Splitting functions

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Workshop *LggF: aN3LO PDFs for Run3 & YR5*, CERN, June 26, 2024

Work in collaboration with:

- *Four-loop splitting functions in QCD – The quark-to-gluon case –*
  G. Falcioni, F. Herzog, S. M., A. Pelloni and A. Vogt arXiv:2404.09701
- *Additional moments and x-space approximations of four-loop splitting functions in QCD*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt arXiv:2310.05744
- *The double fermionic contribution to the four-loop quark-to-gluon splitting function*
  G. Falcioni, F. Herzog, S. M., J. Vermaseren and A. Vogt arXiv:2310.01245
- *Four-loop splitting functions in QCD – The gluon-to-quark case –*
  G. Falcioni, F. Herzog, S. M., and A. Vogt arXiv:2307.04158
- *Four-loop splitting functions in QCD – The quark-quark case –*
  F. Herzog, G. Falcioni, S. M., and A. Vogt arXiv:2302.07593
- *Low moments of the four-loop splitting functions in QCD*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt arXiv:2111.15561
- *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt arXiv:1805.09638
- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt arXiv:1707.08315
- FHMRUVV + P collaboration $\rightarrow$ FHMPRUVV 2017 – ...

Parton evolution

- Evolution equations for parton distributions
  - non-singlet valence PDFs $q_{\rm ns}^{\rm v} = \sum_f (q_f - \bar{q}_f)$
  - flavor asymmetries $q_{{\rm ns},ff'}^{\pm} = (q_f \pm \bar{q}_f) - (q_{f'} \pm \bar{q}_{f'})$

$$\frac{d}{d\ln\mu^2} q_{\rm ns}^{\pm,{\rm v}} \quad = \quad P_{\rm ns}^{\pm,{\rm v}} \otimes q_{\rm ns}^{\pm,{\rm v}}$$

  - quark-flavor singlet PDFs $q_{\rm s} = \sum_f (q_f + \bar{q}_f)$ and gluon PDF g
  - 2x2 matrix equation

$$\frac{d}{d\ln\mu^2} \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} \quad = \quad \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix}$$

- Splitting functions P up to $\rm N^3LO$ (work in progress)

$$P_{\rm ij} \quad = \quad \underbrace{\alpha_s\, P_{\rm ij}^{(0)} + \alpha_s^2\, P_{\rm ij}^{(1)} + \alpha_s^3\, P_{\rm ij}^{(2)}}_{\text{NNLO: standard approximation}} + \alpha_s^4\, P_{\rm ij}^{(3)} + \ldots$$

- Anomalous dimensions (Mellin transform)

$$\gamma_{\rm ij}(N) = -\int_0^1 dx\, x^N\, P_{\rm ij}(x) = \alpha_s\, \gamma_{\rm ij}^{(0)} + \alpha_s^2\, \gamma_{\rm ij}^{(1)} + \alpha_s^3\, \gamma_{\rm ij}^{(2)} + \alpha_s^4\, \gamma_{\rm ij}^{(3)} + \ldots$$

Quark singlet splitting function $P_{\rm qq} = P_{\rm ns}^{+} + P_{\rm ps}$

$$\begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix}$$

Moments of pure-singlet splitting function

- Moments $N = 2, \ldots 20$ for pure-singlet anomalous dimension $\gamma_{\rm ps}^{(3)}(N)$

$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}(N\!=\!2) &= -691.5937093\, n_f + 84.77398149\, n_f^2 + 4.466956849\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N\!=\!4) &= -109.3302335\, n_f + 8.776885259\, n_f^2 + 0.306077137\, n_f^3 \,, \\
&\ldots \\
\gamma_{\rm ps}^{(3)}(N\!=\!20) &= -0.442681568\, n_f + 0.805745333\, n_f^2 - 0.020918264\, n_f^3 \,.
\end{aligned}
$$

- Results $N \leq 8$ agree with inclusive DIS S.M., Ruijl, Ueda, Vermaseren, Vogt '21 (also for $N = 10$ and $N = 12$)
- Quartic color terms $d_R^{abcd} d_R^{abcd}$ agree with S.M., Ruijl, Ueda, Vermaseren, Vogt '18
- Large-n_f parts agree with all-N results Davies, Vogt, Ruijl, Ueda, Vermaseren '17;
- ζ_4 terms in $\gamma_{\rm ps}^{(3)}(N)$ agree with Davies, Vogt '17 based on no-π^2 theorem Jamin, Miravitllas '18; Baikov, Chetyrkin '18
- Checked by n_f^2 terms at all-N Gehrmann, von Manteuffel, Sotnikov, Yang '23

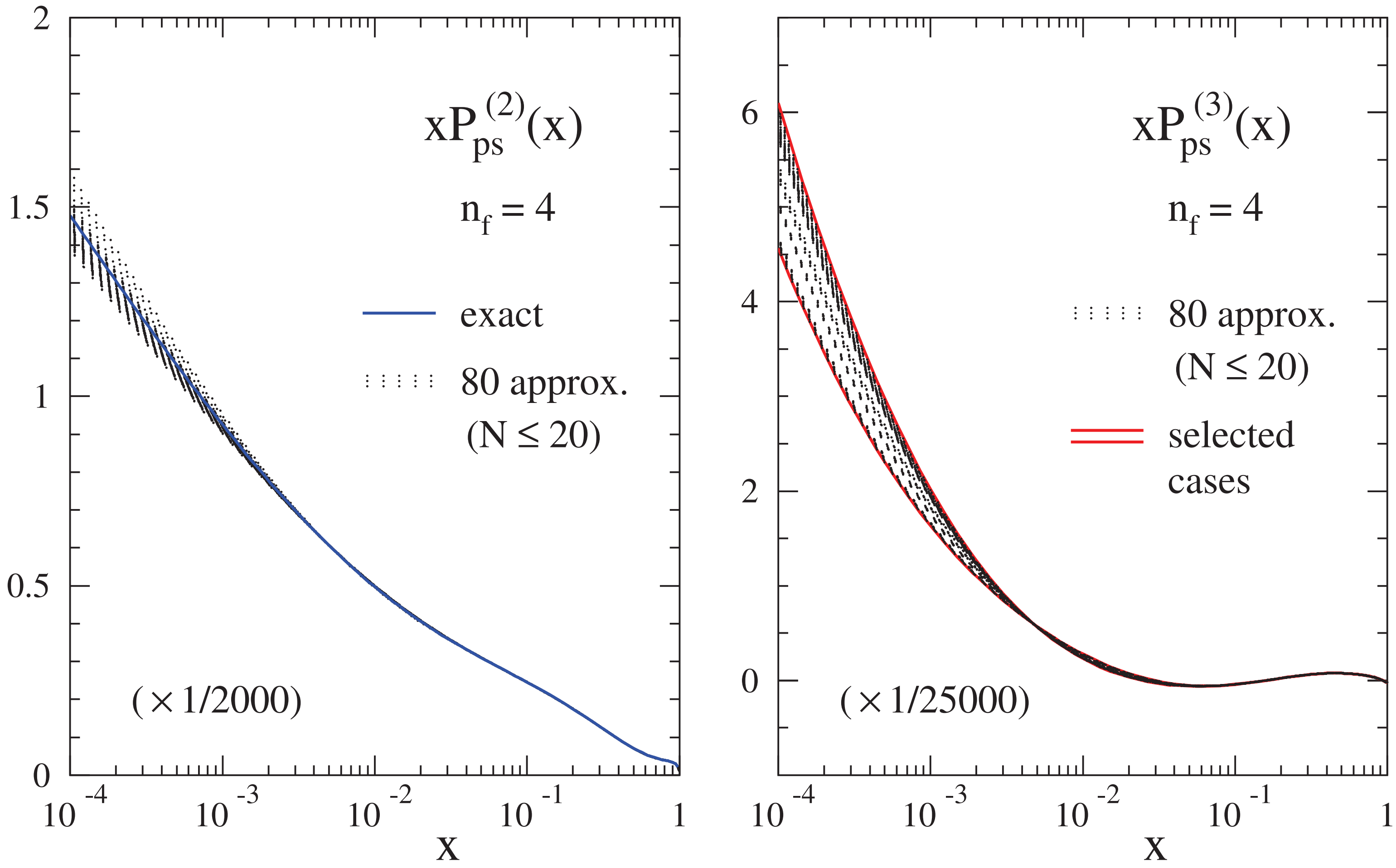
Outlook

- Higher moments $N = 22, \ldots$ to be published

Approximations in x-space

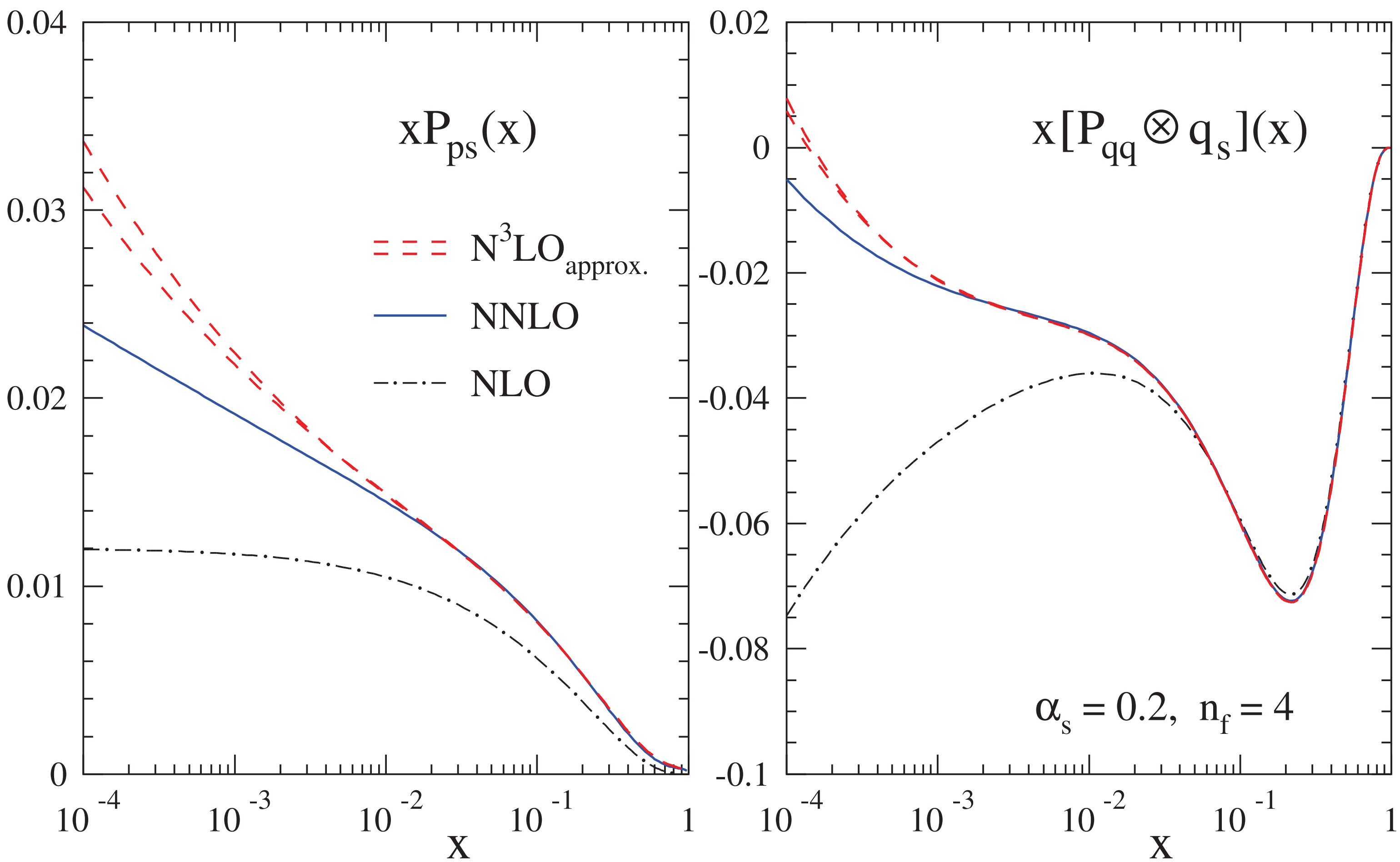
- Large- and small-x information about four-loop splitting function $P_{\rm ps}^{(3)}(x)$
  - leading logarithm $(\ln^2 x)/x$ — Catani, Hautmann '94
  - sub-dominant logarithms $\ln^k x$ with $k = 6, 5, 4$ — Davies, Kom, S.M., Vogt '22
  - leading large-x terms $(1-x)^j \ln^k(1-x)$ with $j \geq 1$ and $k \leq 4$ with $k = 4, 3$ known — Soar, S.M., Vermaseren, Vogt '09
- Approximation of four-loop splitting function $P_{\rm ps}^{(3)}(x)$ with suitable ansatz
  - unknown leading small-x terms: $(\ln x)/x$, $1/x$
  - unknown sub-dominant logarithms: $\ln^k x$ with $k = 3, 2, 1$
  - two remaining large-x terms $(1-x)\ln^k(1-x)$ with $k = 2, 1$
  - different two-parameter polynomials together one function (dilogarithm $\mathrm{Li}_2(x)$ or $\ln^k(1-x)$ with $k = 2, 1$, suppressed as $x \to 1$)
- Approximations for phenomenology with fixed $n_f = 3, 4, 5$
  - easy-to-use
  - no correlations between different n_f dependenct terms accounted for

Pure-singlet splitting function



- Approximations to pure-singlet splitting function $P_{\rm ps}^{(n)}(x)$ at $n_f = 4$ with 80 trial functions
  - left: three-loops ($n = 2$) with comparison to known result
  - right: three-loops ($n = 3$) with remaining uncertainty

Pure-singlet splitting function

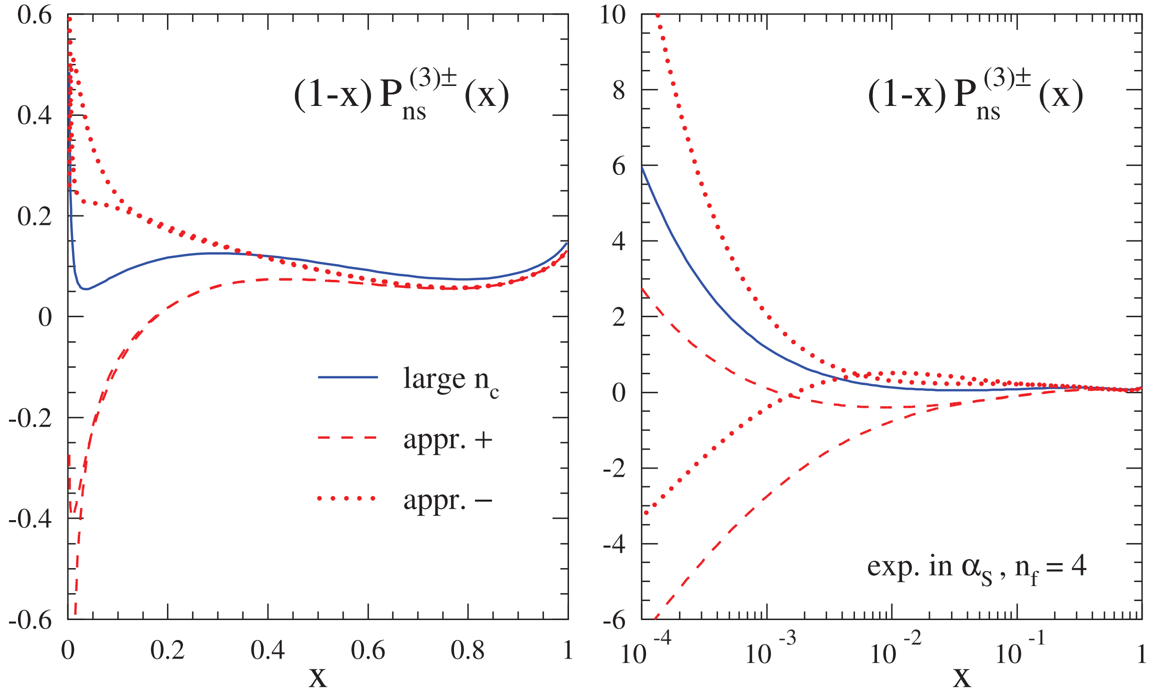


- Left: results for $P_{\rm ps}(x)$ up to N^{3}LO; $\alpha_s \;=\; 0.2$ fixed, $n_f \;=\; 4$
- Right: contribution to evolution kernel $d\ln q_{\rm s}/d\ln\mu_f^2$ up to N^{3}LO for typical quark-singlet shape

$$xq_{\,\rm s}(x,\mu_0^2) \;=\; 0.6\,x^{\,-0.3}(1-x)^{3.5}\,(1+5.0\,x^{\,0.8})$$

Four-loop non-singlet splitting functions

- Four-loop $P_{\rm ns}^{(3)\pm}(x)$ and uncertainty bands beyond large-n_c limit with $n_f = 4$



Analytic results

- contributions to non-singlet splitting functions
  - n_f-terms (n_f^3 Gracey '94; n_f^2 Davies, Vogt, Ruijl, Ueda, Vermaseren '16)
  - leading n_c terms S.M., Vogt, Ruijl, Ueda, Vermaseren '17
  - $n_f C_F^3$ terms Gehrmann, von Manteuffel, Sotnikov, Yang '23

Outlook

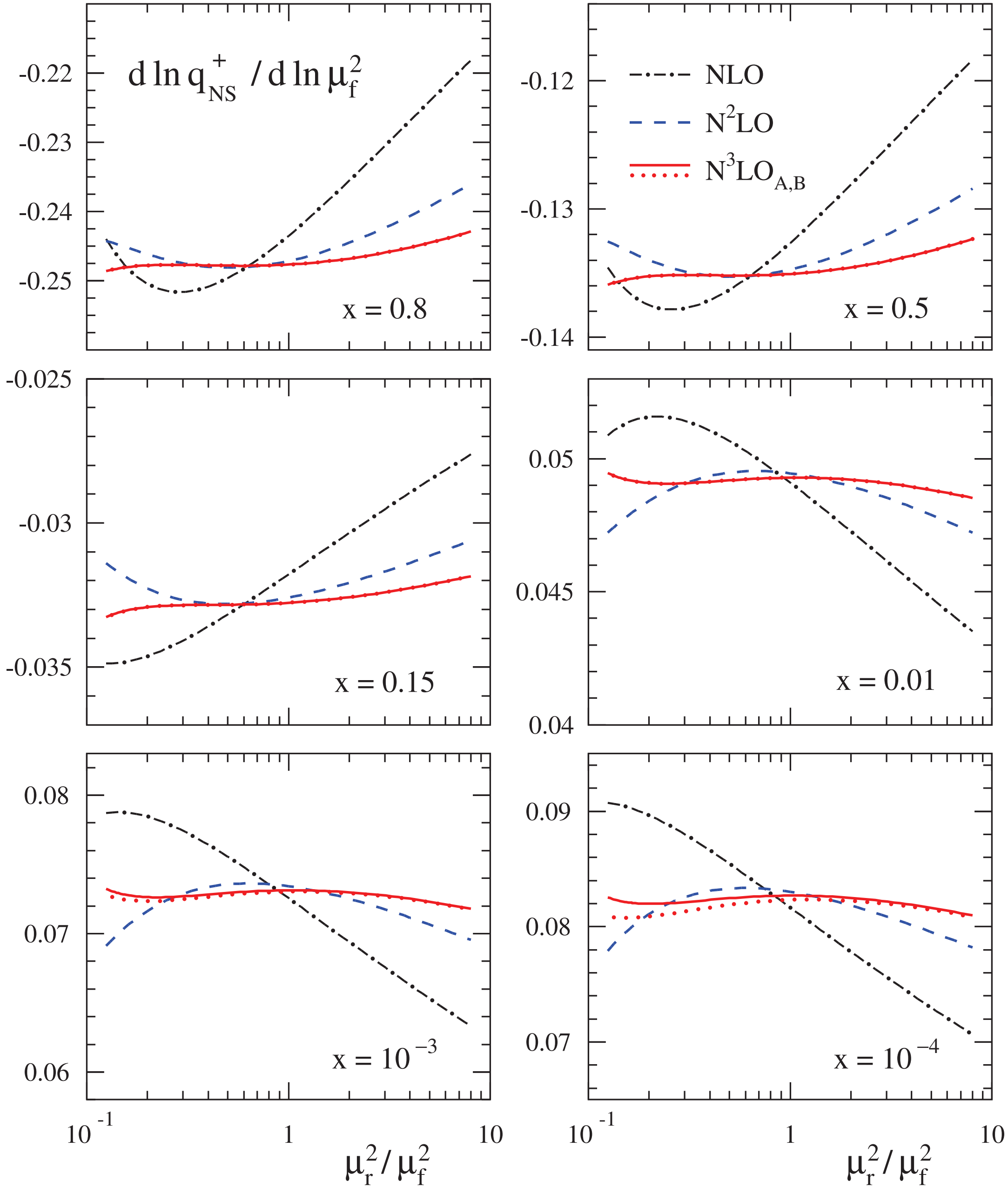
- $P^{(n)\pm}_{\rm ns,x\to 1} = A^{(n)}/(1-x)_+ + B^{(n)}\delta(1-x) + \ldots$ (known $B^{(4)}$ Das, S.M. Vogt '19)
- Higher moments $N = 21, 22, \ldots$ to be published
- Improved approximations to be done

Scale stability of evolution

- Renormalization scale dependence of evolution kernel $d \ln q_{\rm ns}^{+} / d \ln \mu_f^2$
  - non-singlet shape

$$x q_{\rm ns}^{+}(x, \mu_0^2) \; = \; x^{0.5} (1 - x)^3$$

- NLO, NNLO and N^{3}LO predictions
  - remaining uncertainty of four-loop splitting function $P_{\rm ns}^{(3)+}$ almost invisible



Gluon-to-quark splitting function P_{qg}

$$\begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix}$$

Moments of gluon-to-quark splitting function

- Moments $N = 2, \dots 20$ for gluon-to-quark anomalous dimension $\gamma_{\rm qg}^{(3)}(N)$

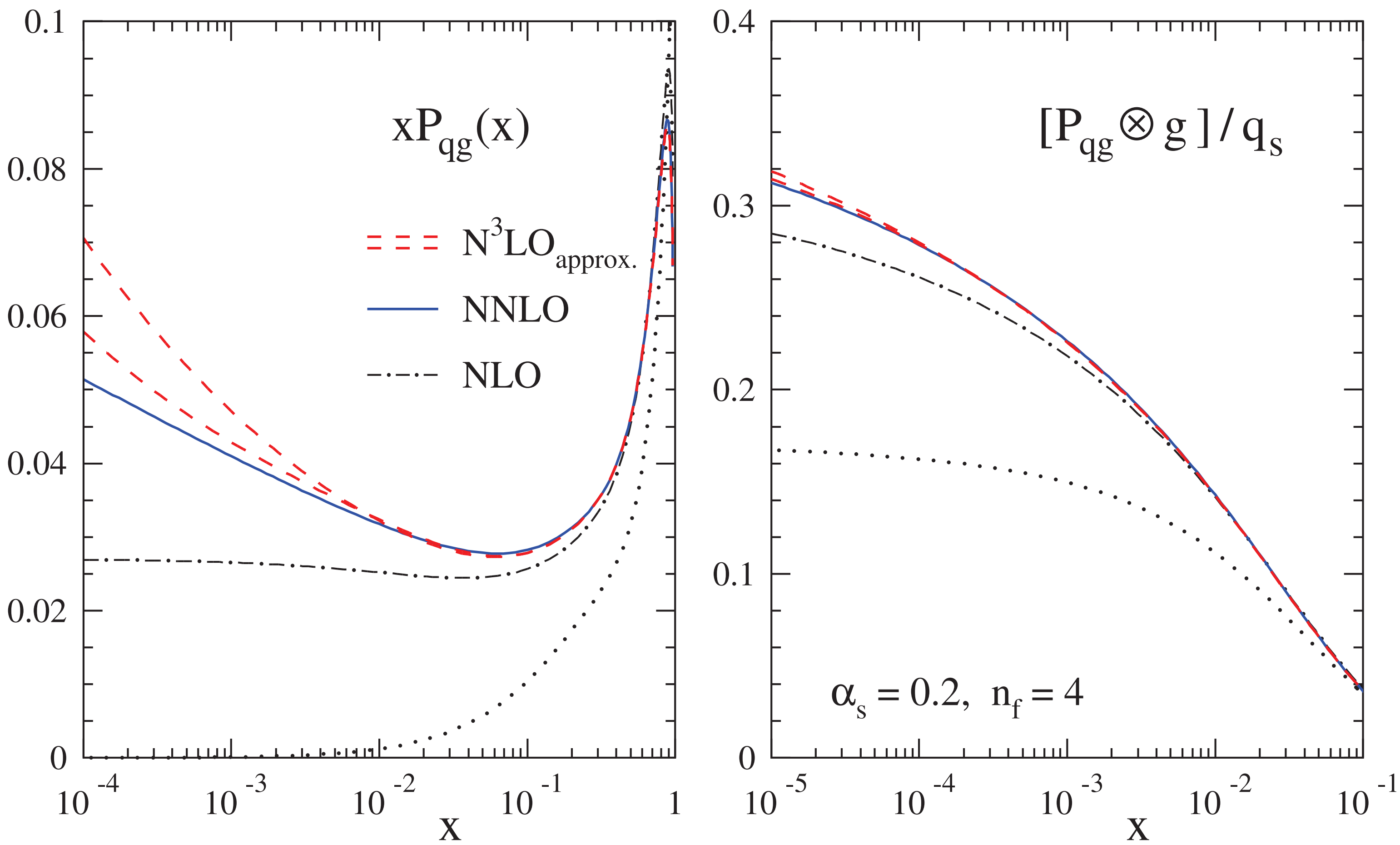
$$\begin{aligned}
\gamma_{\rm qg}^{(3)}(N=2) &= -654.4627782\, n_f + 245.6106197\, n_f^2 - 0.924990969\, n_f^3 \,, \\
\gamma_{\rm qg}^{(3)}(N=4) &= 290.3110686\, n_f - 76.51672403\, n_f^2 - 4.911625629\, n_f^3 \,, \\
\gamma_{\rm qg}^{(3)}(N=6) &= 335.8008046\, n_f - 124.5710225\, n_f^2 - 4.193871425\, n_f^3 \,, \\
\gamma_{\rm qg}^{(3)}(N=8) &= 294.5876830\, n_f - 135.3767647\, n_f^2 - 3.609775642\, n_f^3 \,, \\
&\dots \\
\gamma_{\rm qg}^{(3)}(N=20) &= 52.24329555\, n_f - 109.3424891\, n_f^2 - 2.153153725\, n_f^3 \,.
\end{aligned}$$

- Approximation of four-loop splitting function $P_{\rm qg}^{(3)}(x)$ again with known large- and small-x information and suitable ansatz

Outlook

- Higher moments $N = 22, \dots$ to be published

Gluon-to-quark splitting function



- Left: results for $P_{\rm qg}(x)$ up to N^{3}LO; $\alpha_s \;=\; 0.2$ fixed, $n_f \;=\; 4$
- Right: contribution to evolution kernel $d \ln q_{\rm s} / d \ln \mu_f^2$ up to N^{3}LO for typical gluon shape

$$xg(x, \mu_0^2) \;=\; 1.6\, x^{\,-0.3} (1-x)^{4.5} \;\left(1 - 0.6\, x^{\,0.3}\,\right)$$

Quark-to-gluon splitting function $P_{\rm gq}$

$$\begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix}$$

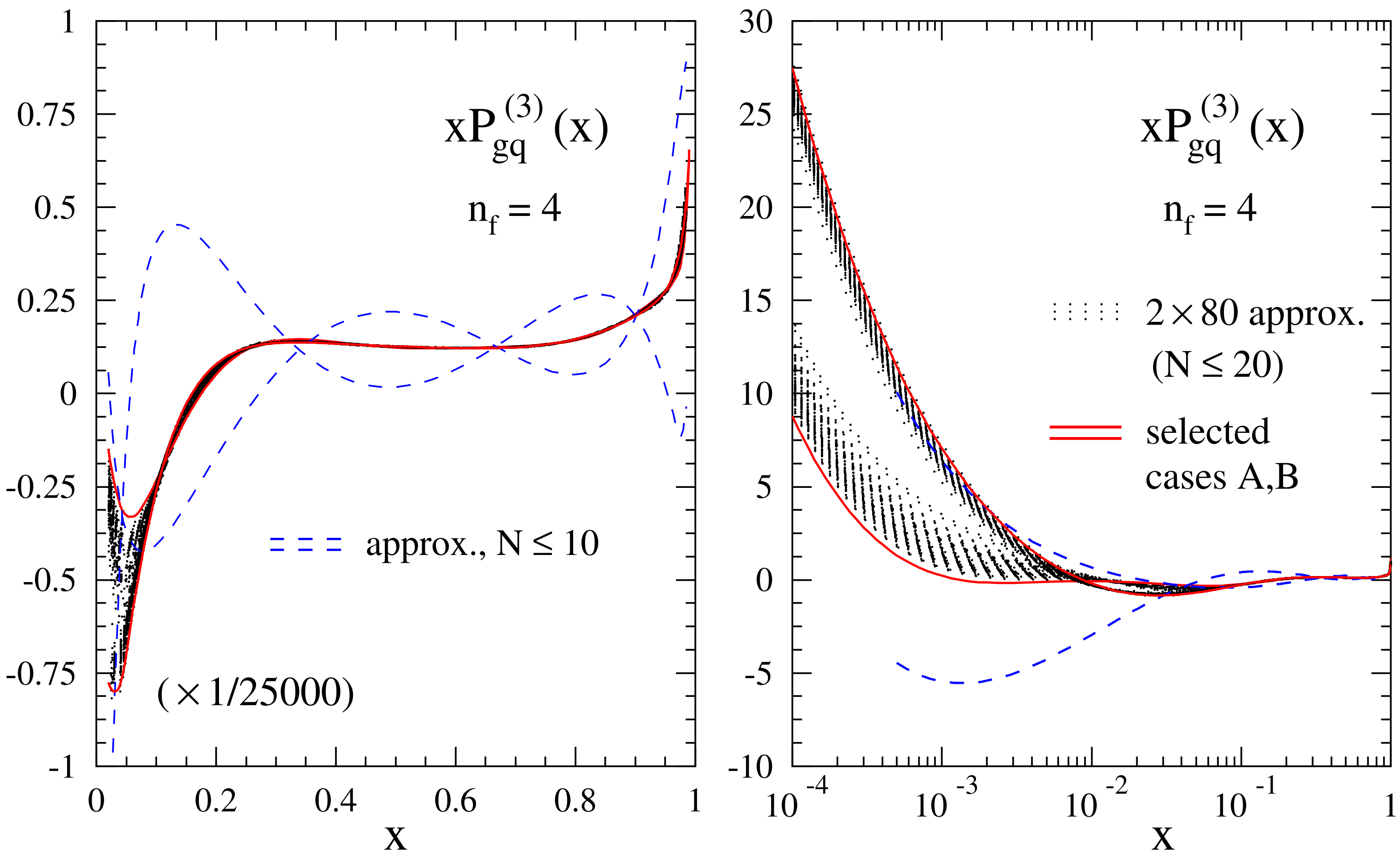
Moments of quark-to-gluon splitting function

- Moments for quark-to-gluon anomalous dimension $\gamma_{\rm gq}^{(3)}(N)$
  - moments $N = 2, \dots 10$ — S.M., Ruijl, Ueda, Vermaseren, Vogt '23
  - *New:* moments $N = 12, \dots 20$ — Falcioni, Herzog, S.M., Pelloni, Vogt '24

$$
\begin{aligned}
\gamma_{\rm gq}^{(3)}(N\!=\!2) &= -16663.2255 + 4439.14375\, n_f - 202.555479\, n_f^{\,2} - 6.37539072\, n_f^{\,3}\,, \\
\gamma_{\rm gq}^{(3)}(N\!=\!4) &= -6565.73145 + 1291.06746\, n_f - 16.1461902\, n_f^{\,2} - 0.83976340\, n_f^{\,3}\,, \\
\gamma_{\rm gq}^{(3)}(N\!=\!6) &= -3937.47937 + 679.718506\, n_f - 1.37207753\, n_f^{\,2} - 0.13979433\, n_f^{\,3}\,, \\
\gamma_{\rm gq}^{(3)}(N\!=\!8) &= -2803.64411 + 436.393057\, n_f + 1.81494624\, n_f^{\,2} + 0.07358858\, n_f^{\,3}\,, \\
&\dots \\
\gamma_{\rm gq}^{(3)}(N\!=\!20) &= -1054.26140 + 105.497994\, n_f + 2.39223577\, n_f^{\,2} + 0.19938504\, n_f^{\,3}\,.
\end{aligned}
$$

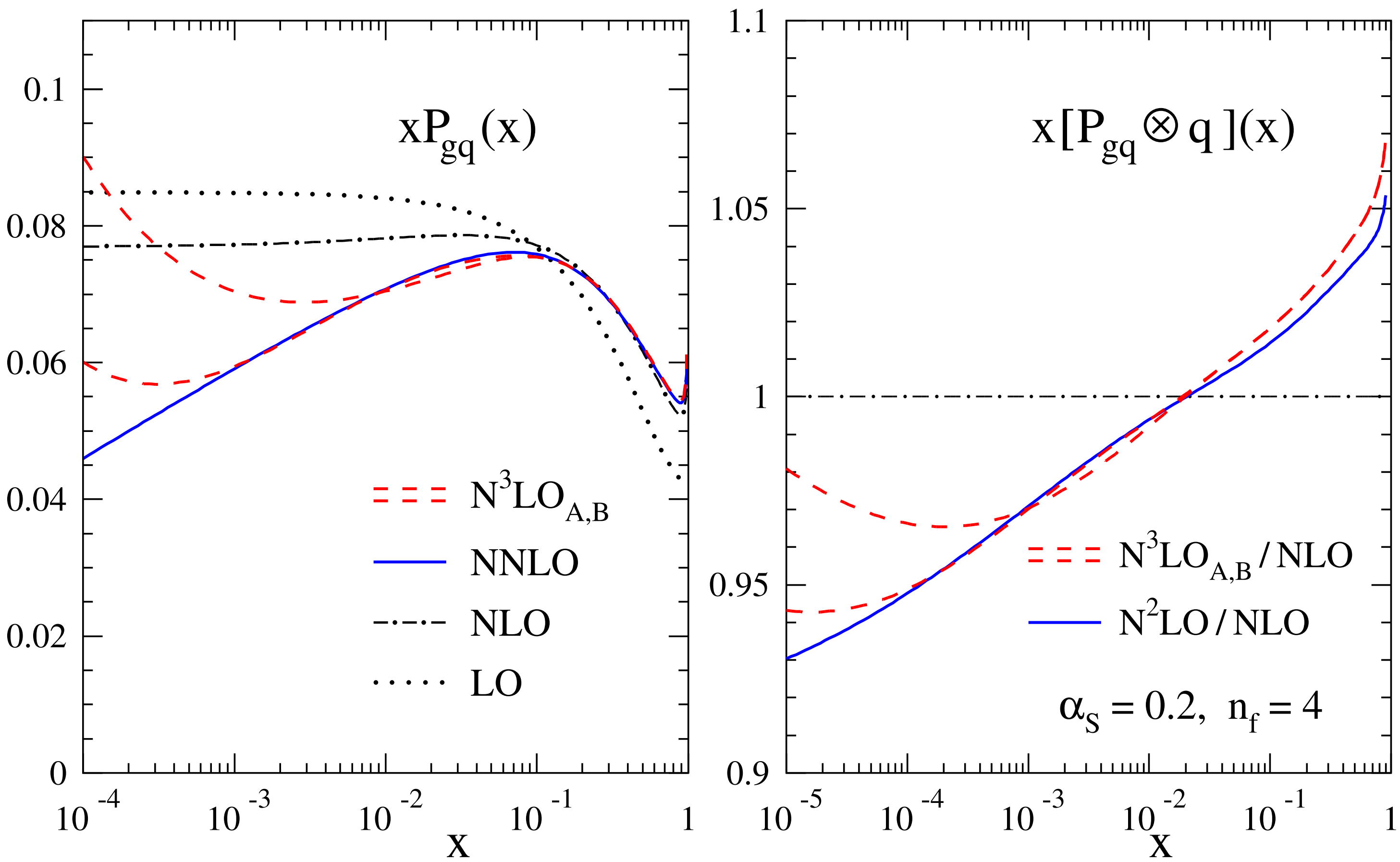
- Approximation of four-loop splitting function $P_{\rm gq}^{(3)}(x)$ again with known large- and small-x information and suitable ansatz

Quark-to-gluon splitting function (I)



- Approximations for $P_{\mathrm{gq}}^{(3)}(x)$ based on moments $N \leq 10$ vs. $N \leq 20$
  - clear improvements at large-x (left) and small-x (right)

Quark-to-gluon splitting function (II)



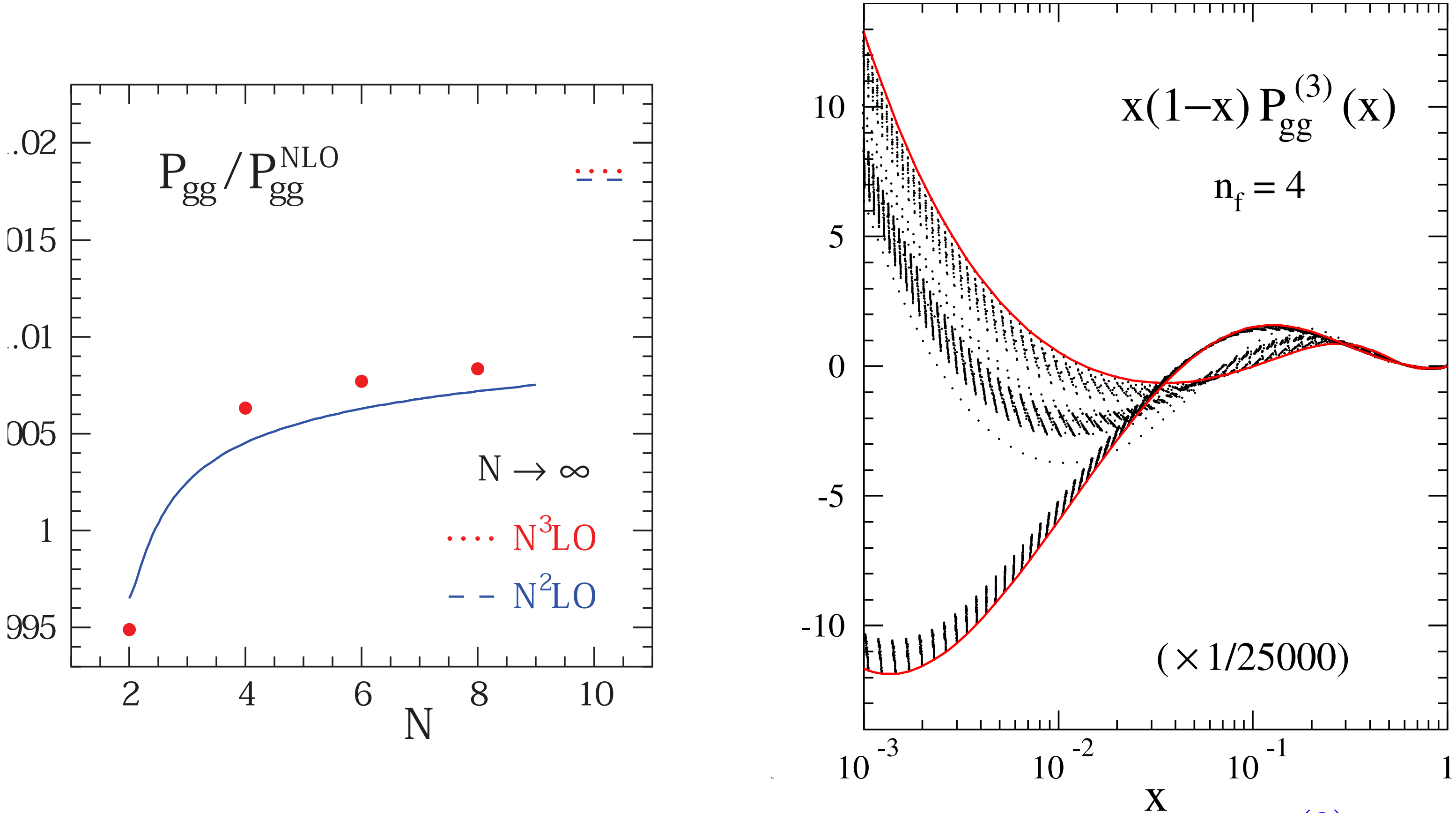
- Left: results for $P_{\rm gq}(x)$ up to N^{3}LO; $\alpha_s \; = \; 0.2$ fixed, $n_f \; = 4$
- Right: contribution to evolution kernel $d \ln g / d \ln \mu_f^2$ up to N^{3}LO for typical quark-singlet shape

$$xq_{\,\rm s}(x, \mu_0^2) \;=\; 0.6\, x^{\,-0.3} (1-x)^{3.5}\, \left(1 + 5.0\, x^{\,0.8}\right)$$

Gluon-gluon splitting function P_{gg}

$$\begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix}$$

Gluon-gluon splitting function (I)

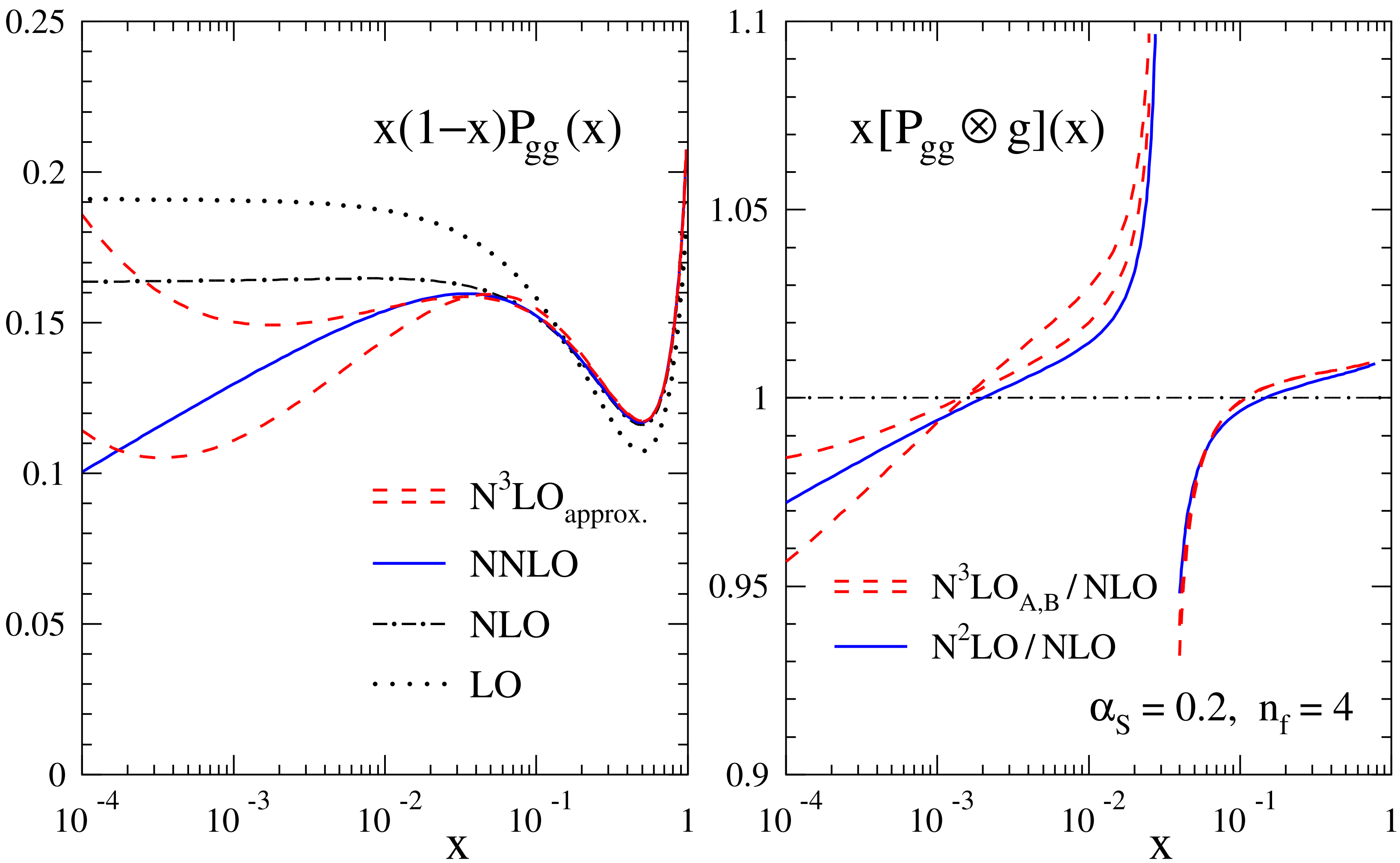


- Moments $N = \leq 8$ for $P_{gg}(x)$ (left) and approximations for $P_{gg}^{(3)}(x)$ based on moments $N \leq 10$ (right)

Outlook

- Comparison to other approximations for $P_{gg}^{(3)}$
  McGowan, Cridge, Harland-Lang, Thorne '22; NNPDF collaboration '24
- Benchmark N^{3}LO evolution to be published

Gluon-gluon splitting function (II)

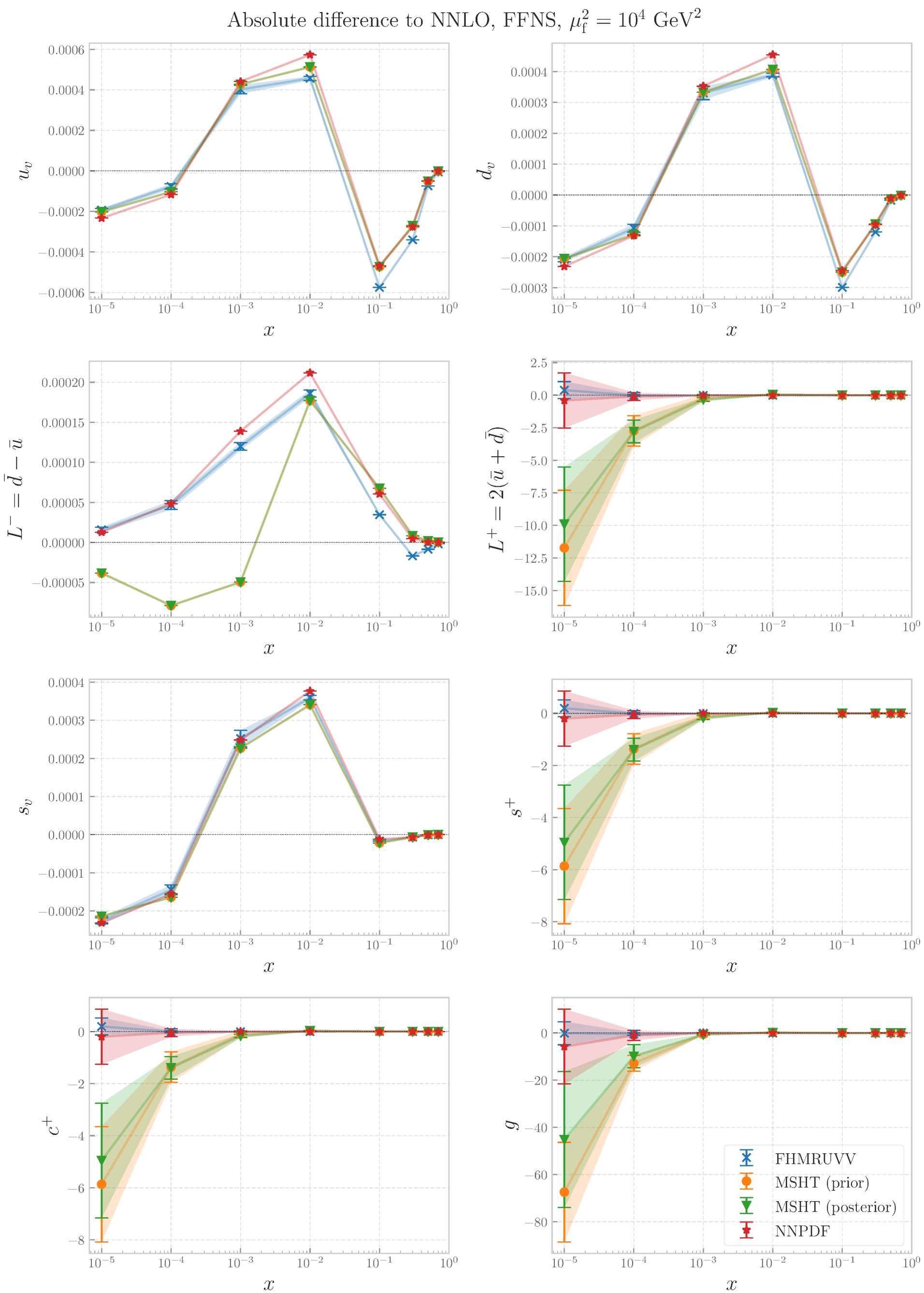


- Left: results for $P_{gg}(x)$ up to N^{3}LO; $\alpha_s = 0.2$ fixed, $n_f = 4$
- Right: contribution to evolution kernel $d \ln g / d \ln \mu_f^2$ up to N^{3}LO for typical gluon shape

$$xg(x, \mu_0^2) = 1.6\, x^{-0.3}(1-x)^{4.5} \left(1 - 0.6\, x^{0.3}\right)$$

Evolution benchmark (I)

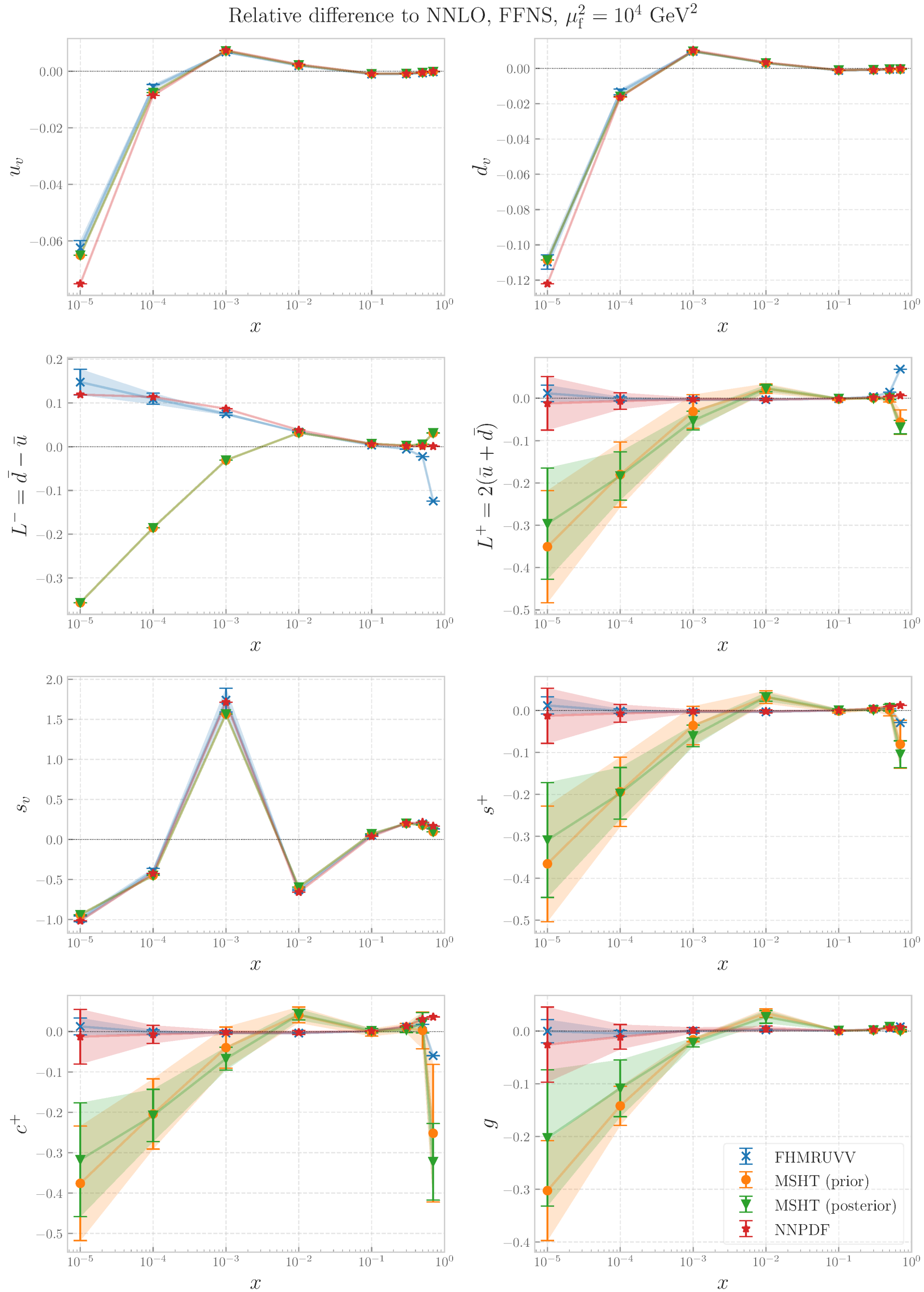
- Absolute difference of aN^{3}LO evolution
  - FFNS scheme
- Approximate spltting functions
  - FHMRUVV, MSHT prior, MSHT posterior and NNPDF
- Toy input parton distributions from Giele et al., `hep-ph/0204316`
  - evolution from $\mu_f = \sqrt{2}$ GeV to $\mu_f = 100$ GeV to



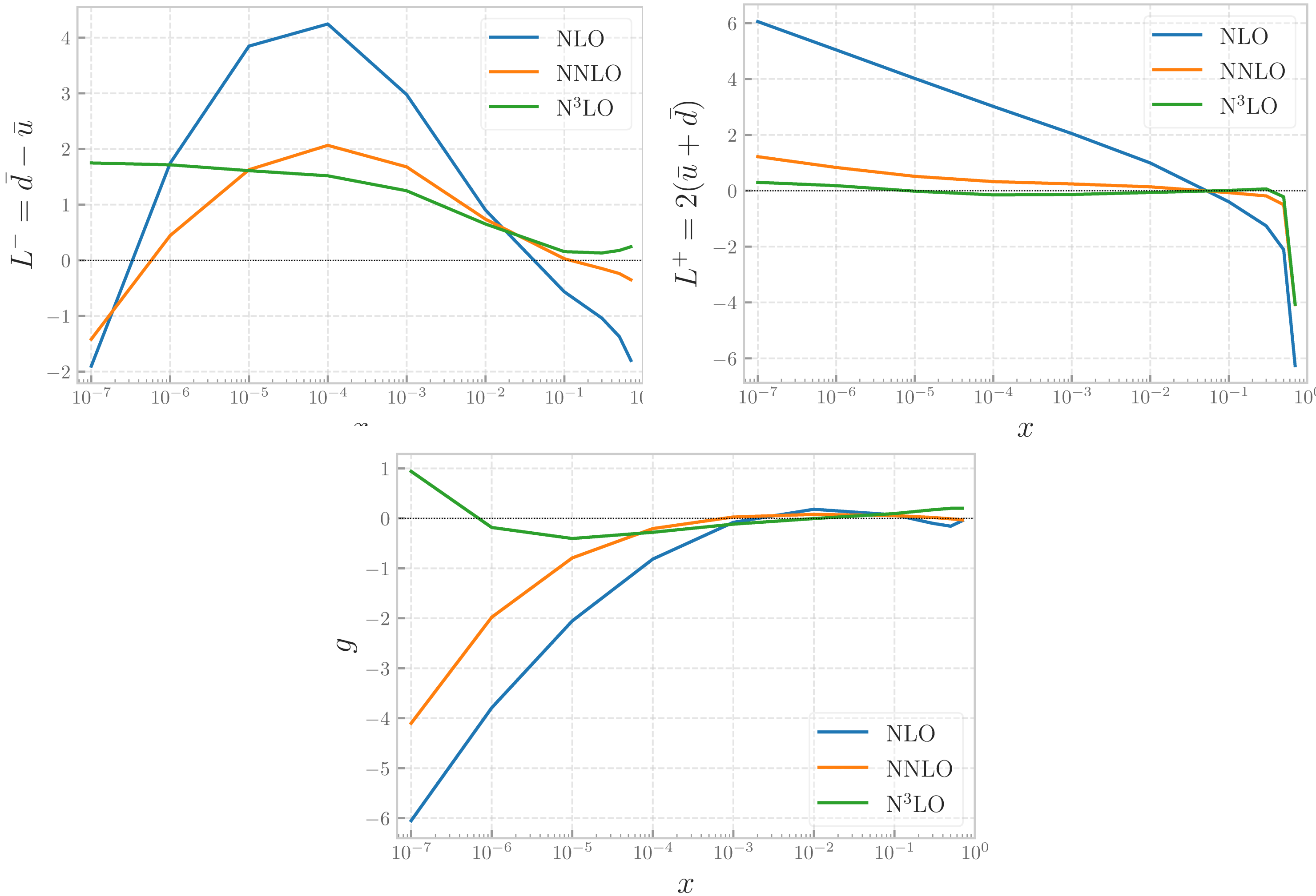
Cooper-Sarkar, Cridge, Giuli, Harland-Lang, Hekhorn, Huston, Magni, S.M., Thorne '24

Evolution benchmark (II)

- Relative difference of aN^{3}LO evolution
  - consistent results for all N^{3}LO approximations for $x \gtrsim 10^{-3}$
  - few percent differences relative to NNLO most (and often less)
- Approximate spltting functions
  - remaining differences in small-x region
- aN^{3}LO evolution in kinematic range of LHC
  - good perturbative convergence
  - significant reduction of residual theoretical uncertainty

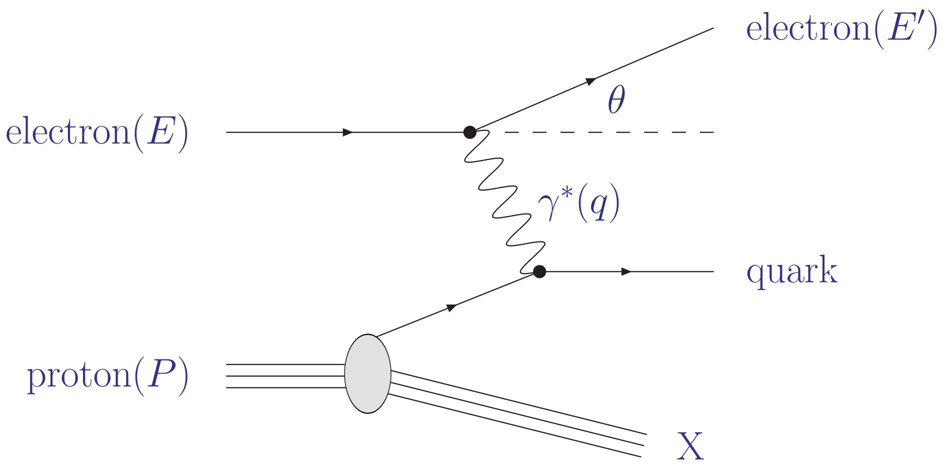


Evolution benchmark (III)



- Exact and truncated solutions of the QCD evolution
  - non-singlet quark combinations L^-, L^+ and gluon g
  - relative difference between at NLO, NNLO, N^{3}LO

Deep-inelastic scattering



Kinematic variables

- momentum transfer $Q^2 = -q^2$
- Bjorken variable $x = Q^2/(2p \cdot q)$

- Structure function F_2^p (up to order $\mathcal{O}(1/Q^2)$)

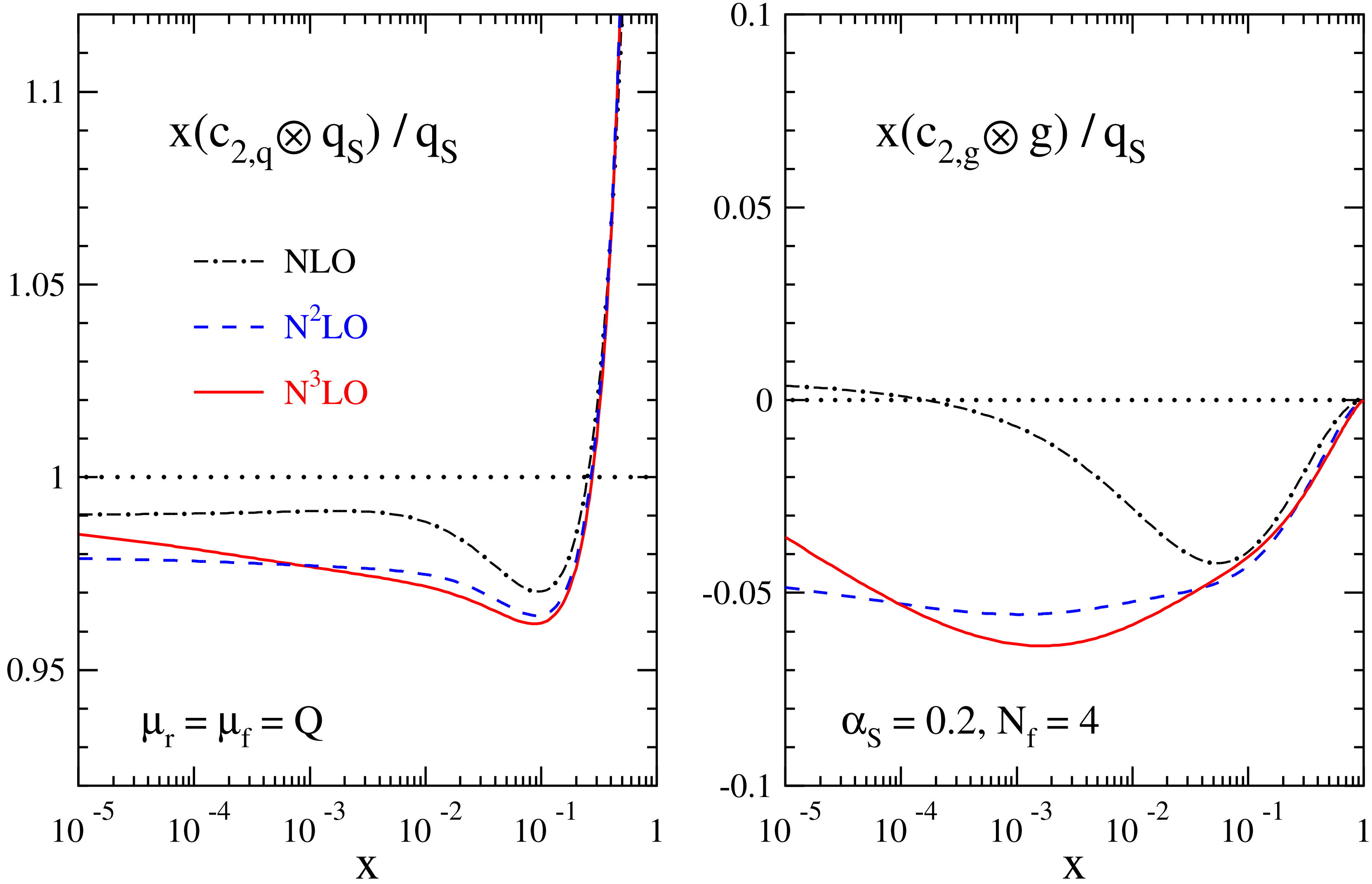
$$x^{-1} F_2^p(x, Q^2) \;=\; \sum_i \int_x^1 \frac{d\xi}{\xi}\, C_{2,i}\left(\frac{x}{\xi}, \alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) f_i^p(\xi, \mu^2)$$

- Coefficient functions

$$C_{a,i} = \alpha_s^n \left(c_{a,i}^{(0)} + \alpha_s c_{a,i}^{(1)} + \alpha_s^2 c_{a,i}^{(2)} + \alpha_s^3 c_{a,i}^{(3)} + \alpha_s^4 c_{a,i}^{(4)} + \dots \right)$$

  - current frontier in perturbation theory N^{4}LO (work in progress)

DIS structure functions at three loops



- Perturbative expansion to N^{3}LO of the quark and gluon contribution
- Perturbative stability of F_2

Vermaseren, Vogt, S.M. '05

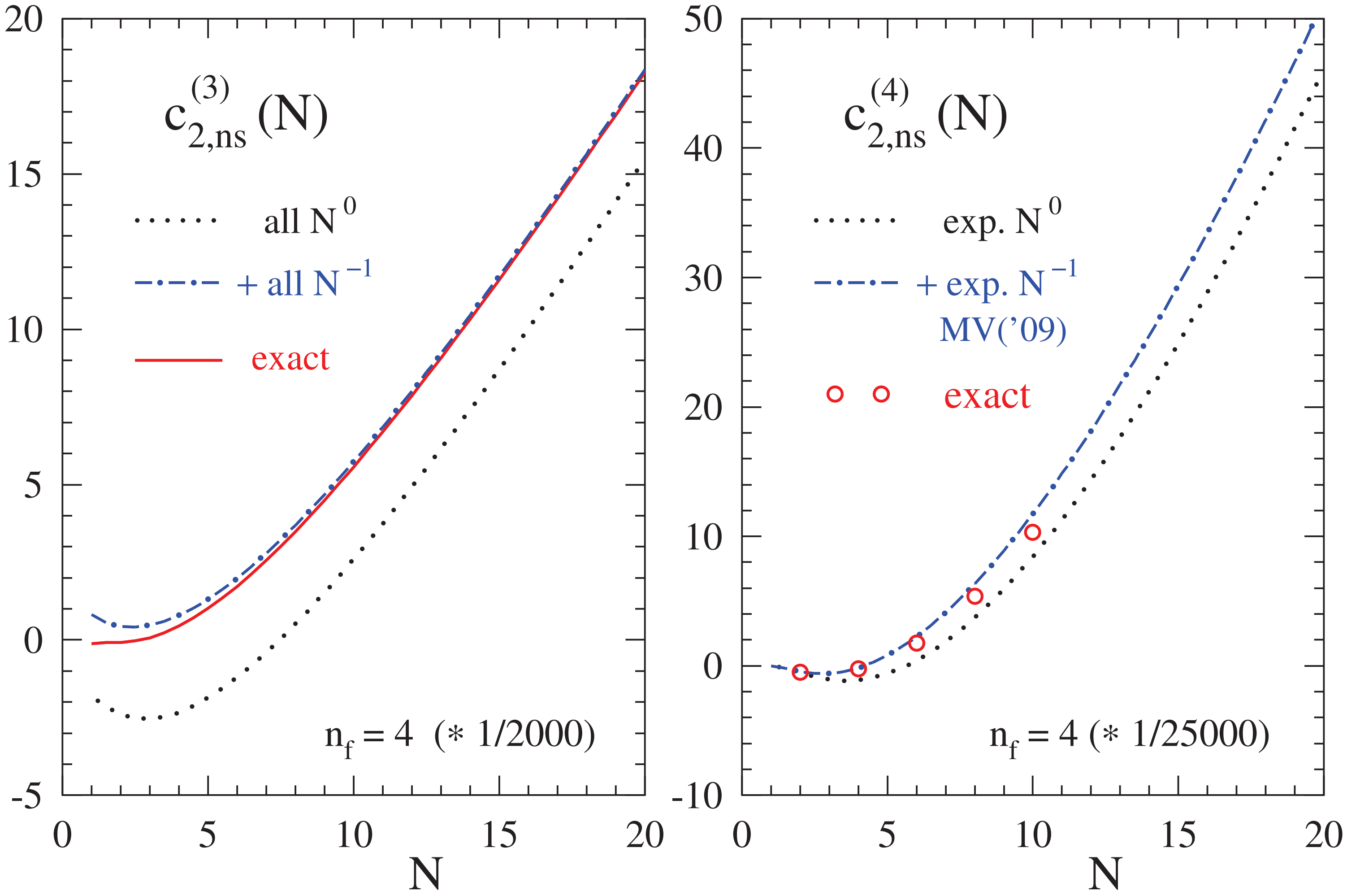
DIS Mellin moments at four loops (I)

- Perturbative expansion of non-singlet coefficient functions
  - Mellin moments $N = 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14$ of $C_{2,\mathrm{ns}}$ and $C_{L,\mathrm{ns}}$ (moments $N = 12,\ 14$ in limit of large n_c)
  - Mellin moments $N = 1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15$ of $C_{3,-}$ (moments $N = 11,\ 13,\ 15$ in limit of large n_c)
- Numerical results for $C_{2,\mathrm{ns}}(N, n_f)$

S.M., Ruijl, Ueda, Vermaseren, Vogt *to appear*

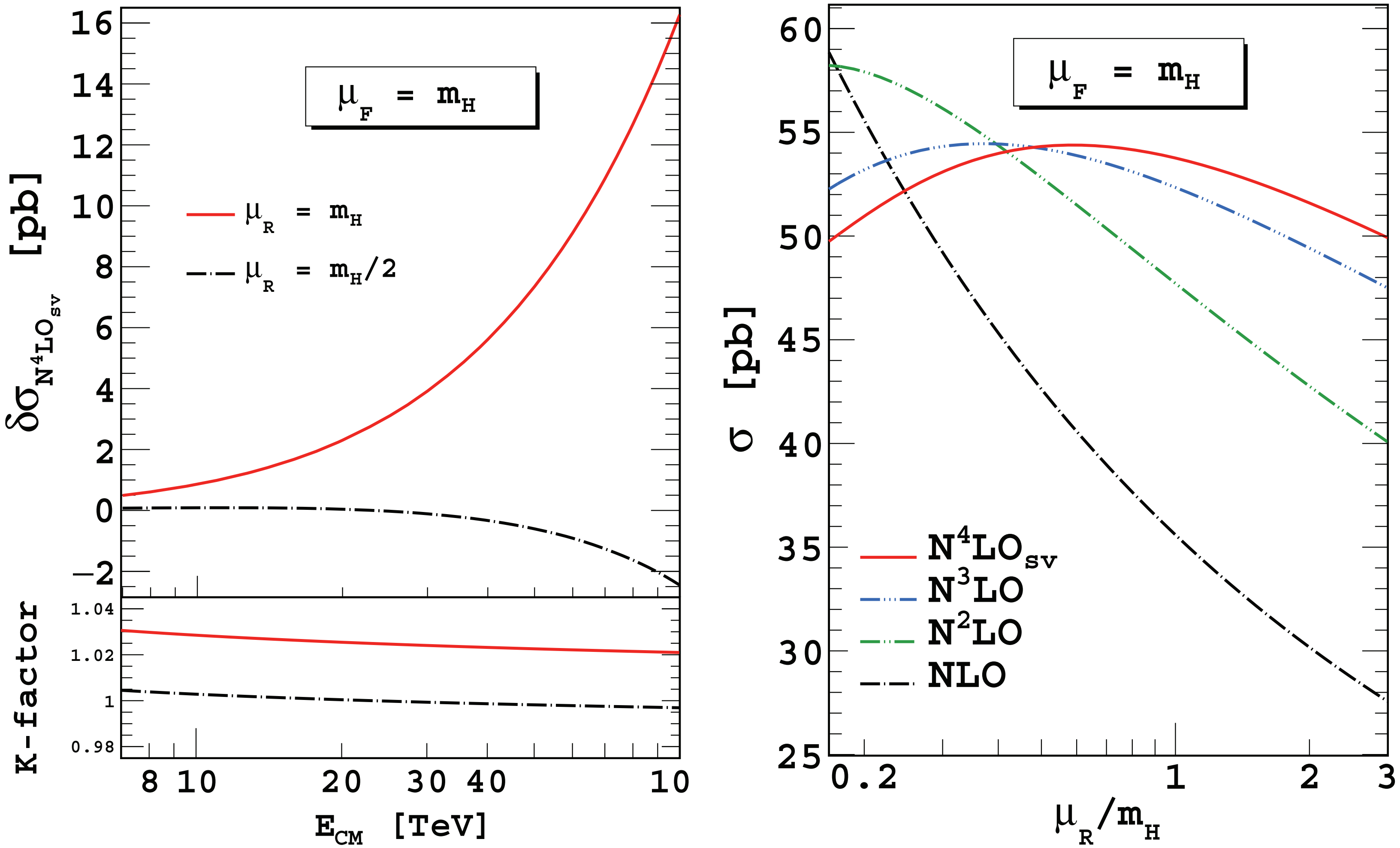
$$
\begin{aligned}
C_{2,\mathrm{ns}}(2,4) &= 1 + 0.0354\ \alpha_s - 0.0231\ \alpha_s^2 - 0.0613\ \alpha_s^3 - 0.4746\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(4,4) &= 1 + 0.4828\ \alpha_s + 0.4711\ \alpha_s^2 + 0.4727\ \alpha_s^3 - 0.2458\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(6,4) &= 1 + 0.8894\ \alpha_s + 1.2054\ \alpha_s^2 + 1.7572\ \alpha_s^3 + 1.7748\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(8,4) &= 1 + 1.2358\ \alpha_s + 2.0208\ \alpha_s^2 + 3.5294\ \alpha_s^3 + 5.3921\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(10,4) &= 1 + 1.5359\ \alpha_s + 2.8608\ \alpha_s^2 + 5.6244\ \alpha_s^3 + 10.324\ \alpha_s^4\ .
\end{aligned}
$$

DIS Mellin moments at four loops (II)



- Exact results for $c^{(3)}_{2,\mathrm{ns}}$ (N^{3}LO) at $n_f = 4$ (rescaled by $2000 \simeq (4\pi)^3$)
- Moments for $c^{(4)}_{2,\mathrm{ns}}$ (N^{4}LO) at $n_f = 4$ (rescaled by $25000 \simeq (4\pi)^4$)
- Comparison with contributions provided by large-N resummations

Higgs cross section in gluon-gluon fusion



- Left: Consistency check with approximate N^{4}LO corrections at two scales $\mu = m_H$ and $\mu = m_H/2$ as function of $\sqrt{s}$
- Right: μ_R scale dependence at approximate N^{4}LO for $\sqrt{s} = 14$ TeV

Das, S.M., Vogt '20

Summary

- Experimental precision of $\lesssim 1\%$ motivates computations at higher order in perturbative QCD
  - theoretical predictions at NNLO in QCD nowadays standard
- Push for theory results at N^{3}LO (and even N^{4}LO)
  - evolutions equations expected to achieve percent-level
  - massive use of computer algebra
- Four-loop splitting functions approximated from moments $N = 2, \ldots 20$
  - residual uncertainties negligible in wide kinematic range of x probed at current and future colliders
  - $P_{\rm qq} = P_{\rm ns}^{+} + P_{\rm ps}$, $P_{\rm qg}$ and $P_{\rm gq}$ done; $P_{\rm gg}$ to come
- N^{3}LO evolution in kinematic range of LHC
  - good perturbative convergence
  - significant reduction of residual theoretical uncertainty