Unification of Conformal Gravity and Internal Interactions

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Introduction

- The dimension of the tangent space is not necessarily equal to the dimension of a curved manifold. *Weinberg, 1984*
- Gravitational theories can be described as gauge theories.
  Utiyama, 1956; Kibble, 1961; MacDowell & Mansouri, 1977; Chamseddine & West, 1977; Ivanov & Niederle, 1980; Kibble & Stelle, 1985
- Particle physics theories are also gauge theories.
- Unification of gravity with internal interactions could be possible by larger gauge groups.

We aim to unify conformal gravity as a gauge theory with internal interactions under one unification gauge group.

Gauge theory of SO(2,3)

- Instead of the Poincaré group - Anti-de Sitter group: SO(2,3)
- Same amount of generators BUT they can be written on equal footing (semisimple group):

$$[\hat{M}_{AB}, \hat{M}_{CD}] = \eta_{AC}\hat{M}_{DB} - \eta_{BC}\hat{M}_{DA} - \eta_{AD}\hat{M}_{CB} + \eta_{BD}\hat{M}_{CA}$$

- η_{AB} is the 5-dim Minkowski metric with two timelike coefficients (1st and 5th) and $A, \ldots, D = 1 \ldots 5$
- Perform a splitting of the indices $A = (a, 5)$
- Define $\hat{M}_{ab} = M_{ab}$ and $\hat{M}_{a5} = \frac{1}{m}P_a$, $[m] = L^{-1}$
- Gauge connection: $A_\mu = \frac{1}{2}\hat{\omega}_\mu{}^{AB}\hat{M}_{AB} = \frac{1}{2}\omega_\mu{}^{ab}M_{ab} + e_\mu{}^a P_a$
- where $\hat{\omega}_\mu{}^{ab} = \omega_\mu{}^{ab}$ and $\hat{\omega}_\mu{}^{a5} = m e_\mu{}^a$
- The same for the field strength tensor $\hat{R}_{\mu\nu}{}^{AB}$:

$$\hat{R}_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} + 2m^2 e_\mu{}^{[a} e_\mu{}^{b]}\,, \quad \hat{R}_{\mu\nu}{}^{a5} = m T_{\mu\nu}{}^a$$

- Consider the following $SO(2,3)$ invariant quadratic action:

$$S = a_{AdS} \int d^4x \Big(m y^E \epsilon_{ABCDE} \frac{1}{4} \hat{R}_{\mu\nu}{}^{AB} \hat{R}_{\rho\sigma}{}^{CD} \epsilon^{\mu\nu\rho\sigma} + \lambda \left(y^E y_E + m^{-2} \right) \Big)$$

- y^E an internal space vector field
- vector taken to be gauge fixed towards the 5-th direction:

$$y = y^0 = \left(0, 0, 0, 0, m^{-1} \right).$$

- the non-vanishing value $y^5(x)$ is responsible for the symmetry breaking of $SO(2,3)$ to the $SO(1,3)$

$$\begin{aligned} S &= \frac{a_{AdS}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \hat{R}_{\mu\nu}{}^{ab} \hat{R}_{\rho\sigma}{}^{cd} \epsilon_{abcd} \\ &= \frac{a_{AdS}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \left(\mathcal{L}_{RR} + m^2 \mathcal{L}_{eeR} + m^4 \mathcal{L}_{eeee} \right) \end{aligned}$$

- $\mathcal{L}_{RR}$: Gauss-Bonnet - no contribution to the e.o.m.
- $\mathcal{L}_{eeR}$: Palatini action (torsionless + Einstein Field Equations)
- $\mathcal{L}_{eeee}$: Plays the role of cosmological constant
- Solution of Einstein Field Equations is the Anti-de Sitter space

Conformal 4d gravity as a gauge theory

- Group parametrizing the symmetry: $SO(2,4)$
- 15 generators: 6 LT M_{ab}, 4 translations, P_a, 4 conformal boosts K_a and the dilatation D
- Following the same procedure one calculates transf of the gauge fields and tensors after defining the gauge connection
- Action is taken of $SO(2,4)$ invariant quadratic form
- Initial symmetry breaks under certain constraints resulting to the *Weyl action* *Kaku,Townsend,Van Nieu/zen '77, Fradkin, Tseytlin '85*
- Initial symmetry breaks spontaneously by introducing a scalar in the adjoint rep fixed in the dilatation direction, or by two scalars in vector reps.

R., Stefas, Zoupanos '24

SSB by using a scalar in the adjoint representation

Gauge connection:

$$A_\mu = \frac{1}{2}\omega_\mu{}^{ab} M_{ab} + e_\mu{}^a P_a + b_\mu{}^a K_a + \tilde{a}_\mu D,$$

Field strength tensor:

$$F_{\mu\nu} = \frac{1}{2} R_{\mu\nu}{}^{ab} M_{ab} + \tilde{R}_{\mu\nu}{}^a P_a + R_{\mu\nu}{}^a K_a + R_{\mu\nu} D,$$

where

$$\begin{aligned}
R_{\mu\nu}{}^{ab} &= \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b - 8 e_{[\mu}{}^{[a} b_{\nu]}{}^{b]} \\
&= R^{(0)ab}_{\mu\nu} - 8 e_{[\mu}{}^a b_{\nu]}{}^{b]}, \\
\tilde{R}_{\mu\nu}{}^a &= \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^{ab} e_{\nu b} - \omega_\nu{}^{ab} e_{\mu b} - 2\tilde{a}_{[\mu} e_{\nu]}{}^a \\
&= T^{(0)a}_{\mu\nu} - 2\tilde{a}_{[\mu} e_{\nu]}{}^a, \\
R_{\mu\nu}{}^a &= \partial_\mu b_\nu{}^a - \partial_\nu b_\mu{}^a + \omega_\mu{}^{ab} b_{\nu b} - \omega_\nu{}^{ab} b_{\mu b} + 2\tilde{a}_{[\mu} b_{\nu]}{}^a \\
&= T^{(0)a}_{\mu\nu}(b) + 2\tilde{a}_{[\mu} b_{\nu]}{}^a, \\
R_{\mu\nu} &= \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu + 4 e_{[\mu}{}^a b_{\nu]a},
\end{aligned}$$

We start with the parity conserving action, which is quadratic in terms of the field strength tensor and introduce a scalar in the rep 15

$$S_{SO(2,4)} = a_{CG} \int d^4x \left[\mathrm{tr}\, \epsilon^{\mu\nu\rho\sigma} m\phi F_{\mu\nu} F_{\rho\sigma} + \left(\phi^2 - m^{-2} \mathbb{1}_4 \right) \right],$$

The scalar expanded on the generators is:

$$\phi = \phi^{ab} M_{ab} + \tilde{\phi}^a P_a + \phi^a K_a + \tilde{\phi} D,$$

We pick the specific gauge in which ϕ is diagonal of the form $diag(1, 1, -1, -1)$. Specifically we choose ϕ to be only in the direction of the dilatation generator D:

$$\phi = \phi^0 = \tilde{\phi} D \xrightarrow{\phi^2 = m^{-2} \mathbb{1}_4} \phi = -2m^{-1} D.$$

The resulting broken action is (after employing anticommutator relations and the traces over the generators):

$$S_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd}$$

The $\tilde{a}_\mu$ is not present in the action, so we can set it equal to zero.

$R_{\mu\nu}$ is also absent so we can also set it equal to zero

$$R_{\mu\nu} = \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu + 4 e_{[\mu}{}^a b_{\nu]a} = 0 \xrightarrow{\tilde{a}_\mu = 0}$$

$$e_\mu{}^a b_{\nu a} - e_\nu{}^a b_{\mu a} = 0$$

We examine two possible solutions of the above equation:

- $b_\mu{}^a = a e_\mu{}^a$, *Chamseddine '03*
- $b_\mu{}^a = -\frac{1}{4}\left(R_\mu{}^a + \frac{1}{6} R e_\mu{}^a\right)$ *Kaku, Townsend, van Nieuwenhuizen, 78*
  Freedman, Van Proyen "Supergravity" '12

The first choice leads to the Einstein-Hilbert action, while the second leads to Weyl action.

Einstein-Hilbert action

- When $b_\mu{}^a = a e_\mu{}^a$, the broken action becomes:

$$S_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4}\int d^4x \epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd} \implies$$

$$S_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4}\int d^4x \epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\Big[R^{(0)ab}_{\mu\nu} R^{(0)cd}_{\rho\sigma} - 16 m^2 a R^{(0)ab}_{\mu\nu} e_\rho{}^c e_\sigma{}^d + \\ + 64 m^4 a^2 e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d \Big]$$

  This action consists of three terms:
- $\mathcal{L}_{RR}$: Gauss-Bonnet - no contribution to the e.o.m.
- $\mathcal{L}_{eeR}$: Palatini action (torsionless + Einstein Field Equations)
- $\mathcal{L}_{eeee}$: Plays the role of cosmological constant
- Solution of Einstein Field Equations is the Anti-de Sitter space, when $a < 0$.

Weyl action

- When $b_\mu{}^a = -\frac{1}{4}(R_\mu{}^a + \frac{1}{6}Re_\mu{}^a)$, the broken action becomes

$$S = \frac{a_{CG}}{4}\int d^4x \epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\Big[R^{(0)ab}_{\mu\nu} - \frac{1}{2}\left(\tilde{e}_\mu{}^{[a}R_\nu{}^{b]} - \tilde{e}_\nu{}^{[a}R_\mu{}^{b]}\right) + \\ + \frac{1}{3}R\tilde{e}_\mu{}^{[a}\tilde{e}_\nu{}^{b]}\Big] \\ \Big[R^{(0)cd}_{\rho\sigma} - \frac{1}{2}\left(\tilde{e}_\rho{}^{[c}R_\sigma{}^{d]} - \tilde{e}_\sigma{}^{[c}R_\rho{}^{d]}\right) + \\ + \frac{1}{3}R\tilde{e}_\rho{}^{[c}\tilde{e}_\sigma{}^{d]}\Big],$$

where $\tilde{e}_\mu{}^a = me_\mu{}^a$ is the rescaled vierbein. The above action is equal to

$$S = \frac{a_{CG}}{4}\int d^4x \epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}C_{\mu\nu}{}^{ab}C_{\rho\sigma}{}^{cd},$$

where $C_{\mu\nu}{}^{ab}$ is the Weyl conformal tensor.

Unification of gravity theories with Internal Interactions

- So far in the gauge theoretic approach of gravity, general relativity is described by gauging the symmetry of the tangent manifold in four dimensions.
- Usually the dimension of the tangent space is considered to be equal to the dimension of the curved manifold. However, the tangent group of a manifold of dimension d is not necessarily SO_d.

Weinberg '84

- It has been suggested that by gauging an enlarged symmetry of the tangent space in four dimensions one could unify gravity with internal interactions.

Chamseddine, Mukhanov '10

- We aim to unify gravities as a gauge theory with internal interactions under one unification gauge group.
- Attempts of unification for the case of Einstein gravity: Chamseddine and Mukhanov, 2010; Percacci, 1991; Konitopoulos, R., Zoupanos, 2023.

Unification group

- Weyl gravity is based on gauging the $SO(2,4)$, while Fuzzy gravity on $SO(2,4) \times U(1)$.
- Internal Interactions by $SO(10)$ (GUT).
- Spontaneous symmetry breakings are used in all cases.

Usually to have a Chiral theory we need a $SO(4n+2)$ group. The smallest unification group in which both Majorana and Weyl condition can be imposed is $SO(2,16)$ from which:

$$SO(2,16) \xrightarrow{SSB} SO(2,4) \times SO(12)$$

and

$$SO(12) \xrightarrow{SSB} SO(10) \times [U(1)].$$

Breakings and branching rules

We start from $SO(2,16) \sim SO(18)$

- For CG we gauge $SO(2,4) \sim SU(2,2) \sim SO(6) \sim SU(4)$
- For FG we gauge $SO(2,4) \times U(1) \sim SO(6) \times U(1) \sim U(4)$
- For internal interactions we require $SO(10)$ GUT.

$$C_{SO(2,16)}(SO(2,4)) = SO(10)$$ and

$$C_{SO(2,16)}(SO(2,4) \times U(1)) = SO(10) \times U(1).$$

Breakings and branching rules (Continued)

$$SO(18) \supset SU(4) \times SO(12)$$

$$\begin{aligned} 18 &= (6,1) + (1,12) && \text{vector} \\ 153 &= (15,1) + (6,12) + (1,66) && \text{adjoint} \\ 256 &= (4,\overline{32}) + (\overline{4},32) && \text{spinor} \\ 170 &= (1,1) + (6,12) + (20',1) + (1,77) && \text{2nd rank symmetric} \end{aligned}$$

VEV in the $\langle 1,1 \rangle$ component of a scalar in 170 leads to $SU(4) \times SO(12)$.

Breakings and branching rules (Continued)

We break the $SO(12)$ down to $SO(10) \times U(1)$ or to $SO(10)$ with the 66 rep or the 77 rep.

$$
\begin{aligned}
&SO(12) \supset SO(10) \times U(1) \\
&\qquad 66 = (1)(0) + (10)(2) + (10)(-2) + (45)(0) \\
&\qquad 77 = (1)(4) + (1)(0) + (1)(-4) + (10)(2) + (10)(-2) + (54)(0)
\end{aligned}
$$

by giving VEV to the $\langle(1)(0)\rangle$ of the 66 rep we obtain $SO(10) \times U(1)$.

by giving VEV to the $\langle(1)(4)\rangle$ of the 77 rep we obtain $SO(10)$.

Breakings and branching rules (Continued)

We break $SU(4)$ in 2 steps:

- First step: Breaking $SU(4) \to Sp_4$:

$$SU(4) \supset Sp4$$
$$4 = 4$$
$$6 = 1 + 5$$

  giving VEV to a scalar in 6 rep in the $\langle 1 \rangle$ component, the $SU(4)$ breaks down to the Sp_4.

- Second step: Breaking $Sp_4 \to SU(2) \times SU(2)$

$$Sp_4 \supset SU(2) \times SU(2)$$
$$5 = (1, 1) + (2, 2)$$
$$4 = (2, 1) + (1, 2).$$

  giving VEV in $\langle 1, 1 \rangle$ of a scalar in the 5 rep we obtain eventually the Lorentz group $SU(2) \times SU(2) \sim SO(1, 3)$.

Fermions

Weyl condition: $\Gamma^{D+1}\psi_\pm = \pm\psi_\pm, \quad D = even.$

Note that since $\Gamma^{D+1} = \gamma^5 \otimes \gamma^{d+1}$, the eigenvalues of γ^5 and γ^{d+1} are interrelated. However the choice of the eigenvalue of Γ^{D+1} does not impose the eigenvalue on γ^5 !

Majorana condition: $\psi = C\bar{\psi}^T$

Weyl-Majorana spinors can exist when $D = 4n + 2$.
Type of spinors of $SO(p, q)$ depends on signature $(p - q) mod 8$.
For $p + q = even$:

- 0 : real rep
- 4 : quaternionic rep
- 2 or 6 : complex rep

Chapline & Slansky, 1982; Polchinski, 1998; D'Auria et al., 2001; Figueroa-O'Farrill, n.d.

Fermions (Continued)

In the case of $SO(2,16)$ the signature is 6 , and imposing the Weyl and Majorana conditions is permitted.

Dirac spinors are defined as direct sum of Weyl spinors and the Weyl condition chooses one of them, say $\sigma_{18} = 256$.
Spinor rep branching rules are:

$$SO(18) \supset SU(4) \times SO(12)$$
$$256 = (4, \bar{32}) + (\bar{4}, 32)$$

Imposing Majorana condition the fermions are in the $(\bar{4}, 32)$. Then

$$SO(12) \supset SO(10) \times [U(1)]$$
$$32 = (\bar{16})(1) + (16)(-1)$$

On the other hand

$$SU(4) \to Sp_4 \to SU(2) \times SU(2)$$
$$4 = 4 = (2,1) + (1,2).$$

Fermions (Continued)

After all the breakings:

$$
\begin{aligned}
&SU(2) \times SU(2) \times SO(10) \times [U(1)] \\
&\{[(2,1) + (1,2)\}\{(16)(-1) + (\bar{16})(1)\} \\
&\qquad = 16_L(-1) + \bar{16}_L(1) + 16_R(-1) + \bar{16}_R(1)
\end{aligned}
$$

and since $\bar{16}_R(1) = 16_L(-1)$ and $\bar{16}_L(1) = 16_R(-1)$,

$$= 2 \times 16_L(-1) + 2 \times 16_R(-1).$$

Finally, keeping only the left-handed part we obtain:

$$2 \times 16_L(-1)$$

Imposing also the Majorana condition in lower dims we obtain

$$16_L(-1) \quad \text{of} \quad SO(10) \times [U(1)]$$

Thank you for your attention!