classical-quantum approximation like $G_{\alpha\beta} = M_{Pl}^{-2} \langle\psi| \hat{T}_{\alpha\beta} |\psi\rangle$
2. Approximation failure after a (***calculable***) finite amount of time

Future directions

1. Explicit ***generalization*** to gravity remains to be seen (e.g., in parametric resonance).
2. ***Long-term behavior*** of entropy ($S \sim 2/3 \ln(E)$) asks for further exploration.

Thank you!
Questions?