Flavour in $SU(5)$ Finite Unified Theories:

Myriam Mondragón

IFUNAM, Mexico

with

Odin Estrada,
Gregory Patellis,
George Zoupanos

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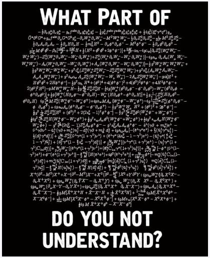
What's going on?

- What happens as we approach the Planck scale? or just as we go up in energy...
- What happened in the early Universe?
- How are the gauge, Yukawa and Higgs sectors related at a more fundamental level?
- How do we go from a fundamental theory to eW field theory as we know it?
- Why there are so many free parameters in the SM?
- How do particles get their very different masses?
- What about flavour?

- **Where is the new physics??**

Mess: SM + BSM



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www.quantumdiaries.org

Search for understanding relations between parameters

addition of symmetries.

$N = 1$ SUSY GUTs.

Complementary approach: look for RGI relations among couplings at GUT scale $\longrightarrow$ Planck scale

$\Rightarrow$ **reduction of couplings**

resulting theory: less free parameters $\therefore$ more predictive

Zimmermann 1985

Remarkable: reduction of couplings provides a way to relate two previously unrelated sectors

gauge and Yukawa couplings

Kapetanakis, M.M., Zoupanos (1993), Kubo, M.M., Olechowski, Tracas, Zoupanos (1995,1996,1997); Oehme (1995); Kobayashi, Kubo, Raby, Zhang (2005); Gogoladze, Mimura, Nandi (2003,2004); Gogoladze, Li, Senoguz, Shafi, Khalid, Raza (2006,2011); M.M., Tracas, Zoupanos (2014)

Reduction of Couplings – RoC

A RGI relation among couplings $\Phi(g_1, \ldots, g_N) = 0$ satisfies

$$\mu\, d\Phi/d\mu \;=\; \sum_{i=1}^{N} \beta_i\, \partial\Phi/\partial g_i \;=\; 0.$$

g_i = **coupling,** β_i **its** β **function**

Finding the $(N-1)$ independent Φ's is equivalent to solve the
reduction equations (RE)

$$\beta_g\,(dg_i/dg) = \beta_i\ ,$$

$i = 1, \cdots, N$

- Reduced theory: only one independent coupling and its β function
- complete reduction: power series solution of RE

$$g_a = \sum_{n=0} \rho_a^{(n)}\, g^{2n+1}$$

- uniqueness of the solution can be investigated at one-loop
  valid at all loops Zimmermann, Oehme, Sibold (1984,1985)
- The complete reduction might be too restrictive, one may use fewer Φ's as RGI constraints
- SUSY is essential for finiteness

finiteness: absence of ∞ renormalizations

$$\Rightarrow \quad \beta^N = 0$$

may be achieved through RE

RoC + SUSY = finiteness

- SUSY no-renormalization theorems
- $\Rightarrow$ only study one and two-loops
- RoC guarantees that is gauge and reparameterization invariant to all loops

Reduction of couplings: the Standard Model

It is possible to make a reduced system in the Standard Model in the matter sector:
solve the REs, reduce the Yukawa and Higgs in favour of α_S gives

$$\alpha_t/\alpha_s = \frac{2}{9}\,; \quad \alpha_\lambda/\alpha_s = \frac{\sqrt{689}-25}{18} \simeq 0.0694$$

border line in RG surface, Pendleton-Ross infrared fixed line.

But including the corrections due to non-vanishing gauge couplings up to two-loops, changes these relations and gives

$$M_t = 98.6 \pm 9.2 GeV$$

and

$$M_h = 64.5 \pm 1.5 GeV$$

Both out of the experimental range, but pretty impressive

Kubo, Sibold and Zimmermann, 1984, 1985

SUSY and RoC

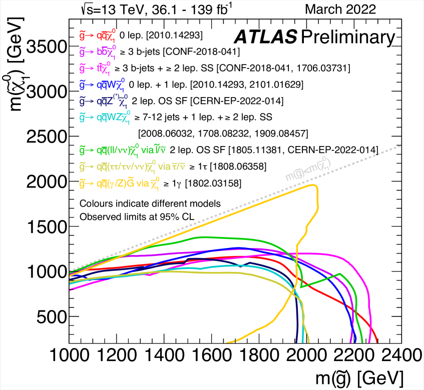
Many of the reduced systems imply SUSY, even if it was not assumed a priori

Moreover: adding SUSY improves predictions $\Rightarrow$

SUSY + reduction of couplings natural

non SUSY 2HDM possible and interesting

see Patellis' talk



- Light SUSY in various SUSY models incompatible with LHC data
- **BUT** Different assumptions on parameters of MSSM or NMSSM lead to different predictions

https://atlas.web.cern.ch/Atlas/GROUPS/PHYSICS/PUBNOTES/ATL-PHYS-PUB-2022-013/

Predictions in $SU(5)$ FUTs – only 3rd generation

$M^{th}_{top} \sim \mathbf{178}$ GeV		large $\tan\beta$	1993
$M^{exp}_{top} = \mathbf{176 \pm 18}$			1995

$M^{th}_{top} \sim \mathbf{174}$	$M^{exp}_{top} = 175.6 \pm 5.5$	heavy s-spectrum	1998
$M^{th}_{top} \sim \mathbf{174}$	$M^{exp}_{top} = 174.3 \pm 5.1$GeV	$M^{th}_{Higgs} \sim \mathbf{115} \sim \mathbf{135}$ GeV	2003

constraints on M_h and $b \to s\gamma$ already push up the s-spectrum > 300 GeV

$M^{th}_{top} \sim \mathbf{173}$	$M^{exp}_{top} = 172.7 \pm 2.9$ GeV	$M^{th}_{Higgs} \sim \mathbf{122} \sim \mathbf{126}$ GeV	2007
		$M^{exp}_{Higgs} = \mathbf{126 \pm 1}$	2012
$M^{th}_{top} \sim \mathbf{173}$	$M^{exp}_{top} = 173.3 \pm 0.9$ GeV	$M^{th}_{Higgs} \sim \mathbf{121} - \mathbf{126}$ GeV	2013

Constraints from Higgs and B physics$\Rightarrow$ s-spectrum > 1 TeV.

More analyses, phenomenological and theoretical, encouraged (and done)

MM, Kapetanakis, Zoupanos 1992; MM, Heinemeyer, Kalinowski, Kotlarski, Kubo, Ma, Olechowski, Patellis, Tracas, Zoupanos 1993-2023

Finiteness

Finiteness = absence of divergent contributions to renormalization parameters $\Rightarrow \beta = 0$

Possible in SUSY due to improved renormalization properties

A chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g has a superpotential

$$W = \frac{1}{2}\, m^{ij}\, \Phi_i\, \Phi_j + \frac{1}{6}\, C^{ijk}\, \Phi_i\, \Phi_j\, \Phi_k \ ,$$

Requiring one-loop finiteness $\beta_g^{(1)} = 0 = \gamma_i^{j(1)}$ gives the following conditions:

$$\sum_i T(R_i) = 3C_2(G)\,, \qquad \frac{1}{2} C_{ipq} C^{jpq} = 2\delta_i^j g^2 C_2(R_i)\,.$$

$C_2(G)$ quadratic Casimir invariant, $T(R_i)$ Dynkin index of R_i, C_{ijk} Yukawa coup., g gauge coup.

- **restricts the particle content of the models**
- **relates the gauge and Yukawa sectors**

All-loop finiteness

- One-loop finiteness $\Rightarrow$ two-loop finiteness

Jones, Mezincescu and Yao (1984,1985)

- Cannot be applied to the Minimal SUSY Standard Model (MSSM): $C_2[U(1)] = 0$
- The finiteness conditions allow only soft supersymmetry breaking terms (SSB) terms

It is possible to achieve all-loop finiteness $\beta^n = 0$:

Lucchesi, Piguet, Sibold

1. One-loop finiteness conditions must be satisfied

   restricts irreps and relates gauge and Yukawa couplings

2. The Yukawa couplings must be a formal power series in g, which is solution (isolated and non-degenerate) to the reduction equations

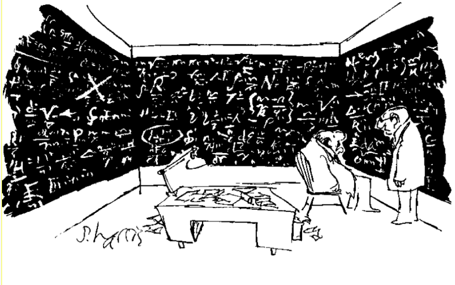
SUSY breaking soft terms

Supersymmetry is essential. It has to be broken, though. . .

$$-\mathcal{L}_{\rm SB} = \frac{1}{6}\, h^{ijk}\, \phi_i\phi_j\phi_k + \frac{1}{2}\, b^{ij}\, \phi_i\phi_j + \frac{1}{2}\,(m^2)^j_i\, \phi^{*\,i}\phi_j + \frac{1}{2}\, M\,\lambda\lambda + \text{H.c.}$$

h trilinear couplings (A), b^{ij} bilinear couplings, m^2 squared scalar masses, M unified gaugino mass

Introduce over 100 new free parameters



"Whatever happened to *elegant* solutions?"

RGI in the Soft Supersymmetry Breaking Sector

The RGI method has been extended to the SSB of these theories.

- One- and two-loop finiteness conditions for SSB have been known for some time
  Jack, Jones, et al.
- It is also possible to have all-loop RGI relations in the finite and non-finite cases
  Kazakov; Jack, Jones, Pickering
- SSB terms depend only on g and the unified gaugino mass M
  universality conditions

$$h = -MC, \qquad m^2 \propto M^2, \qquad b \propto M\mu$$

  but charge and colour breaking vacua ☹

- Possible to extend the universality condition to a sum-rule for the soft scalar masses

  $\Rightarrow$ better phenomenology

  Kawamura, Kobayashi, Kubo; Kobayashi, Kubo, M.M., Zoupanos

Soft scalar sum-rule for the finite case

Finiteness implies

$$C^{ijk} = g \sum_{n=0} \rho^{ijk}_{(n)} g^{2n} \quad \Rightarrow \quad h^{ijk} = -MC^{ijk} + \cdots = -M\rho^{ijk}_{(0)}\, g + O(g^5)$$

If lowest order coefficients $\rho^{ijk}_{(0)}$ and $(m^2)^i_j$ satisfy diagonality relations

$$\rho_{ipq(0)}\rho^{jpq}_{(0)} \propto \delta^j_i \,, \qquad (m^2)^i_j = m^2_j \delta^i_j$$

for all p and q.

The following soft scalar-mass sum rule is satisfied, also to all-loops

$$(\; m_i^2 + m_j^2 + m_k^2 \;)/MM^\dagger = 1 + \frac{g^2}{16\pi^2}\,\Delta^{(2)} + O(g^4)$$

for $i,\; j,\; k$ with $\rho^{ijk}_{(0)} \neq 0$, where $\Delta^{(2)}$ is the two-loop correction =0 for universal choice

Kobayashi, Kubo, Zoupanos

based on developments by Kazakov et al; Jack, Jones et al; Hisano, Shifman; etc

Also satisfied in certain class of orbifold models, where massive states are organized into $N = 4$ supermultiples

Several aspects of Finite Models have been studied

- $SU(5)$ Finite Models studied extensively

  Rabi et al; Kazakov et al; López-Mercader, Quirós et al; M.M, Kapetanakis, Zoupanos; etc
- One of the above coincides with a non-standard Calabi-Yau $SU(5) \times E_8$ Greene et al; Kapetanakis, M.M., Zoupanos
- Finite theory from compactified string model also exists (albeit not good phenomenology) Ibáñez
- Criteria for getting finite theories from branes Hanany, Strassler, Uranga
- $N = 2$ finiteness Frere, Mezincescu and Yao
- Models involving three generations Babu, Enkhbat, Gogoladze
- Some models with $SU(N)^k$ **finite** $\Longleftrightarrow$ **3 generations,** good phenomenology with $SU(3)^3$ Ma, M.M, Zoupanos
- Relation between commutative field theories and finiteness studied

  Jack and Jones
- Proof of conformal invariance in finite theories Kazakov
- Inflation from effects of curvature that break finiteness

  Elizalde, Odintsov, Pozdeeva, Vernov

$SU(5)$ Finite Models only third generation

Example: two models with $SU(5)$ gauge group. The matter content is

$$3\,\overline{\mathbf{5}} + 3\,\mathbf{10} + 4\,\{\mathbf{5} + \overline{\mathbf{5}}\} + \mathbf{24}$$

The models are finite to all-loops in the dimensionful and dimensionless sector. In addition:

- The soft scalar masses obey a sum rule
- At the M_{GUT} scale the gauge symmetry is broken $\Rightarrow$ MSSM
- At the same time finiteness is broken
- Assume two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs $\{\mathbf{5} + \overline{\mathbf{5}}\}$ coupled mainly to the third generation

The difference between the two models is the way the Higgses couple to the **24**

Kapetanakis, Mondragón, Zoupanos; Kazakov et al.

The finiteness relations give at the M_{GUT} scale

3 generation models

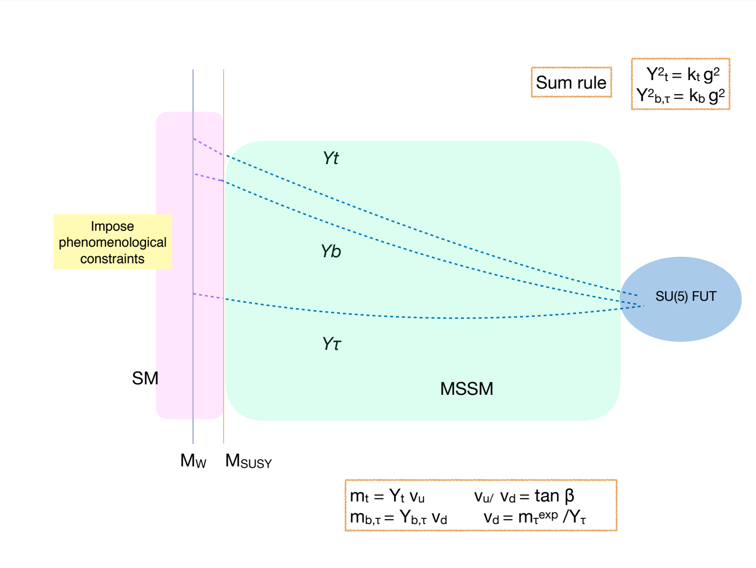
Model A

- $g_t^2 = \frac{8}{5}\, g^2$
- $g_{b,\tau}^2 = \frac{6}{5}\, g^2$
- $m_{H_u}^2 + 2m_{\mathbf{10}}^2 = M^2$
- $m_{H_d}^2 + m_{\overline{\mathbf{5}}}^2 + m_{\mathbf{10}}^2 = M^2$
- **3 free parameters:** M, $m_{\overline{\mathbf{5}}}^2$ and $m_{\mathbf{10}}^2$

Model B

- $g_t^2 = \frac{4}{5}\, g^2$
- $g_{b,\tau}^2 = \frac{3}{5}\, g^2$
- $m_{H_u}^2 + 2m_{\mathbf{10}}^2 = M^2$
- $m_{H_d}^2 - 2m_{\mathbf{10}}^2 = -\frac{M^2}{3}$
- $m_{\overline{\mathbf{5}}}^2 + 3m_{\mathbf{10}}^2 = \frac{4M^2}{3}$
- **2 free parameters:** M, $m_{\overline{\mathbf{5}}}^2$

FUTs at work



Sum rule
$Y^2_t = k_t\, g^2$
$Y^2_{b,\tau} = k_b\, g^2$
Impose phenomenological constraints
Yt
Yb
Yτ
SU(5) FUT
SM
MSSM
M_W
M_{SUSY}
$m_t = Y_t\, v_u$
$v_u / v_d = \tan\beta$
$m_{b,\tau} = Y_{b,\tau}\, v_d$
$v_d = m_\tau^{exp} / Y_\tau$

Interplay with phenomenology

The gauge symmetry is broken below M_{GUT} $\Rightarrow$
Boundary conditions of the form $C_i = \kappa_i g$, $h = -MC$ and the sum rule at M_{GUT}

$\Rightarrow$ MSSM.

- Fix the value of $m_\tau \Rightarrow \tan\beta \Rightarrow M_{top}$ and m_{bot}
- Assume a unique susy breaking scale
- The LSP is neutral
- The solutions should be compatible with radiative electroweak breaking
- No fast proton decay

We also

- Allow 5% variation of the Yukawa couplings at GUT scale due to threshold corrections
- Include radiative corrections to bottom and tau, plus resummation **(very important!)**
- Estimate theoretical uncertainties

Tob, Bottom, and Higgs mass: Predictions

Predictions:

- **FUTB:** $M_{top} \sim 172 \sim 174\ GeV$
  Theoretical uncertainties $\sim 4\%$
- large $\tan\beta$
- $M_H =\sim 121 - 126$ GeV
- LSP neutral

- Radiative eW symmetry breaking
- Δb and $\Delta\tau$ included resummation done.
  Depend mainly on $\tan\beta$ and unified gaugino mass M.
- LSP as CDM very constrained

Now constraints

Facts of life:

- Right masses for top and bottom
- Higgs mass also in experimental range
- B physics observables

$\mathrm{BR}(b \to s\gamma) SM/MSSM : |BRbsg - 1.089| < 0.27$

$\mathrm{BR}(B_u \to \tau\nu) SM/MSSM : |BRbtn - 1.39| < 0.69$

$\Delta M_{B_s} SM/MSSM : 0.97 \pm 20$

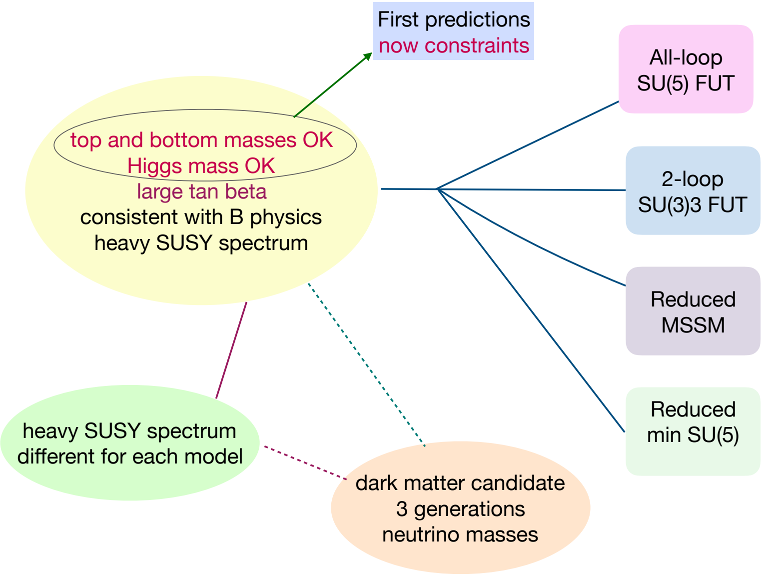
$\mathrm{BR}(B_s \to \mu^+\mu^-) = (2.9 \pm 1.4) \times 10^{-9}$

Results:
Heavy s-spectrum

Explore possibilities of detection
$\Rightarrow$ s-spectrum challenging even for FCC

Heinemeyer, Kalinowski, Kotlarski, MM, Patellis, Tracas, Zoupanos; Heinemeyer, MM, Tracas Zoupanos, Phys.Rept. (2021)

GYU from reduction of couplings at work



FUTs

- Finiteness provides us with an UV completion of our QFT
- Boundary conditions for RGE of the MSSM
- RGI takes the flow in the right direction for the third generation and Higgs masses
  Taking into account experimental constraints
  $\Rightarrow$ susy spectrum high
- Experimentally challenging

Heinemeyer, Kalinowski, Kotlarski, MM, Patellis, Tracas, Zoupanos, (2021)

- Are there other finite models?

  Different finite and reduced (non-finite) models analyzed: minimal and finite $SU(5)$, $SU(3)^3$, $SO(10)$...
- Can it give us insight into the flavour structure?
- Can we have successful reduction of couplings in a SM-like theory?

$SU(5)$ models with three generations

Models with 3 generations?

- First obvious step: include all generations
- Not easy, 2 ways:
  Rotate to MSSM
  Keep all Higgses
- First very simple approach: get diagonal solution for quark masses, no SUSY breaking

- Rotation of Higgs sector $\Rightarrow$ impacts proton decay and doublet-triplet splitting
- Then include off-diagonal terms $\Rightarrow$
  again need discrete symmetries, but possible to get interesting "textures"

$m_u\,(M_Z)$	$m_c\,(M_Z)$	$m_t(M_Z)$	$m_d\,(M_Z)$	$m_s\,(M_Z)$	$m_b\,(M_Z)$	$m_\tau\,(M_Z)$	$\tan\beta$	$\chi^2_{r_{min}}$
$0.0012GeV$	$0.626GeV$	$171.8GeV$	$0.00278GeV$	$0.0595GeV$	$2.86GeV$	$1.74623GeV$	57.4	0.152

Estimation: heavy triplets $\simeq$ 1.25 GUT scale, possible to avoid proton decay.

L.O. Estrada-Ramos, MM, G. Patellis, G. Zoupanos, arXiv:2406.17702

General form of $SU(5)$ FUT matrices

The general form of the $SU(5)$ FUT up and down quark mass matrices, before the rotation to the MSSM:

$$M_u = \begin{pmatrix} g_{11a}\langle\mathcal{H}_a^5\rangle & g_{12a}\langle\mathcal{H}_a^5\rangle & g_{13a}\langle\mathcal{H}_a^5\rangle \\ g_{21a}\langle\mathcal{H}_a^5\rangle & g_{22a}\langle\mathcal{H}_a^5\rangle & g_{23a}\langle\mathcal{H}_a^5\rangle \\ g_{31a}\langle\mathcal{H}_a^5\rangle & g_{32a}\langle\mathcal{H}_a^5\rangle & g_{33a}\langle\mathcal{H}_a^5\rangle \end{pmatrix}$$

$$M_d = \begin{pmatrix} \bar{g}_{11a}\langle\bar{\mathcal{H}}_{a5}\rangle & \bar{g}_{12a}\langle\bar{\mathcal{H}}_{a5}\rangle & \bar{g}_{13a}\langle\bar{\mathcal{H}}_{a5}\rangle \\ \bar{g}_{21a}\langle\bar{\mathcal{H}}_{a5}\rangle & \bar{g}_{22a}\langle\bar{\mathcal{H}}_{a5}\rangle & \bar{g}_{23a}\langle\bar{\mathcal{H}}_{a5}\rangle \\ \bar{g}_{31a}\langle\bar{\mathcal{H}}_{a5}\rangle & \bar{g}_{32a}\langle\bar{\mathcal{H}}_{a5}\rangle & \bar{g}_{33a}\langle\bar{\mathcal{H}}_{a5}\rangle \end{pmatrix}$$

- FUT conditions lead to coupled system of equations among Yukawa couplings
- Parametric solutions $\Rightarrow$ two-loop finite solutions
- Unique solutions $\Rightarrow$ all-loop finite solutions

- Previously A_4 and Q_6 explored

  Babu, Enkhbat, Gogoladze (2003); Babu, Kobayashi, Kubo (2003) ; E. Jiménez-Ramos, MM (2014)

- Let's try the smallest non-Abelian group S_3.
  Accomodates very well quark mass matrices at low energies:

  1st and 2nd generation and 2 pairs Higgs fields in **2**, 3rd generation in $\mathbf{1_S}$, 1 pair of Higgs fields in $\mathbf{1_S}$, 1 pair of Higgs fields in $\mathbf{1_A}$,

In our FUT case

$$M_u = \begin{pmatrix} g_{113}\langle\mathcal{H}_3^5\rangle & 0 & g_{131}\langle\mathcal{H}_1^5\rangle \\ 0 & g_{113}\langle\mathcal{H}_3^5\rangle & g_{131}\langle\mathcal{H}_2^5\rangle \\ g_{131}\langle\mathcal{H}_1^5\rangle & g_{131}\langle\mathcal{H}_2^5\rangle & 0 \end{pmatrix}, \quad (1)$$

$$M_d = \begin{pmatrix} \bar{g}_{113}\langle\bar{\mathcal{H}}_{35}\rangle & 0 & \bar{g}_{131}\langle\bar{\mathcal{H}}_{15}\rangle \\ 0 & \bar{g}_{113}\langle\bar{\mathcal{H}}_{35}\rangle & \bar{g}_{131}\langle\bar{\mathcal{H}}_{25}\rangle \\ \bar{g}_{311}\langle\bar{\mathcal{H}}_{15}\rangle & \bar{g}_{311}\langle\bar{\mathcal{H}}_{25}\rangle & 0 \end{pmatrix}. \quad (2)$$

Too restrictive... leads to two of the masses almost degenerate

Classification of $SU(5)$ FUTs with vanishing off-diagonal γ done already

Fermions coupled to 3 or 4 pairs of Higgs boson fields

$$V_3^{(1)} = \begin{pmatrix} g_{111}\langle\mathcal{H}_1^5\rangle & g_{123}\langle\mathcal{H}_3^5\rangle & g_{132}\langle\mathcal{H}_2^5\rangle \\ g_{213}\langle\mathcal{H}_3^5\rangle & g_{222}\langle\mathcal{H}_2^5\rangle & g_{231}\langle\mathcal{H}_1^5\rangle \\ g_{312}\langle\mathcal{H}_2^5\rangle & g_{321}\langle\mathcal{H}_1^5\rangle & g_{333}\langle\mathcal{H}_3^5\rangle \end{pmatrix}, \quad V_3^{(2)} = \begin{pmatrix} g_{112}\langle\mathcal{H}_2^5\rangle & g_{121}\langle\mathcal{H}_1^5\rangle & 0 \\ g_{211}\langle\mathcal{H}_1^5\rangle & g_{223}\langle\mathcal{H}_3^5\rangle & g_{232}\langle\mathcal{H}_2^5\rangle \\ 0 & g_{322}\langle\mathcal{H}_2^5\rangle & g_{333}\langle\mathcal{H}_3^5\rangle \end{pmatrix},$$

$$V_3^{(3)} = \begin{pmatrix} g_{113}\langle\mathcal{H}_3^5\rangle & g_{121}\langle\mathcal{H}_1^5\rangle & 0 \\ g_{211}\langle\mathcal{H}_1^5\rangle & g_{223}\langle\mathcal{H}_3^5\rangle & g_{232}\langle\mathcal{H}_2^5\rangle \\ 0 & g_{322}\langle\mathcal{H}_2^5\rangle & g_{333}\langle\mathcal{H}_3^5\rangle \end{pmatrix}, \quad V_3^{(4)} = \begin{pmatrix} g_{111}\langle\mathcal{H}_1^5\rangle & 0 & 0 \\ 0 & g_{223}\langle\mathcal{H}_3^5\rangle & g_{232}\langle\mathcal{H}_2^5\rangle \\ 0 & g_{322}\langle\mathcal{H}_2^5\rangle & g_{333}\langle\mathcal{H}_3^5\rangle \end{pmatrix},$$

$$V_4^{(1)} = \begin{pmatrix} g_{111}\langle\mathcal{H}_1^5\rangle & g_{124}\langle\mathcal{H}_4^5\rangle & g_{132}\langle\mathcal{H}_2^5\rangle \\ g_{214}\langle\mathcal{H}_4^5\rangle & g_{222}\langle\mathcal{H}_2^5\rangle & g_{231}\langle\mathcal{H}_1^5\rangle \\ g_{312}\langle\mathcal{H}_2^5\rangle & g_{321}\langle\mathcal{H}_1^5\rangle & g_{333}\langle\mathcal{H}_3^5\rangle \end{pmatrix}, \quad V_4^{(2)} = \begin{pmatrix} g_{112}\langle\mathcal{H}_2^5\rangle & g_{121}\langle\mathcal{H}_1^5\rangle & 0 \\ g_{211}\langle\mathcal{H}_1^5\rangle & g_{222}\langle\mathcal{H}_2^5\rangle & g_{234}\langle\mathcal{H}_4^5\rangle \\ 0 & g_{324}\langle\mathcal{H}_4^5\rangle & g_{333}\langle\mathcal{H}_3^5\rangle \end{pmatrix},$$

$$V_4^{(3)} = \begin{pmatrix} g_{113}\langle\mathcal{H}_3^5\rangle & g_{121}\langle\mathcal{H}_1^5\rangle & g_{132}\langle\mathcal{H}_2^5\rangle \\ g_{211}\langle\mathcal{H}_1^5\rangle & g_{222}\langle\mathcal{H}_2^5\rangle & g_{234}\langle\mathcal{H}_4^5\rangle \\ g_{312}\langle\mathcal{H}_2^5\rangle & g_{324}\langle\mathcal{H}_4^5\rangle & g_{333}\langle\mathcal{H}_3^5\rangle \end{pmatrix}, \quad V_4^{(4)} = \begin{pmatrix} g_{113}\langle\mathcal{H}_3^5\rangle & g_{121}\langle\mathcal{H}_1^5\rangle & g_{132}\langle\mathcal{H}_2^5\rangle \\ g_{211}\langle\mathcal{H}_1^5\rangle & g_{223}\langle\mathcal{H}_3^5\rangle & g_{234}\langle\mathcal{H}_4^5\rangle \\ g_{312}\langle\mathcal{H}_2^5\rangle & g_{324}\langle\mathcal{H}_4^5\rangle & g_{333}\langle\mathcal{H}_3^5\rangle \end{pmatrix}$$

Matrices obtained exchanging the Higgs indices fall under this classification.

Babu, Enkhbat, Gogoladze (2003)

Parametric solutions: 2-loop finiteness

- We looked only for solutions where M_u and M_d have the same texture, other solutions are possible
- Most solutions found are parametric, i.e. not isolated and non-degenerate
- This implies only 2-loop finiteness

  $\Rightarrow$ some Yukawa couplings are determined exactly, some others within a range of values

  More freedom in these models to find viable mass textures
- Taking the limiting values makes some Yukawa couplings zero $\Rightarrow$ symmetry

Example: 2-loop FUT

$V_4^{(1)}$ for both mass matrices

Z_n	$\bar\Psi_1$	$\bar\Psi_2$	$\bar\Psi_3$	X_1	X_2	X_3	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$	$\bar{\mathcal{H}}_1$	$\bar{\mathcal{H}}_2$	$\bar{\mathcal{H}}_3$	$\bar{\mathcal{H}}_4$	Σ
Z_2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
Z_8	4	3	5	0	7	1	0	2	6	1	4	6	2	5	0

The following parametric solutions are found for this model:

$$|g_{124}|^2 = |g_{214}|^2 = \frac{4}{5} g_5^2 \ , \ \ |g_{222}|^2 = \frac{2}{5} g_5^2 \ , \ \ |g_{231}|^2 = |g_{321}|^2 = \boxed{\frac{1}{10}\left(8 g_5^2 - 5\,|g_{111}|^2\right)} \ ,$$

$$|g_{333}|^2 = \frac{6}{5} g_5^2 \ , \ \ |\bar g_{111}|^2 = |\bar g_{124}|^2 = \boxed{\frac{3}{20}\left(8 g_5^2 - 5\,|g_{111}|^2\right)} \ ,$$

$$|\bar g_{214}|^2 = \frac{3}{4}\,|g_{111}|^2 \ , \ \ |\bar g_{222}|^2 = |\bar g_{231}|^2 = \frac{3}{10} g_5^2 \ , \ \ |\bar g_{321}|^2 = \boxed{-\frac{3}{20}\left(2 g_5^2 - 5\,|g_{111}|^2\right)} \ ,$$

$$|\bar g_{333}|^2 = \frac{9}{10} g_5^2 \ , \ \ |f_{22}|^2 = \frac{3}{4} g_5^2 \ , \ \ |f_{33}|^2 = \frac{g_5^2}{4} \ , \ \ |p|^2 = \frac{15}{7} g_5^2 \ ,$$

$$|g_{132}|^2 = |g_{312}|^2 = |\bar g_{132}|^2 = |\bar g_{312}|^2 = |f_{11}|^2 = |f_{44}|^2 = 0 \ .$$

Positivity conditions lead to

$$\boxed{\frac{2}{5} g_5^2 \leq |g_{111}|^2 \leq \frac{8}{5} g_5^2 \ .}$$

All-loop FUT

$V_4^{(1)}$ for both mass matrices similar to Babu et al model, with different symmetries, and with phases

Z_n	$\bar{\Psi}_1$	$\bar{\Psi}_2$	$\bar{\Psi}_3$	X_1	X_2	X_3	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$	$\bar{\mathcal{H}}_1$	$\bar{\mathcal{H}}_2$	$\bar{\mathcal{H}}_3$	$\bar{\mathcal{H}}_4$	Σ
Z_2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
Z_3	0	2	0	0	2	0	1	1	0	0	1	1	0	0	0
Z_4	3	3	2	3	3	2	2	3	0	2	2	3	0	2	0

$$|g_{114}|^2 = |g_{121}|^2 = |g_{211}|^2 = |g_{232}|^2 = |g_{322}|^2 = |g_{333}|^2 = \frac{4}{5} g_5^2 \quad ,$$

$$|\bar{g}_{114}|^2 = |\bar{g}_{121}|^2 = |\bar{g}_{211}|^2 = |\bar{g}_{232}|^2 = |\bar{g}_{322}|^2 = |\bar{g}_{333}|^2 = \frac{3}{5} g_5^2 \quad ,$$

$$|f_{33}|^2 = |f_{44}|^2 = \frac{1}{2} g_5^2 \quad , \quad |p|^2 = \frac{15}{7} g_5^2 \quad .$$

Since these solutions are unique, isolated and non-degenerate, the model is all-loop finite.
The sum rules are:

$$m^2_{\tilde{\psi}_1} = m^2_{\tilde{\psi}_3} = \frac{1}{6}\left(-MM^\dagger + 9m^2_{H_3}\right) \quad , \quad m^2_{\tilde{\psi}_2} = \boxed{\frac{1}{6}\left(-MM^\dagger - 6m^2_{H_1} + 15m^2_{H_3}\right)} ,$$

$$m^2_{\tilde{\chi}_1} = m^2_{\tilde{\chi}_3} = \frac{1}{2}\left(MM^\dagger - m^2_{H_3}\right) \quad , \quad m^2_{\tilde{\chi}_2} = \frac{1}{2}\left(MM^\dagger - 2m^2_{H_1} + m^2_{H_3}\right) ,$$

$$m^2_{H_1} = m^2_{H_2} = \frac{1}{3}\left(2MM^\dagger + 3m^2_{H_1} - 6m^2_{H_3}\right) \quad , \quad m^2_{H_3} = m^2_{H_4} = \frac{1}{3}\left(2MM^\dagger - 3m^2_{H_3}\right) ,$$

$$m^2_{H_2} = m^2_{H_1} \quad ; \quad m^2_{H_4} = m^2_{H_3} \quad , \quad m^2_{\phi_\Sigma} = \frac{1}{3}MM^\dagger \quad .$$

All-loop mass matrices

It is possible to determine the minimum amount of phases and their positions

Kusenko, Shrock (1994)

The mass matrices for this model are:

$$M_u = \begin{pmatrix} g_{114}\langle\mathcal{H}_4^5\rangle & g_{121}\langle\mathcal{H}_1^5\rangle & 0 \\ g_{211}\langle\mathcal{H}_1^5\rangle & 0 & g_{232}\langle\mathcal{H}_2^5\rangle \\ 0 & g_{322}\langle\mathcal{H}_2^5\rangle & g_{333}\langle\mathcal{H}_3^5\rangle \end{pmatrix} = \frac{2}{\sqrt{5}} g_5 \begin{pmatrix} \langle\mathcal{H}_4^5\rangle & \langle\mathcal{H}_1^5\rangle & 0 \\ \langle\mathcal{H}_1^5\rangle & 0 & \langle\mathcal{H}_2^5\rangle \\ 0 & \langle\mathcal{H}_2^5\rangle & e^{i\phi_3}\langle\mathcal{H}_3^5\rangle \end{pmatrix},$$

$$M_d = \begin{pmatrix} \bar{g}_{114}\langle\bar{\mathcal{H}}_{45}\rangle & \bar{g}_{121}\langle\bar{\mathcal{H}}_{15}\rangle & 0 \\ \bar{g}_{211}\langle\bar{\mathcal{H}}_{15}\rangle & 0 & \bar{g}_{232}\langle\bar{\mathcal{H}}_{25}\rangle \\ 0 & \bar{g}_{322}\langle\bar{\mathcal{H}}_{25}\rangle & \bar{g}_{333}\langle\bar{\mathcal{H}}_{35}\rangle \end{pmatrix} = \sqrt{\frac{3}{5}} g_5 \begin{pmatrix} \langle\bar{\mathcal{H}}_{45}\rangle & \langle\bar{\mathcal{H}}_{15}\rangle & 0 \\ e^{i\bar{\phi}_1}\langle\bar{\mathcal{H}}_{15}\rangle & 0 & \langle\bar{\mathcal{H}}_{25}\rangle \\ 0 & e^{i\bar{\phi}_2}\langle\bar{\mathcal{H}}_{25}\rangle & e^{i\bar{\phi}_3}\langle\bar{\mathcal{H}}_{35}\rangle \end{pmatrix}.$$

After the rotation in the Higgs sector, the matrices in the MSSM basis are:

$$M_u = \frac{2}{\sqrt{5}} g_5 \begin{pmatrix} \widetilde{\alpha}_4 & \widetilde{\alpha}_1 & 0 \\ \widetilde{\alpha}_1 & 0 & \widetilde{\alpha}_2 \\ 0 & \widetilde{\alpha}_2 & e^{i\phi_3}\widetilde{\alpha}_3 \end{pmatrix} \langle\mathcal{K}_3^5\rangle,$$

$$M_d = \sqrt{\frac{3}{5}} g_5 \begin{pmatrix} \widetilde{\beta}_4 & \widetilde{\beta}_1 & 0 \\ e^{i\bar{\phi}_1}\widetilde{\beta}_1 & 0 & \widetilde{\beta}_2 \\ 0 & e^{i\bar{\phi}_2}\widetilde{\beta}_2 & e^{i\bar{\phi}_3}\widetilde{\beta}_3 \end{pmatrix} \langle\bar{\mathcal{K}}_{35}\rangle,$$

where $\widetilde{\alpha}_i$ and $\widetilde{\beta}_i$ refer to the rotation angles in the up and down sector, respectively.

Phenomenological prospects?

- Possible to have mass matrices with "good" textures, still have to run RGEs to M_Z
- SUSY radiative corrections can be sizeable, especially for large $\tan\beta$
  - How will they affect the rest of the entries?
  - Viable/unviable textures might change at low energies after SUSY breaking and RGE running
- In all-loop 3 gen model (3,3) entries in mass matrices coincide with FUTB model:
  - accurate predictions for top and bottom quark masses, Higgs mass
  - large $\tan\beta$
  - heavy s-spectrum
- Unknowns mainly from Higgs sector and phases...

Parameters

- Before finiteness solutions: 89 free parameters in total $\Rightarrow$
  Yukawa couplings, soft breaking terms, phases, vev's of the Higgs fields
- After finiteness solutions: 33 free parameters
- At GUT scale:
  doublet-triplet splitting, rotation to MSSM basis with constraints over squared sum of angles, rephasing invariants
- At electroweak scale:
  radiative electroweak breaking, m_τ^{exp} and SM vev fix $\tan\beta$
- 12 parameters left:
  The soft breaking terms, the phases, and the rotation angles:

$$\boxed{\widetilde{\alpha}_1,\ \widetilde{\alpha}_2,\ \widetilde{\alpha}_3,\ \widetilde{\beta}_1,\ \widetilde{\beta}_2,\ \widetilde{\beta}_3,\ \phi_3,\ \bar{\phi}_1,\ \bar{\phi}_2,\ \bar{\phi}_3,\ M,\ \mu\ .}$$

  Only 1 combination of the phases will be observable $\Rightarrow$
- 9 free parameters left, very constrained

Open questions

- For quarks: complete running of the RGE's needed
  large $\tan\beta$, constrained soft breaking terms $\Rightarrow$ results??

  Cakir, Solmaz (2008); Xing (2022)

What about neutrinos and charged leptons?

- Neutrino masses might be added by $\not{R}$ for $SU(5)$ well known for many years...
  finite $SU(3)^3$ includes them
- Soft SUSY breaking sector crucial also for $m_s \neq m_\mu$ and $m_d \neq m_e$
  $\not{R}$ could also help here e.g. Bajc, di Luzio (2022)
- Proton decay tight... more easily suppressed in a type of split-SUSY scenario e.g. Hisano (2022)
  or more fine tuning??

Conclusions

- Reduction of couplings: powerful principle implies Gauge Yukawa Unification $\Rightarrow$ predictive models
- Possible SSB terms $\Rightarrow$ satisfy a sum rule among soft scalars
- Finiteness $\Rightarrow$ reduces greatly the number of free parameters
  - completely finite theories $SU(5)$
  - 2-loop finite theories $SU(3)^3$
  - Successful prediction for top quark and Higgs boson mass
  - Large tan β
  - Satisfy BPO constraints (not trivial)
  - Heavy SUSY spectrum, even for FCC

- 3 generations models:

  2-loops: Yukawa couplings determined within a range

  all-loop: Yukawa couplings completely determined
- Can lead to viable mass textures
- Drastic reduction in the number of free parameters
- Free parameters come mainly from Higgs and SSB sectors, and phases
- Flavour in FUTs $\Rightarrow$ more fundamental theory?

Outlook

Some open questions and future work in reduction of couplings

- Are there more finite and reduced models? Yes...
- Do all fermions acquire masses the same way? ??
- Is it possible to include three generations in a reduced or finite model? Yes...
- How to incorporate flavour? possible, points towards symmetries
  $\Rightarrow$ What will be the impact at low energies?
- How to include neutrino masses? perhaps $\not{R}$ for $SU(5)$, natural for $SU(3)^3$
- Is it indispensible to have SUSY for successful reduced theories?
  Yes for finite theories, but non-SUSY multi-Higgs are possible
- How to make better use symmetries $\Leftrightarrow$ reduction of couplings? ?