THE AVERAGING PROBLEM IN THE TELEPARALLEL EQUIVALENT TO GENERAL RELATIVITY?

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ABSTRACT

The construction of an averaged theory of gravity based on Einstein's General Relativity is challenging due in one part to the difficulty in defining a mathematically precise covariant averaging procedure for tensor fields over differentiable manifolds. Even if one is able to address the first problem, a second problem has to deal with the non-linear nature of the gravitational field equations. Together, these two ideas have been called the averaging problem. The Teleparallel Equivalent to GR offers us a promising alternative.

THE COSMOLOGICAL QUESTION

Could inhomogeneities that are ignored in the standard Spatially Homogeneous and Isotropic (SHI) model in GR, lead in some part to the divergence between model and observation and provide an alternative explanation to **Dark Matter, Dark Energy** and explain the **Hubble tension**?

THE MATHEMATICAL QUESTION

Does a fully inhomogeneous model of the Universe evolve “**on average**” like a spatially homogenous and isotropic model?

AND

Does a fully inhomogeneous model of the Universe evolve “**on average**” to provide possible explanations for **DM** and **DE**? Could it provide an explanation for the Hubble tension?

THE DETAILS OF PROBLEM IN GR

- Einstein's Field Equations for GR (EFEs);

$$G_{\alpha\beta}[g] = \kappa T_{\alpha\beta}$$

  - g is the given metric
  - $G_{\alpha\beta}$ is the Einstein Tensor calculated from metric
  - $T_{\alpha\beta}$ is the Energy Momentum Tensor
- In cosmology,
  - RHS (the matter) is commonly modeled as a perfect fluid
- **The Big Problem**: an averaging/smoothing procedure has been employed on RHS without a corresponding averaging/smoothing procedure on the LHS.

THE BIG PROBLEM: PROBLEM A

- The EFE are tensorial equations on a manifold.
- **PROBLEM A**: How does one average tensor fields on an affine metric manifold?

THE BIGGER PROBLEM: PROBLEM C

Assuming one is able to average both sides of EFE`s,

$$\langle G_{\alpha\beta}[g] \rangle = \kappa \langle T_{\alpha\beta} \rangle = \kappa T_{\alpha\beta}^{\,Fluid}$$

How can one relate $\langle G_{\alpha\beta}[g] \rangle$ with $G_{\alpha\beta}[\langle g \rangle]$?

Can we simply assume $\langle G_{\alpha\beta}[g] \rangle = G_{\alpha\beta}[\langle g \rangle]$?

NO, due to non-linearity of the Einstein tensor

Solution ???: Introduce a Gravitational Correlation

$$C_{\alpha\beta} = \langle G_{\alpha\beta}[g] \rangle - G_{\alpha\beta}[\langle g \rangle]$$

in which case

$$\langle G_{\alpha\beta}[g] \rangle = G_{\alpha\beta}[\langle g \rangle] + C_{\alpha\beta} = \kappa T_{\alpha\beta}^{\,Fluid}$$

but ...

THE BIGGER PROBLEM: PROBLEM C

Assuming a solution to Problem A, we define the Gravitational Correlation

$$C_{\alpha\beta} = \langle G_{\alpha\beta}[g] \rangle - G_{\alpha\beta}[\langle g \rangle]$$

What then is the nature of $C_{\alpha\beta}$?

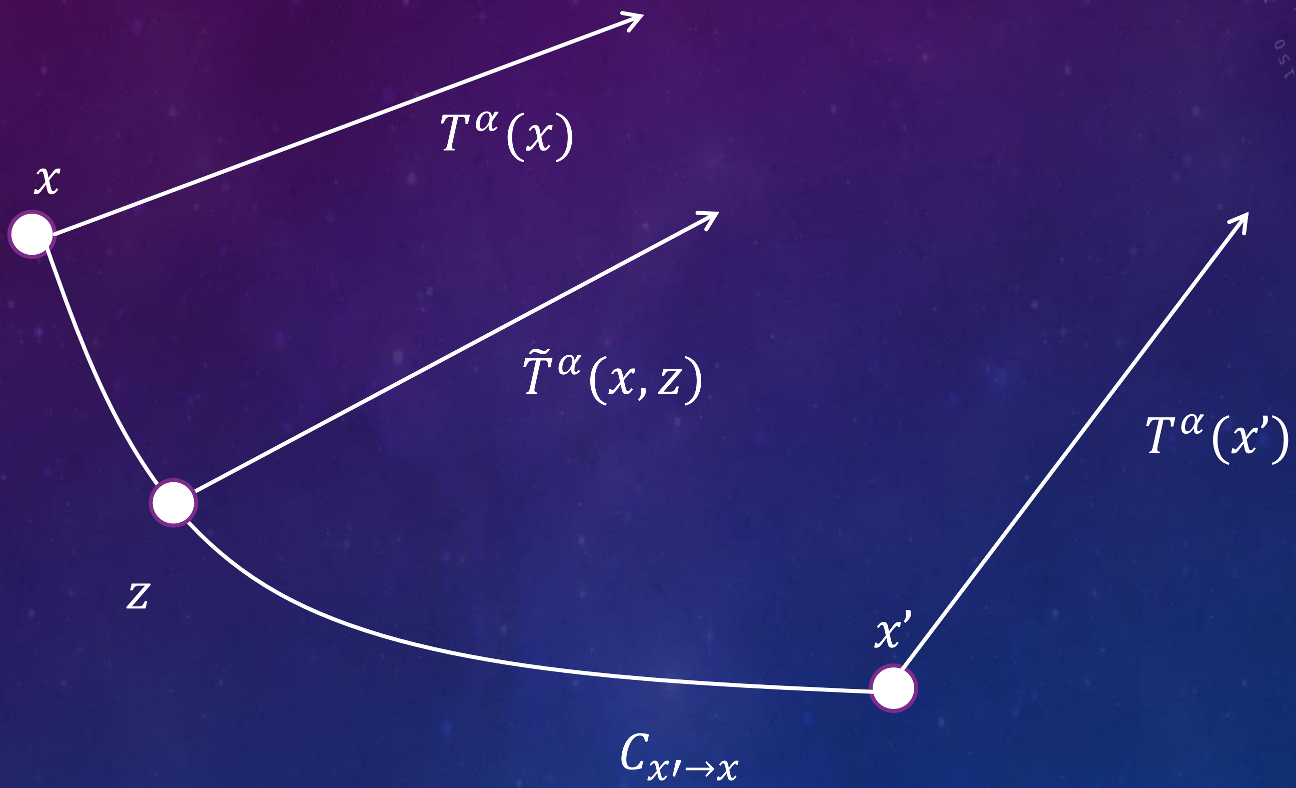
Problem C: Assuming GR is the gravitational theory on small scales and isolated bodies, what is the nature of the gravitational correlation $C_{\alpha\beta}$?

SOLUTION TO PROBLEM A

PROBLEM A: How does one average tensor fields on an affine metric manifold?

Before resolving how to average tensors, we need to review how to add tensors located at different points x and x'

THE TRANSPORT PROBLEM



Introduce curve $C_{x'\to x}$ connecting the two points and parallel transport $T^\alpha(x)$ to obtain the transported image $\tilde{T}^\alpha(x,x')$ which is a vector at x'.

THE TRANSPORT PROBLEM

The value of the translated tensor at x' after parallel transportation from x is dependent

- upon the value of the tensor at x
- the path $C_{x \to x'}$, it takes to go from x to x' and
- the rules for the transportation of the tensor, Parallel Transport
  - i.e., the affine connection $\Gamma^{\alpha}{}_{\beta\gamma}$ defines the rules,

OUR OPTIONS

Two options in developing a well defined covariant averaging procedure.

- Option 1: Levi-Civita connection and geodesic
- Option 2: Flat connection and path independent

LEVI-CIVITA AND GEODESIC

- Levi-Civita connection is the unique symmetric metric compatible connection,
- Selection of Unique Curve: Geodesic connecting x and x'
  - appears ``natural'', as there are no other ``natural'' curves that connect x and x'
  - in Riemannian space, the geodesic is the shortest and straightest path connecting points x and x'
- the **elementary parallel propagators** no longer depend on an arbitrary curve and are functions of the endpoints x and x'
- these special **parallel propagators** are denoted with a lower case $g^{\alpha'}{}_{\alpha}(x', x)$

LEVI-CIVITA AND GEODESIC

- Mathematically challenging to implement
  - Requires a solution to a DE that defines the geodesic
  - Not all pairs of points can be connected by a geodesic
- Therefore, averaging GR using a Levi-Civita connection and geodesics is problematic

FLAT AND PATH INDEPENDENT

- Parallel Transport is independent of path iff curvature of connection is zero
- Parallel transport with respect to a flat connection
  - The connection still has torsion (anti-symmetric part)
  - The transporters are products of the tetrad field
  - Tele-parallel geometries
- Perhaps more natural than geodesic transport (independent of metric)
- these **parallel propagators** are denoted with as $P^{\alpha'}{}_{\alpha}(x', x)$ which can be show to be equal $P^{\alpha'}{}_{\alpha}(x', x) = e_a^{\alpha'}(x')\, e^{a}{}_{\alpha}(x)$ where $e_a(x)$ is a basis for the tangent space.

AVERAGING PROCEDURE

Let M be a simply connected differentiable affine manifold. Let $T^{\alpha}(x)$ be a continuous vector field defined on some simply connected region R. Let $\Sigma_{x'}$ be a compact subset of R at supporting point x'. We define the average of the tensor field $T^{\alpha}(x)$ denoted as $\bar{T}^{\alpha}$ as the definite integral at supporting point x'

$$\langle T^{\alpha}(x')\rangle = \bar{T}^{\alpha}(\boldsymbol{x}') = \frac{1}{V_{\Sigma_{x'}}}\int_{\boldsymbol{\Sigma}_{x'}} \tilde{T}^{\alpha'}(\boldsymbol{x},\boldsymbol{x}')\sqrt{-\boldsymbol{g}(\boldsymbol{x})}\boldsymbol{d}^4\boldsymbol{x}$$

where

$$V_{\Sigma_{x'}} = \int_{\boldsymbol{\Sigma}_{x'}} \sqrt{-\boldsymbol{g}(\boldsymbol{x})}\boldsymbol{d}^4\boldsymbol{x} \quad \text{and}$$

$$\tilde{T}^{\alpha'}(\boldsymbol{x},\boldsymbol{x}') = P^{\alpha'}{}_{\alpha}(x',x)\,T^{\alpha}(x)$$

AVERAGING THE METRIC ?

- In both the geodesic transport or path-independent transport,

$$\langle g_{\alpha\beta} \rangle = g_{\alpha\beta}$$

  Since the metric is covariantly constant

- So, does it make sense to average the metric?

- Which geometrical object of the micro geometry (metric, connection, curvature, torsion, other) when averaged, yields information about the macro geometry?
- GR is based on the metric, so averaging GR in this way may be problematic

COMMENTS ON AVERAGING PROCEDURE

- Averaging procedure as presented is
  - Covariant,
  - Linear
  - Respects tensor contractions and
  - Maintains algebraic symmetries
  - Can be generalized to tensor densities of arbitrary rank
  - Can be applied to teleparallel theories of gravity

THE TELEPARALLEL EQUIVALENT TO GR

- TEGR - Teleparallel Equivalent to GR
  - A theory of gravity that is dynamically equivalent to GR
  - But conceptually different
  - TEGR -- Based on a metric compatible connection that has zero **curvature**,
  - GR -- Based on a metric compatible connection that has zero **torsion**,
  - The Lagrangian for TEGR differs to that of GR by a total derivative

COVARIANT FIELD EQUATIONS OF TEGR

- Let $\{x^{\mu}\}$ be the coordinates
- TEGR is characterized by
  - Four co-frame fields $h^{a} = h^{a}{}_{\mu} dx^{\mu}$
  - Metric compatible, zero curvature spin connection ,

$$\omega^{a}{}_{b} = \omega^{a}{}_{b\mu} dx^{\mu}$$

  - Principle of Relativity implies we can apply a linear transformation and have the option of working in the Proper Frame $\omega^{a}{}_{b} = 0$
  - Equation relating spin connection, $\omega^{a}{}_{b\mu}$, with the spacetime affine connection, $\Gamma^{\lambda}{}_{\mu\nu}$

$$\Gamma^{\lambda}{}_{\mu\nu} = h_{a}{}^{\lambda} \partial_{\nu} h^{a}{}_{\mu} + \omega^{a}{}_{c\nu} h^{c}{}_{\mu} h_{a}{}^{\lambda}$$

COVARIANT FIELD EQUATIONS OF TEGR

- Derived quantities
  - Torsion:

$$T^{a}{}_{\mu\nu} = \partial_{\mu} h^{a}{}_{\nu} - \partial_{\nu} h^{a}{}_{\mu} + \omega^{a}{}_{b\mu} h^{b}{}_{\nu} - \omega^{a}{}_{b\nu} h^{b}{}_{\mu}$$

$$T^{\lambda}{}_{\mu\nu} = h_{a}^{\lambda} T^{a}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu} - \Gamma^{\lambda}{}_{\mu\nu}$$

  - Super-potential:

$$S_{a}{}^{\mu\nu} = \frac{1}{2}\left(T_{a}{}^{\mu\nu} + T^{\nu\mu}{}_{a} - T^{\mu\nu}{}_{a}\right) - h_{a}{}^{\nu} T^{\lambda\mu}{}_{\lambda} + h_{a}{}^{\mu} T^{\lambda\nu}{}_{\lambda}$$

  - Torsion Scalar:

$$T = \frac{1}{2} T^{a}{}_{\mu\nu} S_{a}{}^{\mu\nu}$$

COVARIANT FIELD EQUATIONS OF TEGR

- TEGR Field Equation

$$\kappa \Theta_a{}^{\mu} = h^{-1}\partial_\nu\left(h S_a{}^{\mu\nu}\right) + \frac{1}{2} T h_a{}^{\mu} - T^b{}_{a\nu} S_b{}^{\mu\nu} - \omega^b{}_{a\nu} S_b{}^{\mu\nu}$$

  - $\Theta_a{}^{\mu}$ is the canonical Energy Momentum
- Converting all indices to spacetime indices, it can be shown that the RHS is equivalent to Einstein tensor
- TEGR is fully Lorentz invariant

AVERAGING THE TEGR FIELD EQUATIONS

- Gauge Choice = Proper orthonormal frame
- With all spacetime indices, apply covariant averaging operator

$$\kappa\langle\Theta_\nu{}^\mu\rangle = \left\langle h^{-1}\partial_\lambda\left(hS_\nu{}^{\mu\lambda}\right)\right\rangle - \left\langle\Gamma^\rho{}_{\lambda\nu}S_\rho{}^{\mu\lambda}\right\rangle + \frac{1}{2}\langle T\delta_\nu{}^\mu\rangle - \left\langle T^\rho{}_{\nu\lambda}S_\rho{}^{\mu\lambda}\right\rangle$$

- LHS -- Assume $\langle\Theta_\nu{}^\mu\rangle$ is a now perfect fluid "on average"
- RHS
  - First term are derivatives of , $\Gamma^\mu{}_{\upsilon\lambda}$ (and the co-frame)
  - Remaining terms are products of the connection, $\Gamma^\mu{}_{\upsilon\lambda}$ (and the co-frame)

DEFINING AN "AVERAGED" GEOMETRY

In teleparallel geometries, we could define the Averaged Torsion, $\bar{T}^{\lambda}{}_{\mu\nu}$, and investigate whether one can construct a consistent splitting rule for the product of two torsion tensors.

$$U^{\mu}{}_{\nu\lambda}{}^{\rho}{}_{\varphi\sigma} = \langle T^{\mu}{}_{\nu\lambda} T^{\rho}{}_{\varphi\sigma} \rangle - \langle T^{\mu}{}_{\nu\lambda} \rangle \langle T^{\rho}{}_{\varphi\sigma} \rangle$$

- And re-express the averaged Field equations in terms of, , $\bar{T}^{\lambda}{}_{\mu\nu}$, $U^{\mu}{}_{\nu\lambda}{}^{\rho}{}_{\varphi\sigma}$ and its contractions
- Assuming higher order correlations are zero, imposes a set of algebraic and differential constraints on $U^{\mu}{}_{\nu\lambda}{}^{\rho}{}_{\varphi\sigma}$

DEFINING AN "AVERAGED" GEOMETRY

In teleparallel geometries, we could define the Averaged Connection, $\bar{\Gamma}^{\lambda}{}_{\mu\nu}$, and investigate whether one can construct a consistent splitting rule for the product of two connections.

$$Z^{\mu}{}_{\nu\lambda}{}^{\rho}{}_{\varphi\sigma} = \langle \Gamma^{\mu}{}_{\nu\lambda} \Gamma^{\rho}{}_{\varphi\sigma} \rangle - \langle \Gamma^{\mu}{}_{\nu\lambda} \rangle \langle \Gamma^{\rho}{}_{\varphi\sigma} \rangle$$

- And re-express the Field equation in terms of $\bar{T}^{\lambda}{}_{\mu\nu}$, $\bar{\Gamma}^{\lambda}{}_{\mu\nu}$, $Z^{\mu}{}_{\nu\lambda}{}^{\rho}{}_{\varphi\sigma}$ and its contractions
- Assuming higher order correlations are zero, imposes a set of algebraic and differential constraints on $Z^{\mu}{}_{\nu\lambda}{}^{\rho}{}_{\varphi\sigma}$
- This approach is the one taken by Zalaletdinov in his Theory of Macroscopic Gravity, but he applied it to the Field equations of GR

FUTURE WORK

- Averaging the connection
  - Assuming the higher order correlations are zero, places constraints on $Z^{\mu}{}_{\nu\lambda}{}^{\rho}{}_{\varphi\sigma}$
  - Apply the splitting rule to the TEGR field equations to obtain a "different" (??) theory of macroscopic gravity
- Averaging the Torsion
  - Not sure that this will yield a closed system of equations, if it does, then repeat the steps above
  - May be a bit more natural in the sense that we are averaging a tensor.

OVERVIEW

- **DM, DE, and the Hubble Tension** could be explained by an averaged theory of gravity.
- The averaging problem in cosmology can be broken down into two separate problems
  - How to average? (Problem A)
    - Depends on choice of parallel transport and the selection of curve.
    - A fully covariant approach to averaging tensor fields has been briefly summarized.
    - There is a clear advantage in selecting a path Independent and zero curvature (flat) connections
  - Leads to Teleparallel theories of gravity
  - Gravitational Correlation: (Problem C)
    - Have **not** yet averaged the FE's , to determine the gravitational correlation
    - Since we have assumed a path independence and a flat connection, it makes sense to average TEGR to obtain an averaged theory for GR, but what quantity? Connection or Torsion? -- ONGOING

THANK YOU

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