Towards four-loop splitting functions in QCD

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ggF: aN3LO PDFs for Run3 & YR5   26 June, 2024

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QCD factorization and Parton densities

- Parton model

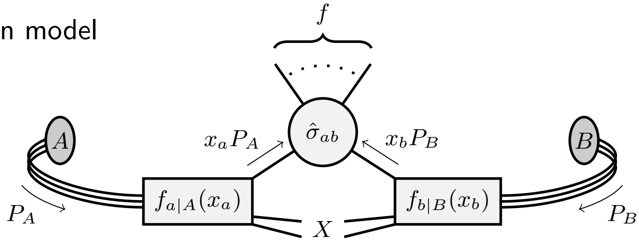
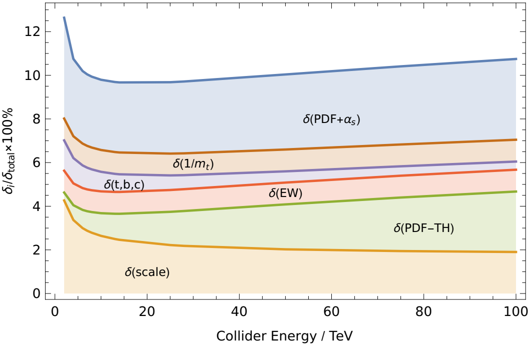


Figure by A. Huss

$$\sigma_{AB} = \sum_{ab} \int_0^1 dx_a \int_0^1 dx_b \, f_{a|A}(x_a) \, f_{b|B}(x_b) \hat{\sigma}_{ab}(x_a, x_b) + \mathcal{O}\big(\Lambda_{\mathrm{QCD}}/Q\big)$$

- $f(x)$ is the parton distribution function (PDF)
- $\hat{\sigma}_{ab}$ is the partonic cross section, perturbatively calculable

Uncertainties for gg $\rightarrow$ H +X



Cummulative uncertainties

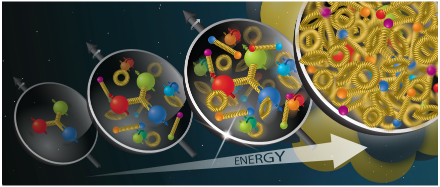
Dulat, Lazopoulos, Mistlberger,2018

- PDF+α_s and PDF-TH form the major uncertainties
- Require (a)N3LO PDF as well as (a)N3LO splitting functions

Evolution of PDFs

- Splitting functions (SFs) govern the DGLAP evolution of PDFs

$$\frac{df_{i|N}}{d\ln\mu} = 2\sum_k P_{ik} \otimes f_{k|N}$$



- Non-singlet

$$\frac{dT_i^{\pm}}{d\ln\mu} = 2P_{\rm ns}^{\pm} \otimes T_i^{\pm}, \quad \frac{d\sum_{k=1}^{n_f} q_k^-}{d\ln\mu} = 2P_{\rm ns}^{V} \otimes \sum_{k=1}^{n_f} q_k^-, \; i = 3, 8, \cdots n_f^2 - 1$$

$$T_3^{\pm} = u^{\pm} - d^{\pm}, T_8^{\pm} = u^{\pm} + d^{\pm} - 2s^{\pm}, \cdots, q_k^{\pm} = q_k \pm \bar{q}_k\,,$$

- Singlet (relevant to ggF production)

$$\frac{d}{d\ln\mu}\begin{pmatrix} \Sigma \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix}, \quad \Sigma = \sum_{k=1}^{n_f} q_k^+$$

Splitting functions & Anomalous dimensions (A.D.)

- Mellin transformation

$$f_q(n) = -\int_0^1 dz\, z^{n-1} f_q(z)\,, \quad \gamma_{ij}(n) = -\int_0^1 dz\, z^{n-1} P_{ij}(z)$$

- DGLAP evolution in n-space

$$\frac{d}{d\ln\mu} f_q(n,\mu^2) = -2\sum_j \gamma_{qj}(n)\, f_j(n,\mu^2)$$

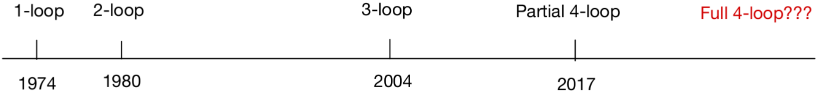
- PDFs in n-space are hadronic operator matrix elements (OMEs)

$$f_q(n) \sim \langle N(P)|\, \bar{\psi}\not{\Delta}(\Delta\cdot D)^{n-1}\psi\, |N(P)\rangle = \langle N(P)|\, O_q\, |N(P)\rangle$$

$$f_g(n) \sim \langle N(P)|\Delta_{\mu_1} G^{\mu_1}_{a,\mu}(\Delta\cdot D)^{n-2}_{ab}\Delta_{\mu_n} G^{\mu_n\mu}_b\, |N(P)\rangle = \langle N(P)|\, O_g\, |N(P)\rangle$$

Towards four-loop splitting functions

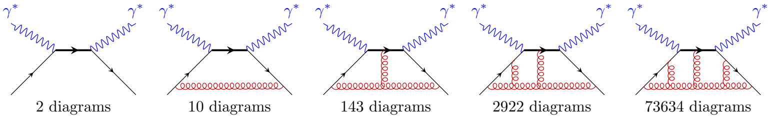
- Timeline of the calculation of splitting functions



- Fixed moments Talk by S. Moch and G. Falcioni
  - Non-singlet $\gamma_{\rm ns}^{(3)}$ with $n \leq 16$[Moch,Ruijl,Ueda,Vermaseren,Vogt,2017]
  - Pure-singlet $\gamma_{\rm ps}^{(3)}$, $\gamma_{qg}^{(3)}$and $\gamma_{gq}^{(3)}$ with $n \leq 20$[Falcioni,Herzog,Moch,Pelloni,A. Vogt,2023, 2024]
  - $\gamma_{gg}^{(3)}$ with $n \leq 10$[Moch,Ruijl,Ueda,Vermaseren,Vogt,2023]
- Exact results with all-n dependence
  - All $\gamma_{ij}^{(3)}$ in the large-N_f limit[Gracey 1994,1996; Davies,Vogt, Ruijl,Ueda,Vermaseren,2016]
  - $\gamma_{\rm ns}^{(3)}$ with leading color[Moch,Ruijl,Ueda,Vermaseren,Vogt,2017]
  - N_f^2 term for $\gamma_{qg}^{(3)}$ [Falcioni,Moch,Ruijl,Ueda,Vermaseren,Vogt,2023]
  - N_f^2 term for $\gamma_{\rm ps}^{(3)}$ and $N_f C_F^3$ term for $\gamma_{\rm ns}^{(3)}$ This talk

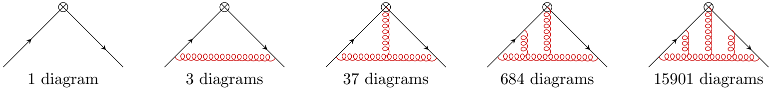
DIS method vs OME method

- Forward DIS (gauge invariant)



Shrinking the heavy lines into effective vertices

- Partonic off-shell OME (fewer diagrams, easier integrals)



- Off-shell OMEs are not gauge invariant, physical operators mix with gauge-variant (GV) operators under renormalization
- Main goal: find all GV operators or their Feynman rules

A new framework to derive GV operators (Feynman rules)

- Generalize the renormalization of O_g

$$O_g^{\mathrm{R}} = Z_{gq} O_q^{\mathrm{B}} + Z_{gg} O_g^{\mathrm{B}} + Z_{gA} O_{ABC}^{\mathrm{B}} + [ZO]_g^{\text{GV, B}}$$

- O_{ABC} : GV operators, $[ZO]_g^{\text{GV, B}}$: collections of counterterm operators
- Consider all-off-shell multi-loop multi-point OMEs

$$\langle j|O_g^{\mathrm{R}}|j+m\,g\rangle_{1\mathrm{PI}}^{\mu_1\cdots\mu_m} = \langle j|(Z_{gq}O_q^{\mathrm{B}} + Z_{gg}O_g^{\mathrm{B}})|j+m\,g\rangle_{1\mathrm{PI}}^{\mu_1\cdots\mu_m} + \langle j|Z_{gA}O_{ABC}^{\mathrm{B}}|j+m\,g\rangle_{1\mathrm{PI}}^{\mu_1\cdots\mu_m} + \langle j|\,[ZO]_g^{\mathrm{GV}}\,|j+m\,g\rangle_{1\mathrm{PI}}^{\mu_1\cdots\mu_m}\,,\; j = q, g \text{ or c}$$

- Renormalization conditions to determine counterterm Feynman rules order by order in α_s

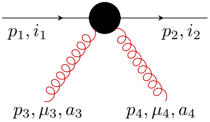
Required OMEs to derive four-loop splitting functions

- 2-point OMEs are used to extract splitting functions
- Multi-point OMEs: determine Feynman rules of GV operators

Loops \ Legs	2	3	4	5	6
0	A.D.	$[ZO]_g^{\mathrm{GV},(3)}$	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_q, O_g (d)
1	$[ZO]_g^{\mathrm{GV},(3)}$	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_g(d)	
2	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_g (ip)		
3	O_{ABC}	O_g			
4	O_q, O_g(ip)				

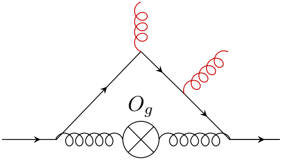
- 3-loop splitting functions (done)
- 1-loop five-point OMEs to extract Feynman rules of O_{ABC} (done)
- 2-loop four-point OMEs to extract $[ZO]_g^{\mathrm{GV},(2)}$ (in progress)
- 4-loop two-point OMEs: focus on $\langle q|O_q|q\rangle^{(4)}$: N_f^2 and $N_f C_F^3$

Sample results: Feynman rules for O_B with four legs



$$\to \frac{1+(-1)^n}{-8} g_s^2 \Delta^{\mu_3} \Delta^{\mu_4} \left(T^{a_3} T^{a_4} - T^{a_4} T^{a_3}\right)_{i_2 i_1} \not{\Delta} \sum_{j_1=0}^{n-3} \Big(3\left(\Delta \cdot (p_1+p_2)\right)^{-j_1+n-3} \left[(-\Delta \cdot p_3)^{j_1} - (-\Delta \cdot p_4)^{j_1} \right] - (-\Delta \cdot p_4)^{j_1} (\Delta \cdot p_3)^{-j_1+n-3} \Big)$$

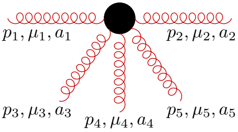
extracted from



all-n Feynman rules

Feynman rules for O_{ABC} with five legs

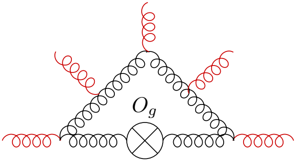
New



$$\to \frac{1+(-1)^n}{2} i g_s^3 \Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4} p_1^{\mu_5} \Bigg[\frac{1}{C_A} f^{aa_1a_2} d_4^{aa_3a_4a_5} \Bigg\{ \frac{3}{32} \sum_{j_1=0}^{n-4} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} (-\Delta\cdot p_1)^{n-4-j_1} (-\Delta\cdot(p_1+p_2))^{j_1-j_2} \times (\Delta\cdot(p_4+p_5))^{j_2-j_3} (\Delta\cdot p_5)^{j_3} + \cdots \Bigg\} + 11 \text{ color structures} \Bigg] + 30 \text{ Lorentz Structures}$$

17074 lines

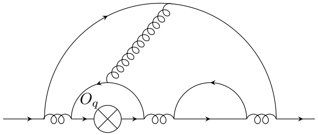
extracted from



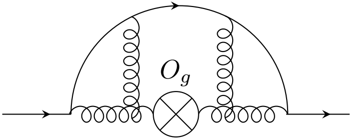
all-n Feynman rules

Sample diagrams for N_f^2 contributions at four-loop order

- OMEs with physical operator insertions

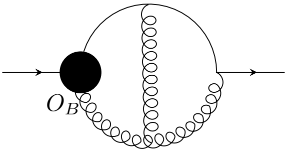


$\langle q|O_q^{\mathsf{B}}|q\rangle^{(4)}$

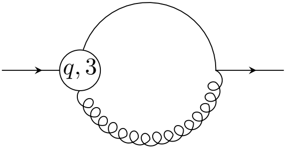


$\langle q|O_g^{\mathsf{B}}|q\rangle^{(3)}$

- OMEs with GV operator or counterterm insertion



$\langle q|O_B^{\mathsf{B}}|q\rangle^{(2)}$



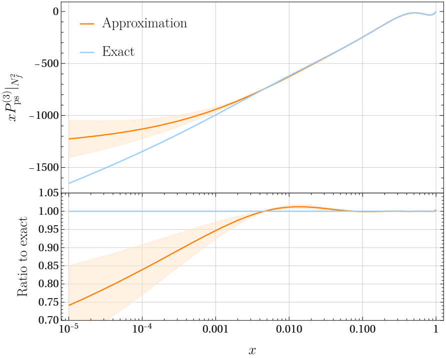
$\langle q|\,[ZO]_q^{\mathrm{GV},\,(3)}\,|q\rangle^{(1)} = 0$

New results for N_f^2 pure-singlet contributions

- New exact result, agree with the $n \leq 20$ results[G. Falcioni et al. 2023]
- Extract small-x result

$$P_{\mathrm{ps}}^{(3)}(x)\Big|_{N_f^2} = \frac{\log(x)^2}{x} \times 0 + \frac{\log(x)}{x}(\text{New results}) + \cdots$$

- Approximation: fitted from $n \leq 20$ results and previously known limits
- Large x-region: agree well with the exact result
- $x \sim 10^{-4}$, derivation $\sim$15%



New $N_f\, C_F^3$ contributions to 4-loop $P_{\rm ns}$

- No mixture with non-physical operators
- Sample Feynman diagrams

- IBP reduction: 54 thousand integrals $\longrightarrow$ 658 master integrals
- Solve the master integrals by DE method analytically
- New exact resut, agree with $n \leq 16$ results from[S. Moch et al. 2017]

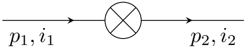
Summary

- For off-shell OMEs, renormalization of physical operators mix with unknown GV operators
- Developed a new framework to infer splitting functions
  - Two-point OMEs are used to extract splitting functions
  - Multi-point (≥ 3) OMEs are required to determine counterterm Feynman rules of the GV operators
- Applied it to derive 3-loop singlet splitting functions and recovered the well-known results in the literature
- Proof of concept: get *exact results* for $\gamma_{\text{ps}}^{(3)}\big|_{N_f^2}$ and $\gamma_{\text{ns}}^{(3)}\big|_{N_f C_F^3}$
- New results: Feynman rules for O_{ABC} with 5 legs

Thanks for your attention!

Computation of off-shell OMEs with all-n dependence

- Non-standard terms appearing in the Feynman rules
- Example: Feynman rules for O_q at lowest order



$$\rightarrow \not{\Delta} (\Delta \cdot p_1)^{n-1}$$

- How to retain all-n dependence?
  - Sum non-standard term into a linear propagator using a tracing parameter t[J. Ablinger et al. 2012]

$$(\Delta \cdot p)^{n-1} \rightarrow \sum_{n=1}^{\infty} t^n (\Delta \cdot p)^{n-1} = \frac{t}{1 - t\Delta \cdot p}$$

  - Parameter-t space $\rightarrow$ n-space, for example

$$H(1,1;t) = \sum_{n=1}^{\infty} t^n \left(-\frac{1}{n^2} + \frac{S(1,n)}{n} \right)$$

Derive Feynman rules from off-shell OMEs

- Consider all-off-shell OMEs with $2j+m$-gluon external states

$$\langle j|O_g^{\mathsf{R}}|j+m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m} = \langle j|(Z_{gq}O_q^{\mathsf{B}} + Z_{gg}O_g^{\mathsf{B}})|j+m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m} + \langle j|Z_{gA}O_{ABC}^{\mathsf{B}}|j+m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m} + \langle j|\,[ZO]_g^{\mathrm{GV}}\,|j+m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m}\,,\ j=q,g \text{ or c}$$

- Expand OMEs order by order in loops and legs

$$\langle j|O|j+m\,g\rangle^{\mu_1\cdots\mu_m} = \sum_{l=1}^{\infty}\left[\langle j|O|j+m\,g\rangle^{\mu_1\cdots\mu_m,\,(l),\,(m)}\right]\left(\frac{\alpha_s}{4\pi}\right)^l g_s^m$$

- Left: UV renormalized and IR finite $\rightarrow$ no poles in ϵ
- Right: Each term is UV divergent, but the sum should be finite
- Requirement of finiteness $\longrightarrow$ counterterm Feynman rules order by order in α_s

Renormalization of O_q to four loops in $q \to q$ channel

- Renormalization of two-point OMEs

$$\langle q|O_q^{\mathsf{R}}|q\rangle = Z_{qq}\,\langle q|O_q^{\mathsf{B}}|q\rangle + Z_{qg}\,\langle q|O_g^{\mathsf{B}}|q\rangle + Z_{qA}\,\langle q|O_{ABC}^{\mathsf{B}}|q\rangle + \langle q|\,[ZO]_q^{\mathrm{GV}}\,|q\rangle,$$

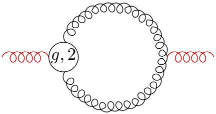
$$Z_{qg} = \mathcal{O}(a_s), Z_{qA} = \mathcal{O}(a_s^2), [ZO]_q^{\mathrm{GV}} = \mathcal{O}(a_s^3)$$

- $\langle q|\,[ZO]_q^{\mathrm{GV}}\,|q\rangle = \mathcal{O}(a_s^5)$
  - Only $\langle q|\,[ZO]_q^{\mathrm{GV},\,(4)}\,|q\rangle^{(0)}$ and $\langle q|\,[ZO]_q^{\mathrm{GV},\,(3)}\,|q\rangle^{(1)}$ are relevant
  - Other operators (O_q, O_g, O_A, O_B) give all possible Lorentz structures of $q\bar{q}$, gg, $q\bar{q}g$ vertex Feynman rules
  - $\to \langle q|\,[ZO]_q^{\mathrm{GV}}\,|q\rangle^{(0)} = 0$, $\langle g|\,[ZO]_q^{\mathrm{GV}}\,|g\rangle^{(0)} = 0$,

$$\langle q|\,[ZO]_q^{\mathrm{GV}}\,|qg\rangle^{(0)} = 0$$

Two-point OMEs with two-loop counterterm insertions

- For a fixed n, normal IBP, but need to reduce integrals with very high numerator degree
- All-n, IBP reduction with polylogarithms?



- Consider a general term of two-loop counterterms with 3-gluon vertex

$$ig_s f^{a_1 a_2 a_3} C_A^2 \Delta^{\mu_1} \Delta^{\mu_2} \Delta^{\mu_3} p_1^2 \sum_{m=0}^{n-3} a_{mn} (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} + \cdots$$

where a_{mn} is known only for fixed m, n

- New idea: replace a_{mn} by another tracing parameter t

$$h(x,t) = \sum_{n=3}^{\infty} x^n \sum_{m=0}^{n-3} t^m (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} = \frac{x^3}{(1 - x\,t\Delta \cdot p_1)(1 - x\Delta \cdot p_2)}$$

- Insert h into two-point diagrams: $\langle g | h(x,t) | g \rangle = \sum_{n=3}^{\infty} x^n \sum_{m=0}^{n-3} t^m c_{mn}$
- $\langle g | \sum_{m=0}^{n-3} a_{mn} (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} | g \rangle = \sum_{m=0}^{n-3} a_{mn} c_{mn}$

Evaluate OMEs to any fixed n efficiently and reconstruct the full-n results

Three-loop results

- Two-point OMEs with two-loop counterterm insertions
  - Compute OMEs to $n = 500$ based on two tracing parameters x, t
  - Reconstruct all-n result to ϵ^0 using the data with n to 440
- Two-point OMEs with the insertion of O_q, O_g, O_{ABC}
  - Compute OMEs with all-x dependence based on a tracing paramter x
  - Expand HPLs to get all-n results in terms of HSs directly
- Results: confirm the ξ-independence explicitly and recover the well known results in the literature
  - non-singlet:

$$\gamma_{\text{ns}}^{(2)} - \gamma_{\text{ns}}^{(2)}[\text{MVV}] = 0$$

  - singlet:

$$\gamma_{qq}^{(2)} - \gamma_{qq}^{(2)}[\text{VMV}] = 0\,, \gamma_{qg}^{(2)} - \gamma_{qg}^{(2)}[\text{VMV}] = 0$$
$$\gamma_{gq}^{(2)} - \gamma_{gq}^{(2)}[\text{VMV}] = 0\,, \gamma_{gg}^{(2)} - \gamma_{gg}^{(2)}[\text{VMV}] = 0$$

Lorentz structures of a twist-two operator

- Based on the following two properties
  - A twist-two operator has spin-n and mass dimension $n+2$
  - Propagator-type Feynman rules like $1/p^2$ can not appear in a vertex
- A twist-2 operator involving quarks or ghosts has one Lorentz structure only

$$\langle q|O|q+m\,g\rangle_{1\text{PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m)} = c_m\Delta^{\mu_1}\cdots\Delta^{\mu_m}$$

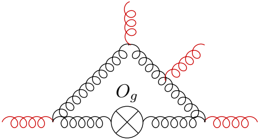
- A twist-two operator involving only gluons
  - Only $1+3/2m(m-1)$ Lorentz structures for m-gluon Feynman rules
  - $m=3$: $a_1\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}+a_2\Delta^{\mu_1}\Delta^{\mu_2}p_1^{\mu_3}+\cdots+a_{10}\Delta^{\mu_3}g^{\mu_1\mu_2}$
  - 19 for $m=4$ and 31 for $m=5$
- Count the mass dimension of a_i: $[a_i]=x_i[\Delta\cdot p_j]+y_i[p_j\cdot p_k](y_i\geq 0)$
  $[a_1]=n-3+y_1[p_j\cdot p_k]=n+2-3\rightarrow y_1=1$(Linear in $p_1^2,\ p_1\cdot p_2\cdots$)
  $[a_2]+[p_1^{\mu_3}]=n-2+y_2[p_j\cdot p_k]+1=n+2-3\rightarrow y_2=0$
- Why not $a_{11}\Delta^{\mu_1}p_1^{\mu_2}p_2^{\mu_3}$

$$[a_{11}]+2+3\geq n-1+y_{11}[p_j\cdot p_k]+2+3=n+4(\text{if } y_{11}=0)$$

where 3 is mass dimension of the external 3 gluons. Twist-4 operators

Computations of single pole for one-loop multi-leg OMEs

- Set all Mandelstam variables $p_1^2,\ p_2^2\ \cdots$ to numerical numbers and reconstruct their linear dependence



- Only two types of integrals are needed, other integrals are finite

$$I_1 = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l-q_1)^2 l^2}, \quad I_2 = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l-q_1)^2 l^2 \big(1 - x\Delta \cdot (l+q_2)\big)}$$

- At most x-dependent logarithms appear in the single pole

$$I_2 = \frac{1}{\epsilon}\left[\frac{\ln(1-x\Delta\cdot q_1 - x\Delta\cdot q_2) - \ln(1-x\Delta\cdot q_2)}{-x\Delta\cdot q_1}\right] + \mathcal{O}(\epsilon^0)$$

- Logarithms in x-space $\rightarrow$ n-space

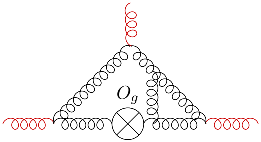
$$\ln(1-x\Delta\cdot p_1 - x\Delta\cdot p_2) = \sum_{n=1}^{\infty} x^n \left[\frac{-1}{n}(\Delta\cdot p_1 + \Delta\cdot p_2)^n\right]$$

- Factoring out the overall factor $Z_{gA}^{(1)} = -\frac{C_A}{\epsilon}\frac{1}{n(n-1)}$

Computations of two-loop three-leg OMEs

- Set all Mandelstam variables p_1^2, p_2^2 $\cdots$ to numerical numbers and

$$\Delta \cdot p_1 = 1,\ \Delta \cdot p_2 = z_1$$



- Derive DE with respect to x
- Difficult to solve DE in terms of special functions
- Expand DE to x^{100} in the limit of $x \to 0$, with the boundary conditions being two-loop three-leg integrals without operator insertions[T. G. Birthwright, E. W. N. Glover, and P. Marquard, 04]

Reconstruct two-loop counterterm Feynman rules

- Obtain two-loop three-leg OMEs to x^{96} or $n = 96$
- For a fixed n, the result is a polynomial in z_1
- Construct full-x or full-n results from data to $n = 76$ based on ansatz
- Polylogarithms to weight-3, generalized Harmonic sums to weight-2

$$G(1,1,1/(1+z_1);x) = \sum_{n=1}^{\infty} x^n \left[\frac{S_1(z_1+1;n)}{n^2} + \frac{S_2(z_1+1;n)}{n} - \frac{S_{1,1}(1,z_1+1;n)}{n} - \frac{(z_1+1)^n}{n^3} \right]$$

where $S_{1,1}(1,z_1+1;n) = \sum_{t_1=1}^{n} \frac{1}{t_1} \sum_{t_2=1}^{t_1} \frac{(1+z_1)^{t_2}}{t_2}$

- Due to the generalized Harmonic sums, impossible to disentangle
  - renormalization constants (no z_1 dependence)
  - operator Feynman rules (no high-weight (≥ 1) functions)

A counterterm Feynman rule & infinite operator Feynman rules ($N_2 = \infty$)