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Modular Bootstrap

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ABSTRACT

One of the most intriguing aspects of theoretical physics is uncovering unexpected connections between seemingly disparate physical systems. These connections often emerge when examining the dynamics of systems at a deeper, more abstract level, revealing fundamental similarities. This thesis explores such a connection through the lens of the modular bootstrap in conformal field theories (CFTs) and its implications for the Hellerman bound. In the realm of quantum gravity, the AdS/CFT correspondence provides a striking example of holography, where quantum gravity in Anti-de Sitter (AdS) space is dual to a conformal field theory on the boundary. This duality underscores the profound impact of conformal symmetry, a central theme in our study.

We begin with an introduction to modular forms and the modular group $SL(2, \mathbb{Z})$, establishing the mathematical foundation necessary for our analysis. We then delve into the principles of conformal symmetry and conformal transformations, highlighting their group structure and significance, particularly in the two-dimensional case. The derivation of the conformal Killing equation and the generators of the 2D conformal algebra is presented, leading to an exploration of the Witt and Virasoro algebras. This framework sets the stage for examining the partition function from a CFT perspective. Leveraging the modular invariance of the partition function, we derive the Hellerman bound, a universal constraint on the energy spectrum of CFTs. We demonstrate how this bound emerges from the invariance of the partition function under modular transformations, providing a rigorous derivation and physical interpretation.

Finally, we address yet another lower bound on the entropy of systems that fall under the Hellerman analysis at a temperature defined by $\beta = 2\pi$. We will also understand why specifically this value for β is important and how it relates to the fixed point of transformation in the modular invariance of the partition function. This thesis not only elucidates the connections between modular forms, conformal symmetry, and CFTs but also illustrates the power of the conformal bootstrap approach in deriving and optimizing universal bounds. These findings are significant for understanding critical phenomena, phase transitions, and the broader implications of holography in theoretical physics.

Keywords: Conformal Field Theory (CFT), Modular Bootstrap, Hellerman Bound, AdS/CFT Correspondence, Modular Forms, SL(2,Z) Group, Conformal Symmetry, Conformal Transformations, Conformal Killing Equation, Witt Algebra, Virasoro Algebra, Partition Function, Modular Invariance, Phase Transitions, Critical Phenomena, Holography, Quantum Gravity, Universal Bounds, Symmetry Transformations.

DEDICATION

To my mom and dad,

for your unwavering support and love.

To the tough times and experiences

that have shaped me.

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My academic journey has been shaped by many struggles, but it has also been a path of considerable growth. These challenges were not always easy to navigate alone, and during these times, I received unconditional support from people who truly cared about my journey and believed in me. It would be extremely selfish not to acknowledge the help and support of these incredible individuals. To them, I owe deep gratitude and an everlasting sentiment.

At the forefront is my mom, from whom I learned the true meaning of resilience in the face of hardships and the importance of being endlessly giving to those you care about. To her, I owe my whole life, not just a piece of acknowledgement. Next, I extend my heartfelt thanks to the rest of my family, who worked tirelessly to provide me with the best they could offer and ensured I had everything I needed, both emotionally and physically.

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1. Preliminaries

1.1 Modular Forms

Modular forms are central to the study of Conformal Field Theories (CFTs) and the conformal bootstrap program. They provide a robust framework for understanding the symmetries and invariances within these theories, particularly in higher dimensions. Modular forms facilitate precise computations and offer deep insights into the mathematical structures underlying CFTs, making them indispensable tools in theoretical physics. In this section, I introduce Modular Forms with the mathematical structure laying behind them.

$SL_2(\mathbb{Z})$ Group

Let $\mathbb{H} = \{x + iy \mid y > 0\}$, the upper half-plane consisting of complex numbers with positive imaginary parts.

$$\text{Define } SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid ad - bc = 1 \right\}$$

This is the special linear group of 2x2 matrices with integer entries and determinant 1.

For $\tau \in \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, the group is defined to act on $\mathbb{H}$ by linear fractional transformations:

$$\gamma\tau \overset{\text{def}}{=} \frac{a\tau + b}{c\tau + d}$$

This transformation maps $\mathbb{H}$ to itself, preserving the structure of the upper half-plane. To show that the image under the transformation remains in $\mathbb{H}$, consider the following:

$$\text{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \text{Im}\left(\frac{(a\tau + b)(c\tau^* + d)}{|c\tau + d|^2}\right) = \text{Im}\left(\frac{ac|\tau|^2 + ad\tau + bc\tau^* + bd}{|c\tau + d|^2}\right) = \text{Im}\left(\frac{ad\tau + bc\tau^*}{|c\tau + d|^2}\right)$$

Since $ac|\tau|^2 + bd \in \mathbb{R}$, it does not affect the imaginary part. We have:

$$= \text{Im}\left(\frac{ad\tau - bc\tau^*}{|c\tau + d|^2}\right) \quad (\text{since } \text{Im}(\tau) = -\text{Im}(\tau^*))$$

$$= \frac{(ad - bc)\,\text{Im}(\tau)}{|c\tau + d|^2} = \frac{\text{Im}(\tau)}{|c\tau + d|^2} \quad (\text{Given } ad - bc = 1)$$

Since $\text{Im}(\tau) > 0$ and $|c\tau + d|^2 > 0$, the imaginary part remains positive, proving the transformation preserves $\mathbb{H}$.

Defining Modular Forms

Definition: Let $k \in \mathbb{Z}$. A modular form of weight k for $SL_2(\mathbb{Z})$ is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying:

1. f is holomorphic.
2. **[Modularity Condition]**:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

3. As $\text{Im}(\tau) \to \infty$, $f(\tau)$ is bounded.

Examples:

Consider the following examples to illustrate the modularity condition:

For $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

$$f(\tau + 1) = f(\tau) \quad \forall \tau \in \mathbb{H}$$

For $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$:

$$f\left(-\frac{1}{\tau}\right) = (-\tau)^k f(\tau) \quad \forall \tau \in \mathbb{H}$$

For $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$:

$$f(\tau) = (-1)^k f(\tau) \quad \forall \tau \in \mathbb{H}$$

Thus, if k is odd:

$$f(\tau) = 0 \quad \forall \tau \in \mathbb{H}$$

(This shows that the only modular form for $SL_2(\mathbb{Z})$ of odd weight is the zero function.)

Claim: $SL_2(\mathbb{Z})$ forms a non-abelian group under matrix multiplication.

Proof:

Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $\beta = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{Z})$.

$$\alpha\beta = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix} \in M_2(\mathbb{Z})$$

The determinant of the product is:

$$\det(\alpha\beta) = \det(\alpha)\det(\beta) = 1$$

So, $\alpha\beta \in SL_2(\mathbb{Z})$. Additionally:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}) \quad \text{(Identity element)}$$

$$\forall\alpha \in SL_2(\mathbb{Z}),\ \exists\alpha^{-1} \text{ such that } \alpha\alpha^{-1} = I \text{ since } \det(\alpha) = 1$$

However, matrix multiplication is not commutative. Therefore:

$$\alpha\beta \neq \beta\alpha$$

Thus, $SL_2(\mathbb{Z})$ forms a non-abelian group.

Conclusion: If the modularity condition holds for a function $f : \mathbb{H} \to \mathbb{C}$ and matrices $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\beta = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{Z})$, then it also holds for the composition $\alpha\beta$.

Generators of $SL_2(\mathbb{Z})$

Claim: The generators of $SL_2(\mathbb{Z})$ are $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Proof:

Let $G = \langle S, T \rangle$ be the subgroup of $SL_2(\mathbb{Z})$ generated by S and T, we need to show that $G = SL_2(\mathbb{Z})$.

First note that:

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}, \quad T^n\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix}$$

Now, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Suppose $c \neq 0$. If $|a| \geq |c|$, then $a = qc + r$, $r = a - qc$.

$$T^{-q}\gamma = \begin{pmatrix} a - qc & b - qd \\ c & d \end{pmatrix} = \begin{pmatrix} r & b' \\ c & d \end{pmatrix}, b' = b - q, |r| < |c|.$$

Applying S:

$$ST^{-q}\gamma = \begin{pmatrix} -c & -d \\ r & b' \end{pmatrix}$$

Again, $|c| \geq |r|$. So, we can apply the same algorithm again and again until we get a matrix $\eta \in SL_2(\mathbb{Z})$ with lower-left entry of zero. Such a matrix should have the form $\begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix}$.

$$ST^{\alpha} \ldots ST^{\beta} ST^{-q}\gamma = g\gamma, \text{ where } g = ST^{\alpha} \ldots ST^{\beta} ST^{-q}. \text{ But } g\gamma = \begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix} = \pm T^m.$$

Thus, $\gamma = \pm g^{-1} T^m$ for some $m \in \mathbb{Z}$. Also, note that $S^2 = -I$. We showed that a general matrix $\alpha \in SL_2(\mathbb{Z})$ was also found $\in G = \langle S, T \rangle$. Hence, $G = SL_2(\mathbb{Z})$, and S and T generate $SL_2(\mathbb{Z})$. Hence, to check $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k, it suffices to check conditions (1) & (3), and check condition (2) only for the S and T matrices. Thus, we can recast the second condition to the following statements:

$$f(S\tau) = f\left(\frac{-1}{\tau}\right) = \tau^k f(\tau), \quad f(T\tau) = f \ \tau + 1) = f(\tau)$$

1.2 Conformal Group

Under coordinate transformations, $x^\mu \longrightarrow x'^\mu$, the metric transforms as:

$$g'_{\alpha\beta}(x') = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}(x)$$

Let us investigate a special type of coordinate transformations called **Conformal transformations**, which keep the metric invariant up to a local scale.

$$g'_{ab}(x') = \Omega^2(x) g_{ab}(x)$$

I will first show that these transformations form a group called the **Conformal Group**, and derive what is called the **Conformal Killing Equation** by considering the infinitesimal version of conformal transformations.

Group Proof

Claim: Conformal transformations form a group $\mathcal{G}$.

Proof:

<u>Closure:</u>

Let $x'^\mu = f(x^\mu), x'^\nu = g(x^\nu)$, for $f, g \in \mathcal{G}$

Under $f : g'^{\mu\nu}(x') = \Omega_1^2(x) g^{\mu\nu}(x)$, and under $g : g'^{\mu\nu}(x') = \Omega_2^2(x) g^{\mu\nu}(x)$

Consider $f \circ g : g'^{\mu\nu}(x') = \Omega_1^2(x)\Omega_2^2(x)g^{\mu\nu}(x) = (\Omega_1(x)\Omega_2(x))^2 g^{\mu\nu}(x)$. Thus, $\eta = f \circ g \in \mathcal{G}$ by definition.

Associativity:

G is trivially associative from the associativity of function composition. Namely, $f \circ (g \circ h)(x) = (f \circ g) \circ h(x)$

The existence of the identity element:

A transformation $\mathcal{I} : g'^{\mu\nu}(x') = g^{\mu\nu}(x)$ is conformal with $\Omega^2(x) = 1$. Hence, the identity $\mathcal{I} \in \mathcal{G}$.

The existence of inverses for each element:

For $f \in \mathcal{G} : g'^{\mu\nu}(x') = \Omega_1^2(x)g^{\mu\nu}(x)$, a corresponding transformation $f^{-1} \in \mathcal{G} : g'^{\mu\nu}(x') = \Omega_1^{-2}(x)g^{\mu\nu}(x)$ exists by definition. Moreover, $f \circ f^{-1} = \mathcal{I}$. Hence, $h = f^{-1} \in \mathcal{G} \forall f \in G$.

This completes the proof that conformal transformations indeed form a group.

Conformal Killing Equation

Now, consider the infinitesimal version of a conformal transformation

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x), \quad \xi^{\mu}(x) \to 0. \quad s.t. \quad g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x)$$

$$\frac{\partial x^{\alpha}}{\partial x'^{\beta}} = \delta^{\alpha}_{\beta} - \frac{\partial \xi^{\alpha}}{\partial x'^{\beta}} = \delta^{\alpha}_{\beta} - \frac{\partial x^{\mu}}{\partial x'^{\beta}} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} = \delta^{\alpha}_{\beta} - (\delta^{\mu}_{\beta} - \frac{\partial \xi^{\mu}}{\partial x'^{\beta}}) \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} = \delta^{\alpha}_{\beta} - \partial_{\beta}\xi^{\alpha} + O(\xi^2)$$

Plugging this in the metric transformation equation yields:

$$g'_{\mu\nu} = (\delta^{\alpha}_{\mu} - \partial_{\mu}\xi^{\alpha})(\delta^{\beta}_{\nu} - \partial_{\nu}\xi^{\beta}) g_{\alpha\beta} = g_{\mu\nu} - \partial_{\nu}\xi_{\mu} - \partial_{\mu}\xi_{\nu} + O(\xi^2)$$

Now, we demand that this is a conformal transformation by imposing $g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x)$.

$$\partial_{\nu}\xi_{\mu} + \partial_{\mu}\xi_{\nu} = (1 - \Omega^2) g_{\mu\nu}$$

$$g^{\mu\nu}(\partial_{\nu}\xi_{\mu} + \partial_{\mu}\xi_{\nu}) = (1 - \Omega^2) g_{\mu\nu} g^{\mu\nu}$$

$$g_{\mu\nu} g^{\mu\nu} = D$$

$$\partial^{\nu}\xi_{\nu} + \partial_{\mu}\xi_{\mu} = 2\partial^{m}\xi_{m} = D(1 - \Omega^2), \quad 1 - \Omega^2 = \frac{2}{D}\partial^{m}\xi_{m} = \frac{2}{D}(\partial \cdot \xi)$$

Hence, we obtain the **Conformal Killing Equation** $\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} = g_{\mu\nu}(\partial \cdot \xi)$.

Case of D = 2 with Flat Euclidean Metric

$$g_{uv} = \delta_{uv}$$

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \delta_{\mu\nu}(\partial \cdot \xi)$$

For $\mu = \nu$:

$$2\partial_1\xi_1 = \partial_1\xi_1 + \partial_2\xi_2, \quad \partial_1\xi_1 = \partial_2\xi_2$$

For $\mu \neq \nu$:

$$\partial_1\xi_2 + \partial_2\xi_1 = 0, \quad \partial_1\xi_2 = -\partial_2\xi_1$$

We observe that these are the Cauchy-Riemann conditions for complex analytic functions. Hence, the conformal transformations for 2D flat euclidean metric are the set of analytic functions $f(z) = \xi_1 + i\xi_2$. This is manifested upon complexifying the coordinates $z = x + iy$, $z' = z + f(z)$.

1.3 Conformal Algebra in 2D Flat Space

We are now ready to investigate the conformal algebra in 2D flat space. In the previous section, we showed that the conformal killing equation in 2D flat euclidean space implies that the set conformal transformations include all the analytic functions. This result could be generalized trivially to 2D Minkowski space. In this section, I will derive the 2D conformal algebra known as **Witt Algebra** and its central extension known as **Virasoro Algebra**. I will also show that the witt algebra admits a sub-algebra, which corresponds to the global conformal transformations.

We showed that in 2 dimensions (x^0, x^1), the infinitesimal conformal transformations are:

$$x'^0 = x^0 + \xi^0, \quad x'^1 = x^1 + \xi^1, \quad s.t. \quad f = \xi^0 + i\xi^1 \quad analytic$$

It is then convenient to work in the complex coordinates:

$$z = x^0 + ix^1, \quad z' = z + f(z)$$

Recall from complex analysis that the only bounded and entire function is the constant function. So, in general an analytic function shall admit singularities if it's bounded. Hence, in general, we can laurent expand the function $f(z) = -\sum_{n\in\mathbb{Z}} a_n z^{n+1}$, where the negative sign and the $n+1$ in the power are technical conventions. Now, consider the m-th term in this expansion, for which $z = z - a_m z^{m+1}$, we need to get the generator of this term:

$$e^{a_m \ell_m} z = z - a_m z^{m+1}, \quad (1 + a_m \ell_m + O(\ell_m^2))z = z - a_m z^{m+1}$$

$$z + a_m \ell_m z = z - a_m z^{m+1}, \quad \ell_m z = -z^{m+1}$$

$$\ell_m = -z^{m+1} \partial_z$$

Now, consider the lie algebra of this set of generators

$$[\ell_n, \ell_m] f(z) = z^{n+1} \partial_z (z^{m+1} \frac{\partial f}{\partial z}) - z^{m+1} \partial_z (z^{n+1} \frac{\partial f}{\partial z}) = (m+1) z^{n+m+1} \frac{\partial f}{\partial z} - (n+1) z^{n+m+1} \frac{\partial f}{\partial z} = -(n-m) z^{m+n+1} \partial_z f = (n-m) \ell_{n+m} f$$

Hence, we obtain what is called the Witt algebra:

$$[\ell_n, \ell_m] = (n - m)\ell_{n+m}$$

$$[\overline{\ell}_n, \overline{\ell}_m] = (n - m)\overline{\ell}_{n+m}$$

$$[\ell_n, \overline{\ell}_m] = 0$$

It is noticeable that this algebra is infinite-dimensional, and is not defined everywhere on the Riemann Sphere which is the correct compactification of $\mathbb{R}$. However, this algebra admits a sub-algebra generated by: ℓ_{-1}, ℓ_0, ℓ_1. To see this, consider the following:

$$[\ell_0, \ell_{-1}] = \ell_{-1}$$

$$[\ell_1, \ell_{-1}] = 2\ell_0$$

$$[\ell_1, \ell_0] = \ell_1$$

This sub-algebra however is globally defined and corresponds to the global conformal gorup. To argue this, we observe that on $\mathbb{R}^2 \sim \mathbb{C}$ the generators are not everywhere defined. Of course, we should probably be working on the Riemann sphere $S^2 \sim \mathbb{C} \cup \mathbb{R}$, as it is the conformal compactification of $\mathbb{R}$. Even here, however, some generators are not well defined. The generators ℓ_n are non-singular at $z = 0$ only for $n \geq -1$. By performing the change of variables $z = \frac{1}{w}$, we can also see that ℓ_n are non-singular as $w \longrightarrow 0$ for $n \leq 1$. Therefore globally defined conformal transformations on the Riemann sphere are generated by ℓ_{-1}, ℓ_0, ℓ_1.

Now, we notice that ℓ_{-1} and $\overline{\ell}_{-1}$ generate translations on the complex plane, while ℓ_1 and $\overline{\ell}_1$ generate special conformal transformations. Let us investigate the action of ℓ_0. $\ell_0 = -z\partial_z$ can be written in terms of the complex polar coordinates

($z = re^{i\theta}$) as follows:

$$\ell_0 = -\frac{1}{2}r\partial_r + \frac{i}{2}\partial_\theta, \quad \overline{\ell}_0 = -\frac{1}{2}r\partial_r - \frac{i}{2}\partial_\theta$$

Then the useful linear combinations are easily observed to be

$$\ell_0 + \overline{\ell}_0 = -r\partial_r, \quad i(\ell_0 - \overline{\ell}_0) = -\partial_\theta$$

The first generator corresponds to dilitations and the second one corresponds to the generator of rotations. Let us confirm this:

$$e^{a(\ell_0+\overline{\ell}_0)}re^{i\theta} = e^{-ar\partial_r}re^{i\theta} = (1 - ar\partial_r + O(\partial_r^2))re^{i\theta} = (1-a)re^{i\theta}$$

$$e^{\phi i(\ell_0-\overline{\ell}_0)}re^{i\theta} = e^{-\phi\partial_\theta}re^{i\theta} = (1 - \phi\partial_\theta + O(\partial_\theta^2))re^{i\theta} = (1-i\phi)re^{i\theta}$$

These are the infinitesimal versions, to get the finite transformations, we let $A = Na$, and $\Phi = N\phi$, and take the limit as $N \to \infty$ and compose the transformations:

$$\lim_{N\to\infty}(1 - \frac{A}{N})^N re^{i\theta} = e^A re^{i\theta}$$

$$\lim_{N\to\infty}(1 - i\frac{\Phi}{N})^N re^{i\theta} = e^{i\Phi}re^{i\theta} = re^{i(\theta+\Phi)}$$

Indeed, these operators generate dilitations and rotations respectively.

These operators generate infinitesimal transformations of the form $z \to \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}$. This is exactly the definition of the conformal group $PSL(2,\mathbb{C})$. It's similar to the Modular Group $SL(2,\mathbb{Z})$ except that we now allow complex entries. This also the general definition of the mobius transformations which are the nicest invertible transformations on the whole complex plane (including the point at infinity) that map the plane to itself, and this is why they are globally conformal.

The Witt algebra admits what is called a central extension to a different kind of algebra known as **Virasoro Algebra** which is depicted below. The detailed derivation is beyound the scope of this thesis.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

with a corresponding formula for $\overline{L}$, and $\overline{c}$. Notice that the central extension does not affect our finite subalgebra of conformal transformations.

2. Introduction

Conformal Field Theories (CFTs) have emerged as a cornerstone in theoretical physics, particularly in the realms of statistical mechanics and string theory. These theories are instrumental in understanding critical phenomena, phase transitions, and the behavior of systems at large scales. At the heart of CFTs lies the concept of conformal symmetry, which extends the usual Poincaré symmetry of spacetime by including transformations that preserve angles but not necessarily distances. This additional symmetry imposes powerful constraints on the physical theories, allowing for exact results that are often out of reach in other contexts.

2.1 Conformal Field Theories and Statistical Mechanics

The relationship between CFTs and statistical mechanics is profound and multifaceted. In statistical mechanics, systems at critical points exhibit scale invariance, and their behavior can be described by CFTs. These critical points, where phase transitions occur, are characterized by the absence of a characteristic length scale, leading to universal behavior that CFTs can capture with remarkable precision. For example, in two dimensions, the Ising model at its critical temperature is described by a CFT with central charge $c = \frac{1}{2}$. This correspondence allows for the exact calculation of critical exponents and correlation functions, providing deep insights into the nature of phase transitions.

2.2 Modular Forms in Conformal Field Theories

Modular forms play a pivotal role in the study of CFTs, especially in the context of the modular bootstrap program. These mathematical objects are functions on the upper half-plane that transform in specific ways under the action of the modular group $SL_2(\mathbb{Z})$. The modular group, consisting of 2x2 matrices with integer entries and determinant 1, encapsulates symmetries that are crucial for understanding the behavior of CFTs under transformations such as the modular transformation $\tau \to \frac{a\tau+b}{c\tau+d}$, where τ is a complex number in the upper half-plane.

The properties of modular forms provide a robust framework for understanding the symmetries and invariances within CFTs. They facilitate precise computations and offer deep insights into the mathematical structures underlying these theories. In higher-dimensional CFTs, modular forms help in understanding the partition functions and correlation functions, which are central to the study of these theories.

2.3 Motivating the Modular Bootstrap Program

The modular bootstrap program is a powerful approach that leverages the symmetries of CFTs to constrain and solve these theories. By analyzing the constraints imposed by modular invariance, one can derive consistency conditions that any CFT must satisfy. This program has its roots in the pioneering work on the conformal bootstrap in the 1970s, which aimed to determine the spectrum of primary operators and their correlation functions using the symmetries of the conformal group.

In two dimensions, the modular bootstrap program has led to remarkable successes, such as the classification of minimal models and the exact solution of numerous CFTs. The central idea is to use the modular invariance of the partition function on the torus to derive powerful constraints on the theory. By demanding that the partition function transforms appropriately under the modular group, one can derive constraints that severely limit the possible spectra and operator dimensions in the theory.

Moreover, the modular bootstrap has profound implications for the AdS/CFT correspondence, a duality that relates CFTs to theories of gravity in higher-dimensional spacetimes. This correspondence has opened new avenues for using CFT techniques to gain insights into quantum gravity and string theory.

In conclusion, the study of conformal field theories through the lens of modular forms and the modular bootstrap program not only enhances our understanding of statistical mechanics and critical phenomena but also provides a deep and rich mathematical structure that underpins many of the most exciting developments in theoretical physics today.

3. CFTs on a torus and the modular bootstrap

3.1 Complex tori

In this section, we will consider conformal field theories on a torus. From a statistical physics point of view, this consideration is natural because more often than not we work with periodic boundary conditions which is equivalent to working on a torus, as shown in figure 3.1. Moreover, in condensed matter physics, the torus structure appears when we consider a one-dimensional quantum system at finite temperature. At an abstract level, studying CFTs on a torus yields deep physical insights.

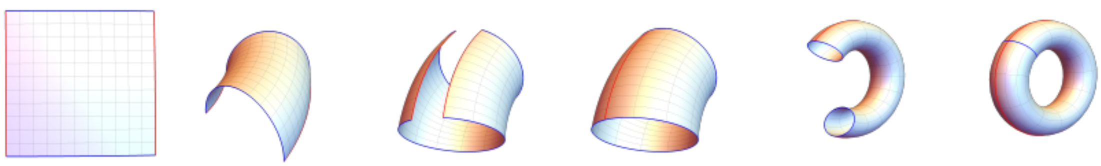

Figure 3.1: The square grid with periodic boundary condition is a torus

3.1.1 Complex tori and moduli space

From the point of view of topology, a torus $\mathbb{T}$ is the Cartesian product of two circles $\mathcal{S}^1 x \mathcal{S}^1$. This means that as a smooth surface, the torus is equivalent to the $\mathcal{R}^2$ plane quotiented by the group $\mathbb{Z}^2$ such that the points $(x, y) \to (x+1, y)$ and $(x, y) \to (x, y+1)$ are identified. Hence, a conveinent way to describe a complex torus is as a quotient of the plane by a lattice Λ. So, given two linearly independent complex lattice vectors ω_1 and ω_2, the lattice Λ generated by them is the subset

$$\Lambda = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}$$

as shown in figure 3.2. ω_1 and ω_2 are called the periods of the lattice. Moreover, if we have two scaled, translated, or rotated tori from each other, we consider them conformally equivalent because we are working with a CFT. In the preliminaries section, we showed that the conformal group includes rotations, dilitations, translations, and special conformal transformations. Here, we study a CFT on a torus, so, any conformally related tori will give the same physics. Our goal then is to study the implications of conformal symmetry on the invariants of our physical objects. In particular, the lattice $\omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}$ is equivalent to

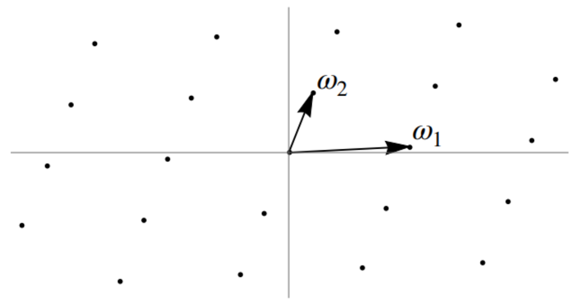

Figure 3.2: The lattice Λ generated by ω_1 and ω_2

$$\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}, \quad \tau = \frac{\omega_1}{\omega_2}$$

τ is called the complex structure of the torus. Note also that we can assume without loss of generality that $Im(\tau) > 0$; for if it is negative, we can just transform $\omega_1 \to -\omega_1$. Furthermore, the bases (ω_1, ω_2) and (ω_1', ω_2') generate the same lattice if $\omega_1' = \omega_1 + n$ and $\omega_2' = \omega_2 + m$, where $n, m \in \mathbb{Z}$. Because rotations and translations of the torus generate conformally equivalent tori, the bases (ω_1, ω_2) and (ω_1', ω_2') are equivalent if and only if they are related by a unimodular matrix A, *i.e.* an element of the modular group $\mathrm{SL}(2, \mathbb{Z})$ discussed in the preliminaries section. Namely,

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1$$

In terms of the complex structure τ, this means

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

which we immediately recognize as the action of the $\mathrm{SL}(2, \mathbb{Z})$ on the upper-half plane $\mathcal{H}$ as discussed in the preliminaries section. Hence, a torus with a complex structure τ in the upper-half plane is conformally equivalent to a torus with a complex structure τ' in the upper-half plane and related to τ by an element in $\mathrm{SL}(2, \mathbb{Z})$. We also discussed in the preliminaries section that $\mathrm{SL}(2, \mathbb{Z})$ is generated by the S, and T matrices whose action in this case is:

$$T : \tau \to \tau + 1, \quad S : \tau \to \frac{-1}{\tau}$$

3.2 Torus partition function and modular invariance

From our discussion, the complex torus $\mathbb{T}_\tau$ is a parallelogram, spanned by 1 and $\tau = \tau_1 + i\tau_2 = \frac{\omega_1}{\omega_2}$ with the opposite sides identified. This procedure rolls the parallelogram into a cylinder of height τ_2 as shown in the figure below.

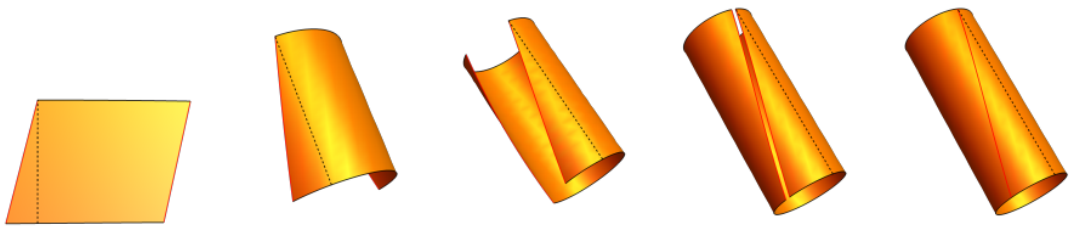

Figure 3.3: The cylinder obtained from the identification

We can then obtain the partition function of a one-dimensional quantum system at a finite temperature in a very ingenious way as follows. First, consider the case when $\tau_1 = 0$, so, $\tau = i\tau_2$, the partition function would coincide exactly with that system if we identify $\beta = \tau_2$, namely

$$Z(i\tau_2) = Tr(e^{-\tau_2 H}) = Tr(e^{-\beta H})$$

where H is the Hamiltonian of the 1d quantum system on a circle of unit length. However, the real part τ_1 of τ requires an additional translation of space on top of the Euclidean time evolution before sewing up (i.e. before taking the trace), therefore

$$Z(\tau) = Tr(e^{-\tau_2 H} e^{-i\tau_1 P})$$

Now, consider the exponential map $w = e^{-i2\pi z}$ where $z = x + it$, so, $w = e^{-i2\pi x} e^{2\pi t}$. We showed in the preliminaries section that the finite action of the operator $L_0 + \overline{L_0} - \frac{c+\bar{c}}{24}$ on any complex number is a dilitation, and that the action of the operator $i(L_0 - \overline{L_0} - \frac{c-\bar{c}}{24})$ is a rotation in the complex plane. Hence, under the exponential map, the operator $L_0 + \overline{L_0} - \frac{c+\bar{c}}{24}$ generate time translation, and that the operator $i(L_0 - \overline{L_0} - \frac{c-\bar{c}}{24})$ generate spatial translation. So, it is therefore straightforward to identify $L_0 + \overline{L_0} - \frac{c+\bar{c}}{24}$ as the Hamiltonian operator H, and $i(L_0 - \overline{L_0} - \frac{c-\bar{c}}{24})$ as the momentum operator P. So, in a compact form, the partition function can be written as

$$Z(\tau, \bar{\tau}) = Tr(q^{L_0 - \frac{c}{24}} \bar{q}^{-L_0 - \frac{\bar{c}}{24}}), \quad q = e^{2\pi i \tau}$$

Under the identification of $\tau = \frac{i\beta}{2\pi}$, $Z(\tau, \overline{\tau}) = Tr(e^{-\beta(L_0 + \overline{L_0} - \frac{c+\overline{c}}{24})}) = Tr(e^{-\beta H})$. Moreover, the partition function shall be invariant under a modular transformation on the complex structure τ. In particular, the S transformation $Z(\tau) = Z(\frac{-1}{\tau})$. Hence, $Z(\beta) = Z(\frac{4\pi^2}{\beta})$ and this is the modular invariance of the partition function that we will rely on moving forward.

3.2.1 Further discussion on the partition function and the exponential map

We stated that the partition function also involves the momentum operator which might not seem obvious at first glance. However, when one looks at fig 3.2 under the identification of ω_1 with space and ω_2 with time, we see that a closed loop in time also involves spatial translations. We are therefore motivated to define the partition function this way. This is essentially the same thing for CFTs as in statistical mechanics: a sum over configurations by some weight which is the Boltzmann factor $e^{-\beta H}$ in statistical mechanics. It also corresponds to the generating functional in Euclidean QFT due to the fact that the thermodynamic expression can be found by compactifying the time on a circle of radius $R = \beta = \frac{1}{T}$. Moreover, the essence of what we did lies in the fact that because of the conformal symmetry and the identification of conformally equivalent tori, we have a whole class of modularly related complex structures which yield the same physics. So, the physics should be insensitive to choice of the modular parameter τ, and this symmetry is immediately inherited in the partition function which encodes the physical information.

The exponential map $w = e^{-i2\pi z}$ where $z = x + it$ maps the complex plane into concentric circles where when we consider points of constant time and evolve the space we find ourselves moving on a circle in the new w plane. Moreover, the time evolution amounts to moving in the radial direction in this plane. It is also important to note that the exponential map takes the point at the infinite past $-\infty$ to the zero point in the w plane which is defined, and this gives us an advantage of dealing rigorously with this point. It also moves the infinite future to the point at infinity in the w plane which is also well-defined because the whole complex plane includes the point at infinity and we can work with it under the usual transformation in complex analysis $w' \rightarrow \frac{1}{w}$ and take the limit as $w \rightarrow \infty$.

4. The Hellerman Bound

4.1 Upper bound on the first excited state energy

The Hellerman bound is a universal constraint on the first excited state energy of any CFT. It states that the first excited state E_1 in the specturm of a CFT is bounded from above by negative the ground state energy which is related to the central charge of the theory $-E_0 = \frac{c}{24}$. Namely,

$$E_1 \leq \frac{c}{24} + O(1)$$

Our objective in this section will be to use the modular invariance of the partition function discussed earlier along with the obvious fact that we are dealing with a CFT to derive the Hellerman bound. Hence, we start off by defining the problem clearly and work towards the derivation.

Problem Statement: Consider the set of conformal 2D quantum mechanical systems with a bounded from below Hamiltonian H and a discrete set of energy eigenvalues E_n at finite temperature T. The thermal partition function of the system can then be defined as

$$Z(\beta) = Tr(e^{-\beta H}) = \sum_n e^{-\beta E_n} = \sum_{E_n} d(n) e^{-\beta E_n}, \quad \beta = \frac{1}{T},$$

where $d(n) \in \mathbb{Z}_+$ is the degeneracy of the eigenvalues E_n. From conformal field theory, we know the following facts by now:

(1) E_0 is negative and equal to a universal constant $E_0 = \frac{-c}{24}$, where $c > 0$. This is because of the anomalies that make the vacuum energy negative and this is why we needed the central extension of the Witt algebra known as Virasoro algebra as discussed in the preliminaries section.

(2) The partition function $Z(\beta)$ is modularly invariant under $\beta \longrightarrow \frac{4\pi^2}{\beta}$. This means

$$Z(\beta) = Z(g(\beta)), \quad g(\beta) = \frac{4\pi^2}{\beta}$$

The task is then to use these conditions to derive the Hellerman bound. First of all, we notice that the point $\beta = 2\pi$ is a fixed point of the transformation in the sense that $g(\beta) = \beta$ for $\beta = 2\pi$. First, we will show that the derivative of the partition function evaluated at this fixed point actually vanishes

$$\frac{d}{d\beta} Z(\beta)|_{\beta=2\pi} = \frac{d}{d\beta} \sum_n e^{-\beta E_n}|_{\beta=2\pi} = -\sum_n E_n e^{-\beta E_n}|_{\beta=2\pi} = -\sum_n E_n e^{-2\pi E_n}$$

$$\frac{d}{d\beta}Z(g(\beta))|_{\beta=2\pi} = \frac{dZ}{d\beta}\frac{dg}{d\beta}|_{\beta=2\pi} = -\frac{4\pi^2}{\beta^2}\frac{dZ}{d\beta}|_{\beta=2\pi} = \frac{4\pi^2}{\beta^2}\sum_n E_n e^{-\beta E_n}|_{\beta=2\pi} = \sum_n E_n e^{-2\pi E_n}$$

But since $g(\beta) = \beta$ for $\beta = 2\pi$, $\frac{d}{d\beta}Z(\beta)|_{\beta=2\pi} = \frac{d}{d\beta}Z(g(\beta))|_{\beta=2\pi}$, this implies

$$-\sum_n E_n e^{-2\pi E_n} = \sum_n E_n e^{-2\pi E_n} = 0 \tag{4.1}$$

Hence, $\frac{d}{d\beta}Z(\beta)|_{\beta=2\pi} = 0$. In fact, it is even more instructive to investigate the action of the operator $(\beta\frac{d}{d\beta})^N$ on $Z(\beta)$ at $\beta = 2\pi$, particularly because $\beta\frac{dg}{d\beta} = -\frac{4\pi^2}{\beta} = -g$. So, let's investigate this by first considering how the object $(\beta\frac{d}{d\beta})^N$ transforms under $\beta \longrightarrow g(\beta) = \frac{4\pi^2}{\beta}$. Under this transformation

$$\frac{d}{d\beta} \longrightarrow -\frac{\beta^2}{4\pi^2}\frac{d}{d\beta}$$

So,

$$(\beta\frac{d}{d\beta})^N \longrightarrow (\frac{4\pi^2}{\beta})^N(-\frac{\beta^2}{4\pi^2}\frac{d}{d\beta})^N = (-1)^N(\beta\frac{d}{d\beta})^N$$

However, at the fixed point of transformation ($\beta = 2\pi$), $(\beta\frac{d}{d\beta})^N Z(\beta)|_{\beta=2\pi} = (\beta\frac{d}{d\beta})^N Z(g(\beta))|_{\beta=2\pi}$. Hence,

$$(\beta\frac{d}{d\beta})^N Z(\beta)|_{\beta=2\pi} = (-1)^N(\beta\frac{d}{d\beta})^N Z(\beta)|_{\beta=2\pi}$$

Thus,

$$(\beta\frac{d}{d\beta})^N Z(\beta)|_{\beta=2\pi} = 0, \quad \forall N \in 2\mathbb{Z}_+ + 1$$

Let us write down explicitly the case $N = 3$.

$$(\beta\frac{d}{d\beta})^3 Z(\beta)|_{\beta=2\pi} = 2\pi[-\sum_n E_n e^{-2\pi E_n} + 2\pi[3\sum_n E_n^2 e^{-2\pi E_n} - 2\pi\sum_n E_n^3 e^{-2\pi E_n}]] = 0$$

But the first term vanishes as shown in (4.1)

$$3\sum_n E_n^2 e^{-2\pi E_n} - 2\pi\sum_n E_n^3 e^{-2\pi E_n} = 0$$

$$\sum_n E_n^2(3 - 2\pi E_n)e^{-2\pi E_n} = \sum_n E_n \alpha(E_n)e^{-2\pi E_n} = 0 \tag{4.2}$$

Where we defined $\alpha(E) = E(3 - 2\pi E)$.

Now, let's identify E_0 from (4.1)

$$\sum_n E_n e^{-2\pi E_n} = 0 \rightarrow \sum_{n=1} E_n e^{-2\pi E_n} = -E_0 e^{-2\pi E_0} \tag{4.3}$$

Now, separate the zero term in (4.2)

$$\sum_{n=1} E_n \alpha(E_n) e^{-2\pi E_n} + E_0^2(3-2\pi E_0)e^{-2\pi E_0} = \sum_{n=1} E_n \alpha(E_n) e^{-2\pi E_n} - E_0(3-2\pi E_0)\sum_{n=1} E_n e^{-2\pi E_n} = 0$$

Where, we have used (4.3)

$$\sum_{n=1} E_n(\alpha(E_n) - \alpha(E_0))e^{-2\pi E_n} = 0 \tag{4.4}$$

It is now instructive to plot $\alpha(E)$ and investigate its axis of symmetry

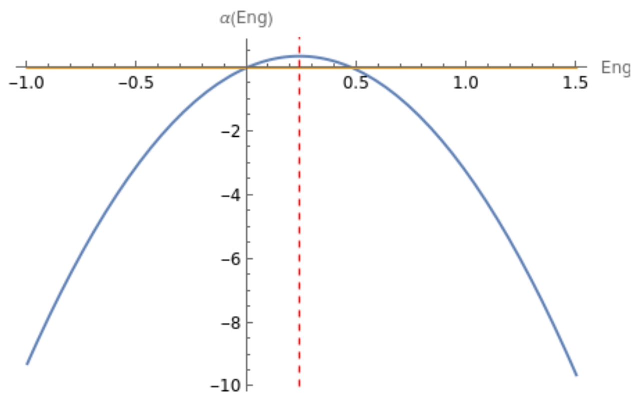

Figure 4.1: $\alpha(Eng)$ vs Eng

The roots of $\alpha(E)$ are $E \in \{0, \frac{3}{2\pi}\}$, and the axis of symmetry of $\alpha(E)$ is $E = \frac{3}{4\pi}$. Hence,

$$\alpha(E) = \alpha(\frac{3}{2\pi} - E) \tag{4.5}$$

Moreover, since $E_0 < 0$, $\alpha(E_0) = \alpha(\frac{3}{2\pi} - E_0) < 0$. This immediately implies that $\alpha(E)$ is monotonically decreasing for all $E > \frac{3}{2\pi} - E_0$. Yet, if $E_1 > \frac{3}{2\pi} - E_0$, then,

$$\alpha(E_n) < \alpha(E_0) \quad \forall n \in \mathbb{Z}_+$$

which contradicts (4.4) because this would imply that we're summing infinite negative terms and getting zero and this is impossible. Hence, this implies by contradiction that

$$E_1 \leq \frac{3}{2\pi} - E_0 = \frac{c}{24} + \frac{3}{2\pi} \tag{4.6}$$

This is the Hellerman bound we were looking for, it turned out that this bound is necessary for the consistency of the theory. In the process, we started out by studying a CFT on a torus and identifying the torus modular parameter with the thermal partition function but because the torus inherited the conformal symmetry in the modular invariance of the modular parameter τ, the physics should be insensitive to equivalent choices of the modular parameter, and this implied an upper bound on the first excited state energy by the central charge for the pure consistency of the physical theory. This result is truly significant and it demonstrates how studying the consistency of CFTs can constraint the physical theories very much using the conformal bootstrap program.

4.2 Lower bound on the entropy

A more straightforward consequence of the modular invariance of the partition function is a lower bound on the thermodynamic entropy of the canonical ensemble at the fixed point of transformation $\beta = 2\pi$. We know from statistical mechanics that the entropy is related to the partition function in the following way

$$\sigma = ln(Z) + \beta\langle E\rangle \tag{4.7}$$

Equation (4.1) is nothing but the expectation value of the energy at $\beta = 2\pi$, $\langle E\rangle|_{\beta=2\pi} = 0$, vanishing the second term. Moreover, because the partition function involves a positive contribution from each term, it's bounded from below by the contribution of the ground state as follows

$$Z|_{\beta=2\pi} = \sum_{n} e^{-2\pi E_n} = e^{-2\pi E_0} + \sum_{n\geq 1} e^{-2\pi E_n} \geq e^{-2\pi E_0} \tag{4.8}$$

This immediately implies

$$\sigma|_{\beta=2\pi} \geq -2\pi E_0 = \frac{\pi c}{12} \tag{4.9}$$

Again, this is a powerful result from employing the modular bootstrap program in 2D CFTs, and it concludes our discussion on the topic.

5. Conclusion

In this work, we have delved into the profound implications of modular invariance in two-dimensional conformal field theories (CFTs), focusing particularly on the Hellerman bound and the lower bound on entropy. By harnessing the modular invariance of the partition function and the intrinsic properties of CFTs, we have derived crucial constraints that any consistent CFT must satisfy.

Firstly, we derived the Hellerman bound, which provides an upper limit on the first excited state energy in the spectrum of a CFT. This bound is expressed as:

$$E_1 \leq \frac{c}{24} + \frac{3}{2\pi}$$

where c is the central charge of the theory. The derivation hinges on the modular invariance of the partition function under the transformation $\beta = 2\pi$ is a fixed point of this transformation. By examining the behavior of the partition function at this fixed point and considering the consistency of the theory, we established that the first excited state energy must indeed be bounded by the central charge.

Secondly, we explored the lower bound on the thermodynamic entropy of the canonical ensemble at the fixed point $\beta = 2\pi$. By expressing the entropy in terms of the partition function and leveraging the fact that the expectation value of the energy vanishes at this point, we concluded that the entropy is bounded from below by the contribution from the ground state energy:

$$\sigma|_{\beta=2\pi} \geq \frac{\pi c}{12}$$

This result underscores the profound influence of modular invariance on the thermodynamic properties of CFTs, providing a universal lower bound on the entropy.

In conclusion, our exploration of the Hellerman bound and the entropy bound highlights the power of modular invariance and the conformal bootstrap program in constraining the physical characteristics of CFTs. These results not only reinforce the consistency conditions required for a well-defined CFT but also illustrate how fundamental symmetries can impose stringent limits on the properties of physical theories. Through such rigorous analyses, we gain deeper insights into the structure and behavior of CFTs, paving the way for further advancements in theoretical physics.

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