Classification of Fermionic RCFTs & Topological Phases Revisited

SUNGJAY LEE

KIAS

INPP Demokritos-APCTP meeting & HOCTOOLS-II mini-workshop

National Centre For Scientific Research Demokritos

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Based on the collaboration with

Jimbeom Bae(Oxford), Kimyeong Lee, Linfeng Li, Zhihao Duan(KIAS)

arXiv:2210.06805, 2112.14130, 2108.01647, 2010.12392

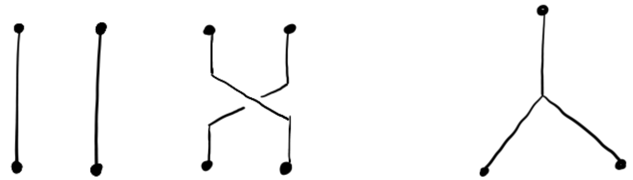
TOPOLOGICAL PHASES OF MATTER

how to characterize them ?

§ Bulk approach

topological phase : gapped phase

$\exists$ anyons (quasi-particle)



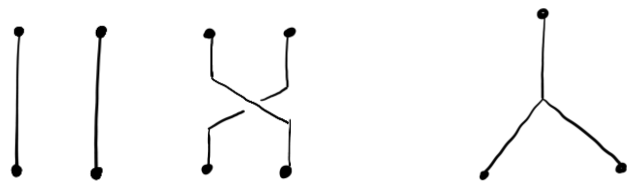
$\underbrace{\text{braiding} \qquad\qquad \text{fusion}}_{\text{consistent conditions}}$

MODULAR TENSOR CATEGORY

TOPOLOGICAL PHASES OF MATTER   how to characterize them ?

§ Bulk approach

$\exists$ anyons (quasi-particle)



braiding   fusion

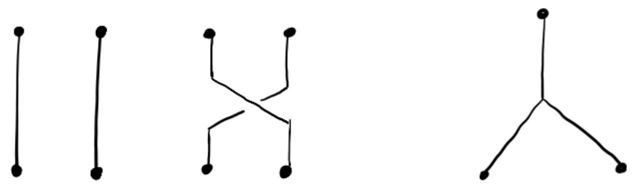
MODULAR TENSOR CATEGORY

$\left\{ \begin{array}{l} \text{classification of} \\ \text{bosonic top'l phases} \end{array} \right\} \longleftarrow \left\{ \begin{array}{l} \text{Search for anyonic systems} \\ \text{consistent with MTC} \end{array} \right\}$

TOPOLOGICAL PHASES OF MATTER   how to characterize them ?

§ Bulk approach

∃ anyons (quasi-particle)



braiding   fusion

$\underbrace{\qquad\qquad\qquad\qquad\qquad}$

MODULAR TENSOR CATEGORY

$\left\{ \begin{array}{l} \text{classification of} \\ \text{fermionic top'l phases} \end{array} \right\} \longleftarrow$?

super - MTC
spin - MTC ?
fermionic - MTC

TOPOLOGICAL PHASES OF MATTER   how to characterize them ?

§ Bulk approach

$\exists$ anyons (pseudo-particle)

braiding   fusion

$\underbrace{\qquad\qquad\qquad\qquad}$

MODULAR TENSOR CATEGORY

{ classification of bosonic top'l phases } ← { Search for anyonic systems consistent with MTC }

§ Boundary approach

$\exists$ edge states on $\partial\mathcal{M}_{(3)}$ ← { their dynamics is controlled by the chiral part of RATIONAL CFT_2 }

TOPOLOGICAL PHASES OF MATTER   how to characterize them ?

§ Bulk approach

$\exists$ anyons (pseudo-particle)

braiding   fusion

MODULAR TENSOR CATEGORY

{ classification of bosonic top'l phases } $\longleftarrow$ { Search for anyonic systems consistent with MTC }

§ Boundary approach

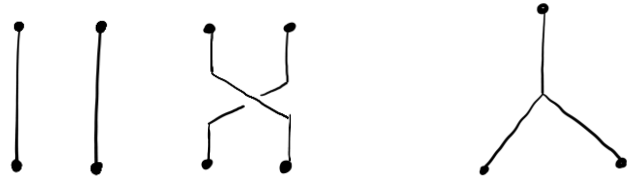
$\exists$ edge states on $\partial\mathcal{M}_{(3)}$ $\longleftarrow$ { their dynamics is controlled by the chiral part of RATIONAL CFT_2 }

{ classification of bosonic top'l phases } $\longleftarrow$ { search for bosonic rational CFT_2 }

TOPOLOGICAL PHASES OF MATTER   how to characterize them ?

§ Bulk approach

$\exists$ anyons (pseudo-particle)



braiding   fusion

MODULAR TENSOR CATEGORY

{ classification of fermionic top'l phases } $\leftarrow$? super - MTC, spin - MTC, fermionic - MTC ?

§ Boundary approach

$\exists$ edge states on $\partial\mathcal{M}_{(3)}$ $\leftarrow$ { their dynamics is controlled by the chiral part of RATIONAL CFT_2 }

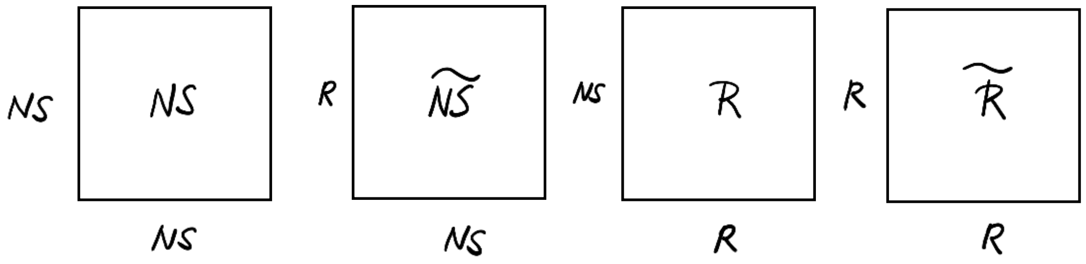
{ classification of fermionic top'l phases } $\leftarrow$ { search for fermionic rational CFT_2 }

FERMIONIC CONFORMAL FIELD THEORY

SPIN STRUCTURE

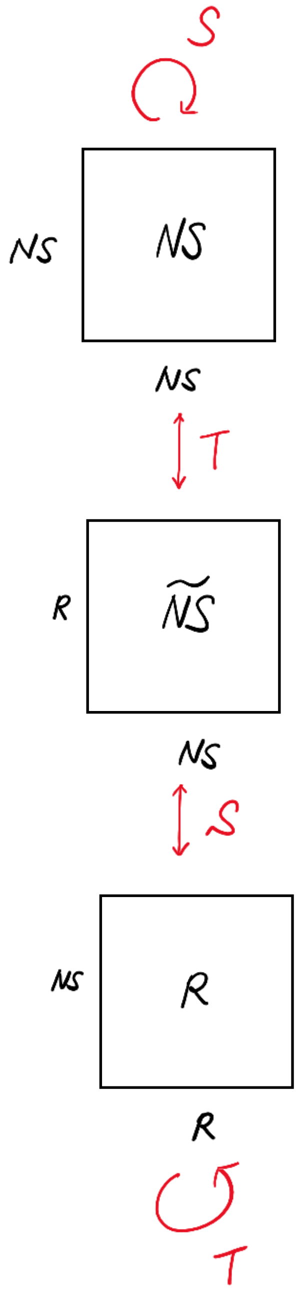
To define a theory of fermion on $\mathcal{M}_{(2)}$ we need to specify boundary condition along each S^1 of $\mathcal{M}_{(2)}$

e.g. on T^2, $\exists$ four different spin structures



Neveu-Schwarz (NS) anti-periodic b.c.

Ramond (R) periodic b.c.



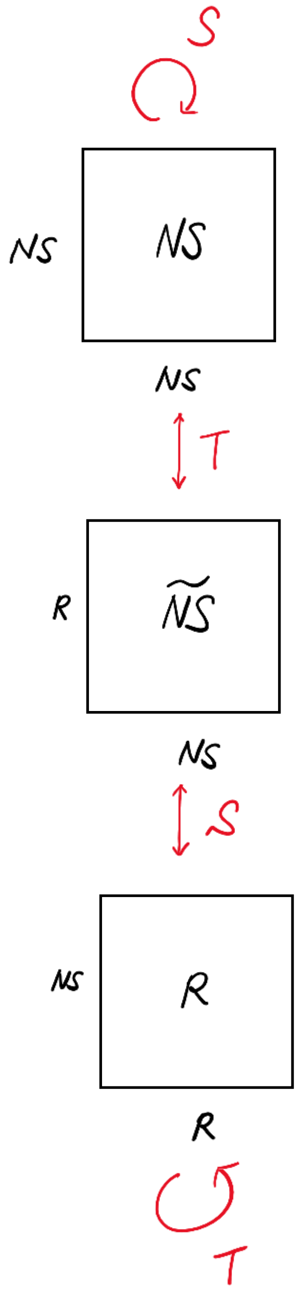
MODULAR SYMMETRY

$$\Gamma_\theta(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) \, , \, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bmod 2 \right\}$$

$$\Gamma^0(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) \, , \, \beta = 0 \bmod 2 \right\}$$

$$\Gamma_0(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) \, , \, \gamma = 0 \bmod 2 \right\}$$

LEVEL-2 CONGRUENCE SUBGROUP OF $SL(2,\mathbb{Z})$



MODULAR SYMMETRY

PARTION FUNCTIONS

$q = e^{2\pi i \tau}$

torus complex parameter

$\Gamma_\theta(2)$

$$Z_{NS} = \mathrm{tr}_{\mathcal{H}_{NS}} \left[q^{L_0 - c/24} \, \bar{q}^{\bar{L}_0 - c/24} \right]$$

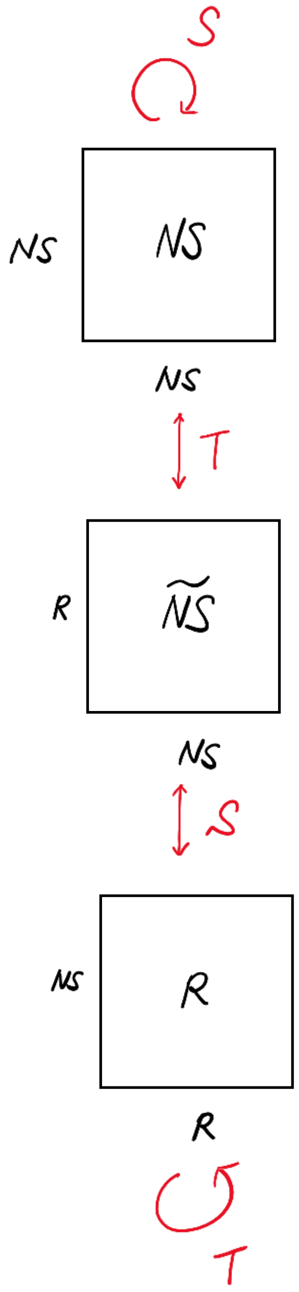
fermion number op.

$\Gamma^0(2)$

$$Z_{\widetilde{NS}} = \mathrm{tr}_{\mathcal{H}_{NS}} \left[(-1)^F q^{L_0 - c/24} \, \bar{q}^{\bar{L}_0 - c/24} \right]$$

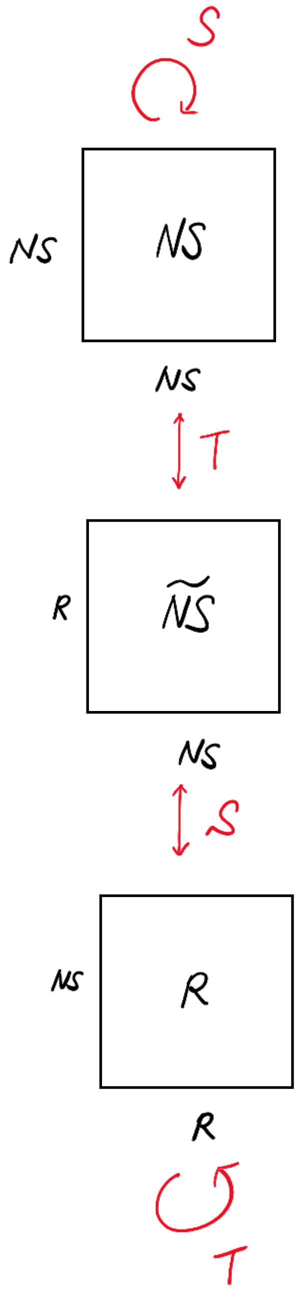
$\Gamma_0(2)$

$$Z_{R} = \mathrm{tr}_{\mathcal{H}_{R}} \left[q^{L_0 - c/24} \, \bar{q}^{\bar{L}_0 - c/24} \right]$$



MODULAR SYMMETRY	PARTION FUNCTIONS	RATIONAL CFTs
$\Gamma_\theta(2)$	$Z_{NS} = \mathrm{tr}_{\mathcal{H}_{NS}}\left[q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - c/24} \right]$	FINITE SUM! $= \sum_{i,j=0}^{d-1} M_{ij}\, \chi_i^{NS}(\tau)\, \bar{\chi}_j^{NS}(\bar{\tau})$ conformal characters
$\Gamma^0(2)$	fermion number op. $Z_{\widetilde{NS}} = \mathrm{tr}_{\mathcal{H}_{NS}}\left[(-1)^F q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - c/24} \right]$	$= \sum_{i,j=0}^{d-1} M_{ij}\, \chi_i^{\widetilde{NS}}(\tau)\, \bar{\chi}_j^{\widetilde{NS}}(\bar{\tau})$
$\Gamma_0(2)$	$Z_{R} = \mathrm{tr}_{\mathcal{H}_{R}}\left[q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - c/24} \right]$	$= \sum_{i,j=0}^{d-1} N_{ij}\, \chi_i^{R}(\tau)\, \bar{\chi}_j^{R}(\bar{\tau})$

$\{ M_{ij},\ N_{ij} : \text{constant matrices} \}$



MODULAR SYMMETRY

RATIONAL CFTs

CONFORMAL CHARACTER

$\Gamma_\theta(2)$

$$Z_{NS} = \sum_{i,j=0}^{d-1} M_{ij}\, \chi_i^{NS}(\tau)\, \bar{\chi}_j^{NS}(\bar{\tau})$$

conformal characters

$\chi_i^{NS}(\tau)$: $\Gamma_\theta(2)$ vector-valued modular form

$$\chi_i^{NS}(\tau+2) = \rho_{NS}(T^2)_{ij}\, \chi_j^{NS}(\tau)$$

$$\chi_i^{NS}\left(-\frac{1}{\tau}\right) = \rho_{NS}(S)_{ij}\, \chi_j^{NS}(\tau)$$

$\rho_{NS}(g)$: d-dim'l rep. of $g \in \Gamma_\theta(2)$

$\Gamma^0(2)$

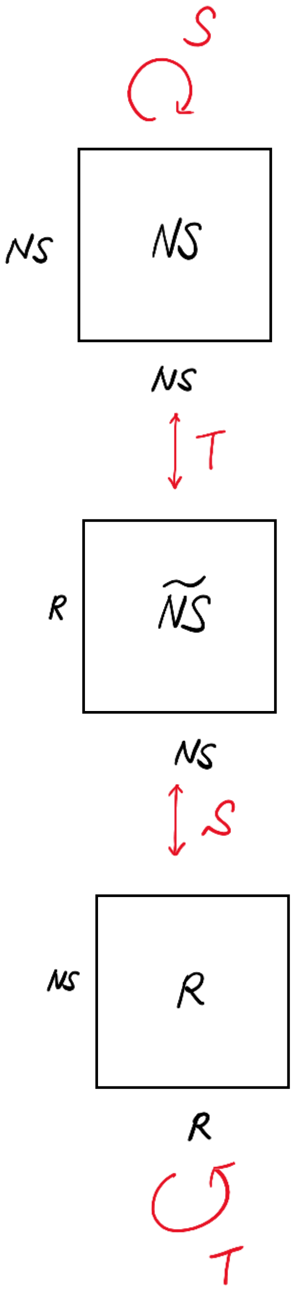
$$Z_{\widetilde{NS}} = \sum_{i,j=0}^{d-1} \widetilde{M}_{ij}\, \chi_i^{\widetilde{NS}}(\tau)\, \bar{\chi}_j^{\widetilde{NS}}(\bar{\tau})$$

$\Gamma_0(2)$

$$Z_{R} = \sum_{i,j=0}^{d-1} N_{ij}\, \chi_i^{R}(\tau)\, \bar{\chi}_j^{R}(\bar{\tau})$$

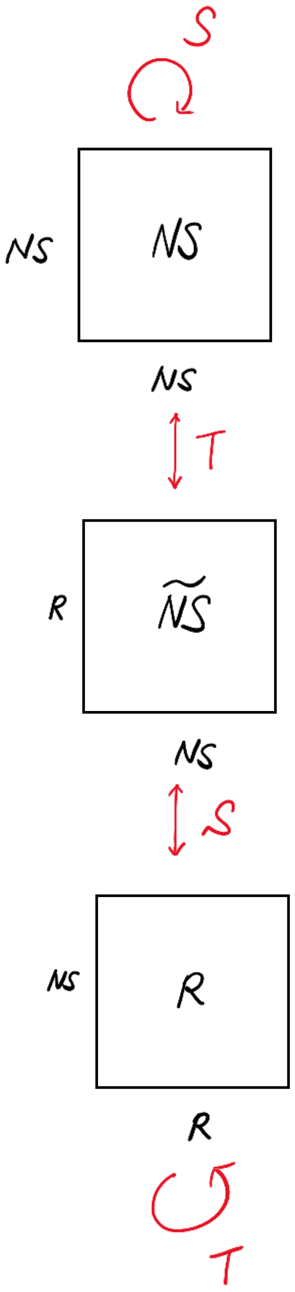
LEVEL-2 CONGRUENCE SUBGROUP OF $SL(2,\mathbb{Z})$

$\{ M_{ij},\ \tilde{M}_{ij},\ N_{ij}$: constant matrices$\}$



MODULAR SYMMETRY	RATIONAL CFTs	CONFORMAL CHARACTER
$\Gamma_\theta(2)$	$Z_{NS} = \sum_{i,j=0}^{d-1} M_{ij}\, \chi_i^{NS}(\tau)\, \bar{\chi}_j^{NS}(\bar{\tau})$ conformal characters	$\chi_i^{NS}(\tau)$: $\Gamma_\theta(2)$ vector-valued modular form
$\Gamma^0(2)$	$Z_{\widetilde{NS}} = \sum_{i,j=0}^{d-1} \widetilde{M}_{ij}\, \chi_i^{\widetilde{NS}}(\tau)\, \bar{\chi}_j^{\widetilde{NS}}(\bar{\tau})$	$\chi_i^{\widetilde{NS}}(\tau)$: $\Gamma^0(2)$ vector-valued modular form
$\Gamma_0(2)$	$Z_R = \sum_{i,j=0}^{d-1} N_{ij}\, \chi_i^{R}(\tau)\, \bar{\chi}_j^{R}(\bar{\tau})$	$\chi_i^{R}(\tau)$: $\Gamma_0(2)$ vector-valued modular form

LEVEL-2 CONGRUENCE SUBGROUP OF $SL(2,\mathbb{Z})$



Conformal Character

$$\chi_i^{NS}(q) \underset{\tau \to i\infty}{\simeq} q^{\alpha_i^{NS}} \left(a_0 + a_{1/2}\, q^{1/2} + a_1 q^1 + \cdots \right)$$

$$\alpha_i^{NS} = h_i^{NS} - c/24$$

central charge

conformal weights

$$\chi_i^{\widetilde{NS}}(q) \simeq q^{\alpha_i^{NS}} \left(a_0 - a_{1/2}\, q^{1/2} + a_1 q^1 - a_{3/2}\, q^{3/2} + \cdots \right)$$

$$\chi_i^{R}(q) \simeq q^{\alpha_i^{R}} \left(b_0 + b_1 q^1 + b_2 q^2 + \cdots \right)$$

$$\alpha_i^{R} = h_i^{R} - c/24$$

conformal weights

' $\{ \alpha_i^{NS}, \alpha_i^{R} \}$ are rational-valued. '

Properties

(i) Integrality

Fourier coefficients of $\chi_i^{NS}, \chi_i^{\widetilde{NS}}, \chi_i^{R}$ are all integer-valued.

(ii) Positivity

$a_n, b_n \geq 0$! operator counting

(iii) Weak Holomorphicity

Conformal characters are hol. inside the fundamental domain

(iv) The Unique Vacuum

$$\chi_{vac}^{NS} \simeq q^{-c/24} \left(1 + a_{\frac{1}{2}}\, q^{\frac{1}{2}} + \cdots \right)$$

unique state $\leftrightarrow$ identity op.

(v) (optional) Unitarity $c, h_i^{NS}, h_i^{R} \geq 0$

$$\left\{ \begin{array}{c} \text{Space of} \\ \text{fermionic} \\ \text{RCFTs} \end{array} \right\} \Leftarrow \left\{ \begin{array}{l} \text{Space of} \\ \text{v.v.m.f.} \\ \text{of } \Gamma_\theta(2), \Gamma_0(2) \end{array} \right\}$$

integrality

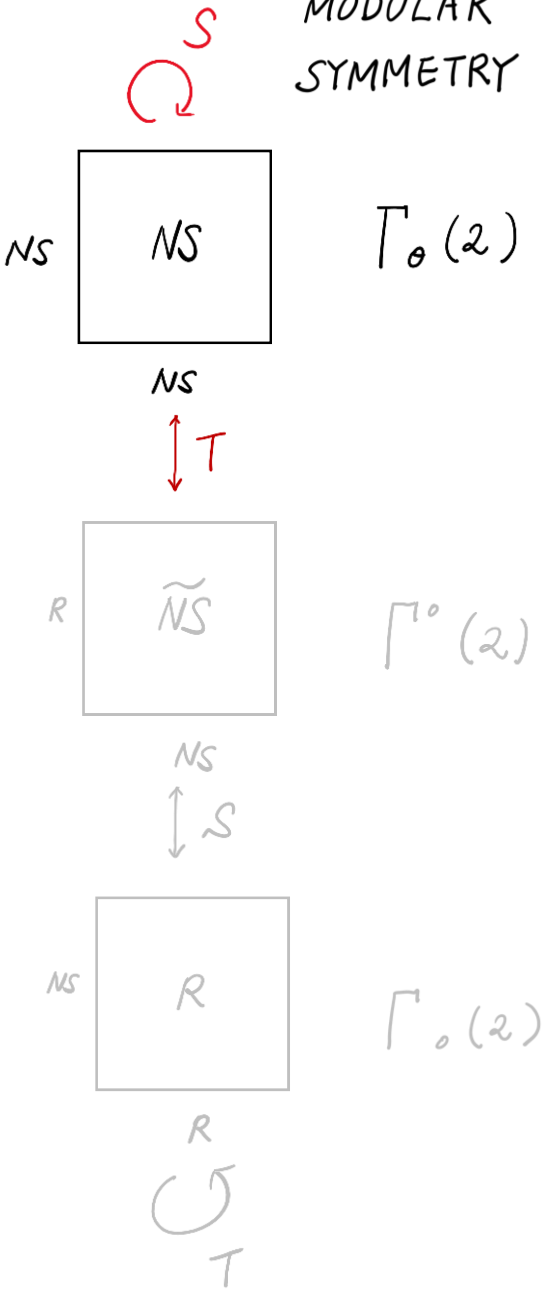
positivity

weak hol.

the unique vacuum

(unitarity)

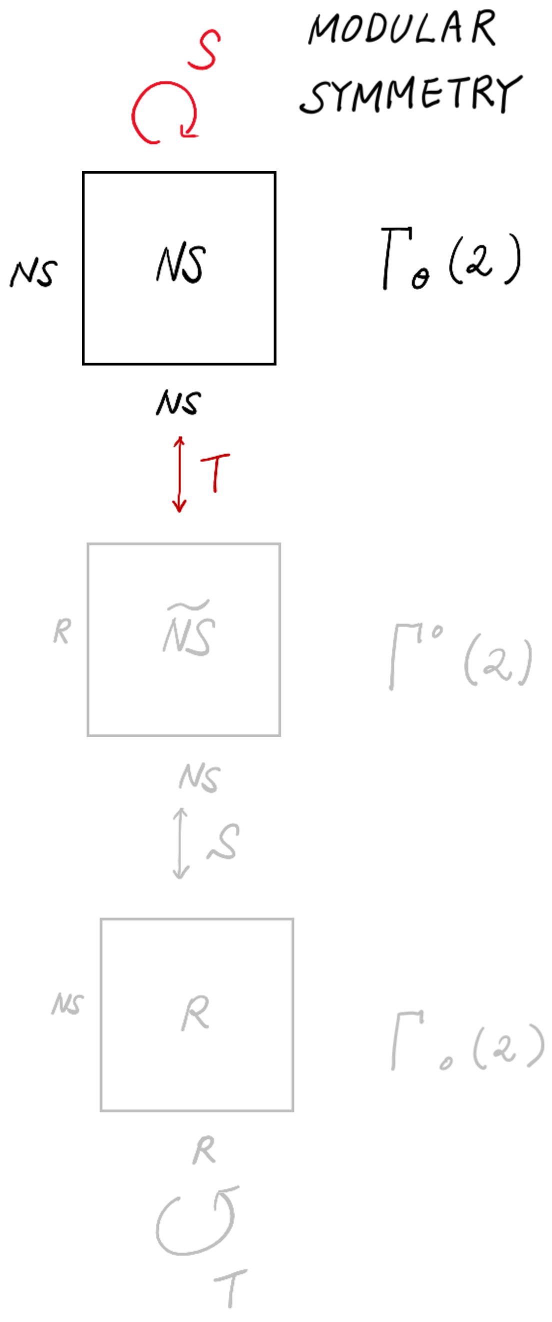
MODULAR SYMMETRY



MODULAR LINEAR DIFFERENTIAL EQUATION

systematic & practical approach to study the space of v.v.m.f.

(Mathur, Mukhi, Sen)



MODULAR LINEAR DIFFERENTIAL EQUATION

$$\underbrace{\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{NS}(\tau) \cdot D_\tau^k \right]}_{\text{invariant under } \Gamma_\theta(2)} f^{NS}(\tau) = 0$$

SERRE DERIVATIVE

$$D_\tau := \frac{1}{2\pi i} \frac{d}{d\tau} - \frac{m}{12} E_2(\tau)$$

weight

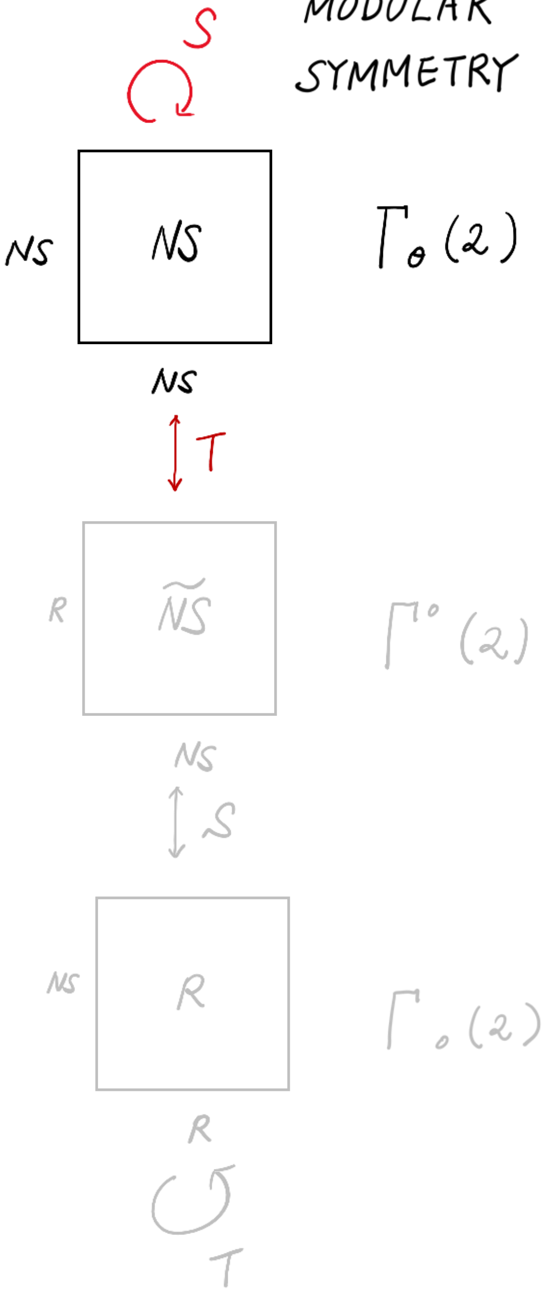
2nd Eisenstein series

$\phi_k^{NS}(\tau)$: modular form of weight $2(d-k)$ w.r.t. $\Gamma_\theta(2)$

$$\left\{ \begin{matrix} d \text{ independent} \\ \text{solutions} \end{matrix} \right\} \xrightarrow{\Gamma_\theta(2)} \left\{ \begin{matrix} d \text{ independent} \\ \text{solutions} \end{matrix} \right\}$$

Solutions are VECTOR-VALUED MODULAR FORMS for $\Gamma_\theta(2)$

MODULAR SYMMETRY



MODULAR LINEAR DIFFERENTIAL EQUATION

$$\left[\mathcal{D}_\tau^d + \sum_{k=0}^{d-1} \phi_k^{NS}(\tau) \cdot \mathcal{D}_\tau^k \right] f^{NS}(\tau) = 0$$

$\phi_k^{NS}(\tau)$: modular form of weight $2(d-k)$ w.r.t. $\Gamma_\theta(2)$

$\chi_i^{NS}(\tau)$: d-independent solutions

$$\phi_k^{NS}(\tau) = \frac{W_k(\tau)}{W_d(\tau)}$$

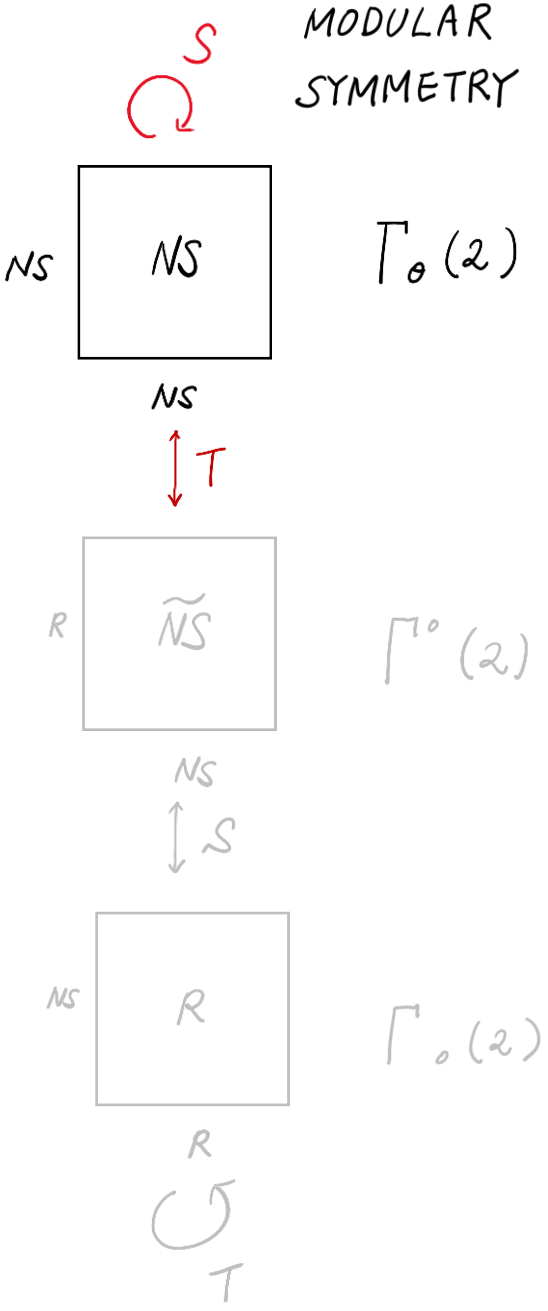
hol. inside $\mathcal{F}$

$\exists$ poles at zeroes of $W_d(\tau)$

$$W_k = \det \begin{pmatrix} \chi_0 & \cdots & \chi_{N-1} \\ \mathcal{D}\chi_0 & \cdots & \mathcal{D}\chi_{N-1} \\ \vdots & & \vdots \\ \mathcal{D}^{k-1}\chi_0 & \cdots & \mathcal{D}^{k-1}\chi_{N-1} \\ \mathcal{D}^{k+1}\chi_0 & \cdots & \mathcal{D}^{k+1}\chi_{N-1} \\ \vdots & & \vdots \\ \mathcal{D}^{N}\chi_0 & \cdots & \mathcal{D}^{n}\chi_{N-1} \end{pmatrix}$$

Weakly holomorphic when χ_i^{NS} are conformal characters

$\frac{\ell}{2} \equiv$ order of poles of $\phi_k^{NS}(\tau)$



MODULAR LINEAR DIFFERENTIAL EQUATION

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{NS}(\tau) \cdot D_\tau^k \right] f^{NS}(\tau) = 0$$

$\phi_k^{NS}(\tau)$: modular form of weight $2(d-k)$ w.r.t. $\Gamma_\theta(2)$

$\frac{\ell}{2} \equiv$ order of poles of $\phi_k^{NS}(\tau)$

To study the space of solutions to MLDE we need to determine the coefficient functions $\phi_k^{NS}(\tau)$ by physical data

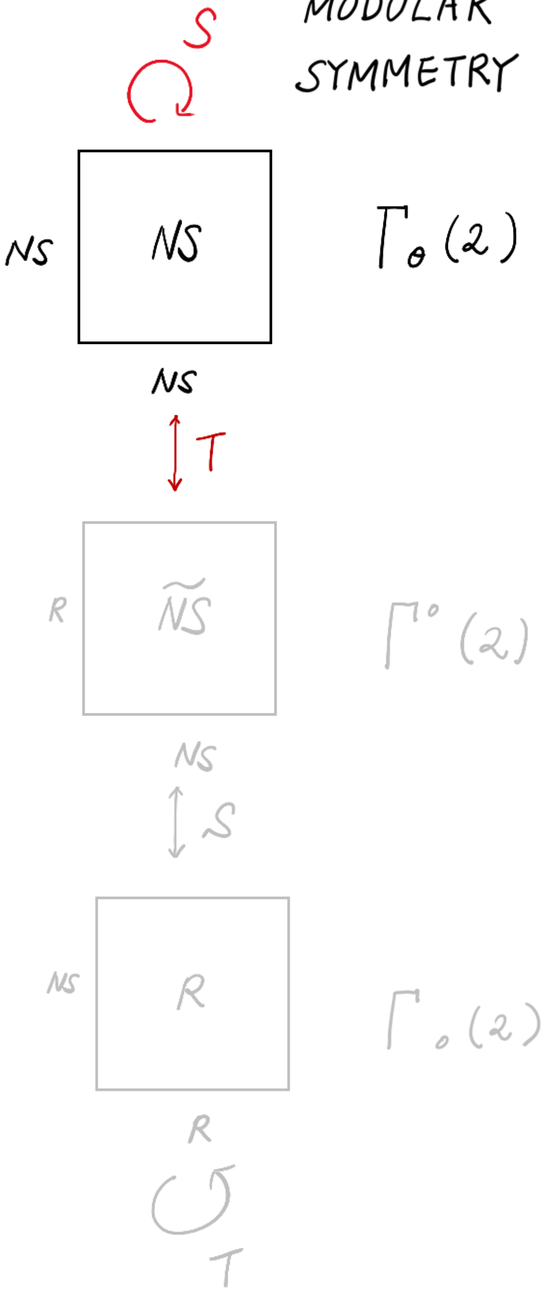
	space of $\phi_k^{NS}(\tau)$
$\ell = 0$	finite dimensional
$\ell = 1$	finite dimensional

space of hol. modular form of weight $2k$

$$\mathcal{M}_{(2k)} = \left\langle (-\vartheta_2^4)^r \vartheta_4^{4s} + (-\vartheta_2^4)^s \vartheta_4^{4r} \,,\ r \le s \ \&\ r+s=k \right\rangle$$

$\phi_k^{NS}(\tau)$ have poles only at $\tau = i$

MODULAR SYMMETRY



MODULAR LINEAR DIFFERENTIAL EQUATION

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{NS}(\tau) \cdot D_\tau^k \right] f^{NS}(\tau) = 0$$

$\phi_k^{NS}(\tau)$: modular form of weight $2(d-k)$ w.r.t. $\Gamma_\theta(2)$

$\frac{\ell}{2} \equiv$ order of poles of $\phi_k^{NS}(\tau)$

e.g. $d=2$, $\ell \leq 1$

$$\left[D_\tau^2 + \phi_1^{NS}(\tau) D_\tau + \phi_0^{NS}(\tau) \right] f^{NS}(\tau) = 0$$

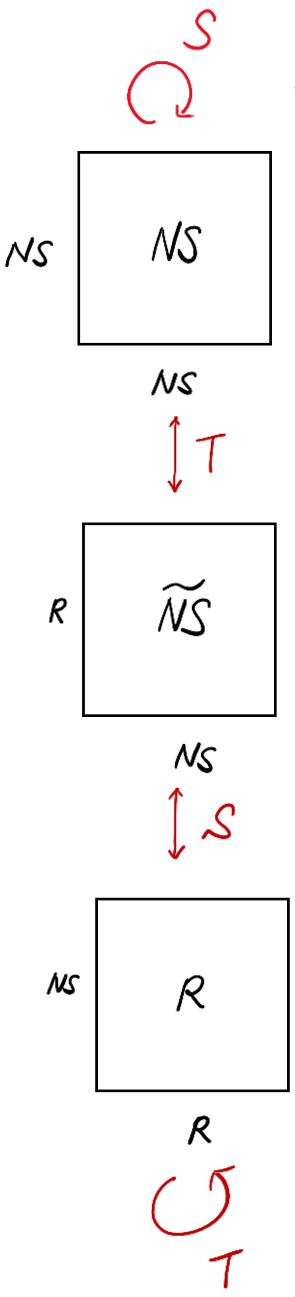
$$\phi_0^{NS}(\tau) = \mu_3 (\vartheta_2^8 + \vartheta_4^8) + \mu_4 \vartheta_2^4 \vartheta_4^4$$

$$\phi_1^{NS}(\tau) = \frac{\mu_1 (\vartheta_2^8 + \vartheta_4^8) + \mu_2 \vartheta_2^4 \vartheta_4^4}{\vartheta_4^4 - \vartheta_2^4}$$

$\exists$ four parameters

$$\mu_a = \mu_a(\alpha_i^{NS}, \alpha_i^R)$$

they are determined by $\{c, h_1^{NS}, h_0^R, h_1^R\}$



MODULAR SYMMETRY

$\Gamma_\theta(2)$

$\Gamma^0(2)$

$\Gamma_0(2)$

MODULAR LINEAR DIFFERENTIAL EQUATION

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{NS}(\tau) \cdot D_\tau^k \right] f^{NS}(\tau) = 0$$

$$\updownarrow T \quad (\vartheta_2^4, \vartheta_3^4, \vartheta_4^4) \to (-\vartheta_2^4, \vartheta_4^4, \vartheta_3^4)$$

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{\widetilde{NS}}(\tau) \cdot D_\tau^k \right] f^{\widetilde{NS}}(\tau) = 0$$

$$\updownarrow S \quad (\vartheta_2^4, \vartheta_3^4, \vartheta_4^4) \to -\tau^2 \cdot (\vartheta_4^4, \vartheta_3^4, \vartheta_2^4)$$

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{R}(\tau) \cdot D_\tau^k \right] f^{R}(\tau) = 0$$

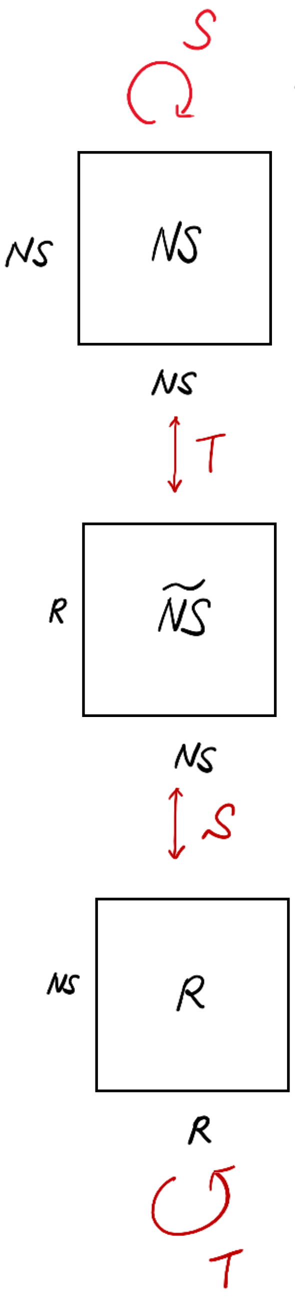
HOLOMORPHIC MODULAR BOOTSTRAP

(i) Insert $\{\alpha_i^{NS}, \alpha_i^{R}\}$ to fix parameters in an MLDE with a given ℓ

(ii) Solve the MLDE

(iii) check if the solutions satisfy the physical conditions

① integrality
② positivity
③ unique vacuum
④ (unitarity)



MODULAR SYMMETRY

MODULAR LINEAR DIFFERENTIAL EQUATION

HOLOMORPHIC MODULAR BOOTSTRAP

$\Gamma_\theta(2)$

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{NS}(\tau) \cdot D_\tau^k \right] f^{NS}(\tau) = 0$$

$$\updownarrow T \quad (\vartheta_2^4, \vartheta_3^4, \vartheta_4^4) \to (-\vartheta_2^4, \vartheta_4^4, \vartheta_3^4)$$

$\Gamma^0(2)$

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{\widetilde{NS}}(\tau) \cdot D_\tau^k \right] f^{\widetilde{NS}}(\tau) = 0$$

$$\updownarrow S \quad (\vartheta_2^4, \vartheta_3^4, \vartheta_4^4) \to -\tau^2 \cdot (\vartheta_4^4, \vartheta_3^4, \vartheta_2^4)$$

$\Gamma_0(2)$

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{R}(\tau) \cdot D_\tau^k \right] f^{R}(\tau) = 0$$

(i) Insert $\{\alpha_i^{NS}, \alpha_i^{R}\}$ to fix parameters in an MLDE with a given ℓ

(ii) Solve the MLDE

NUMERICALLY IMPOSSIBLE TO SCAN ALL POSSIBLE RATIONAL VALUES FOR $\{\alpha_i^{NS}, \alpha_i^{R}\}$

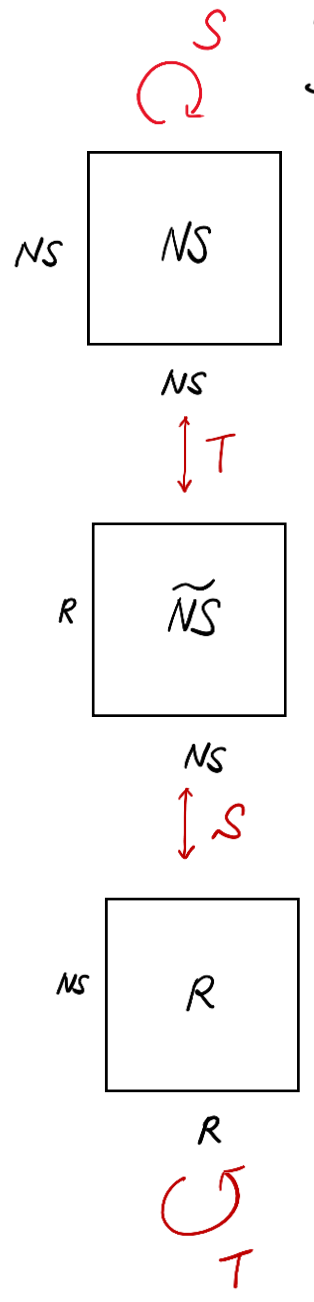
(iii) check if the solutions satisfy physical conditions

① integrality

② positivity

③ unique vacuum

④ (unitarity)



MODULAR SYMMETRY

MODULAR LINEAR DIFFERENTIAL EQUATION

HOLOMORPHIC MODULAR BOOTSTRAP

$\Gamma_\theta(2)$

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{NS}(\tau) \cdot D_\tau^k \right] f^{NS}(\tau) = 0$$

$$\updownarrow T \quad (\vartheta_2^4, \vartheta_3^4, \vartheta_4^4) \to (-\vartheta_2^4, \vartheta_4^4, \vartheta_3^4)$$

$\Gamma^0(2)$

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{\widetilde{NS}}(\tau) \cdot D_\tau^k \right] f^{\widetilde{NS}}(\tau) = 0$$

$$\updownarrow S \quad (\vartheta_2^4, \vartheta_3^4, \vartheta_4^4) \to -\tau^2 \cdot (\vartheta_4^4, \vartheta_3^4, \vartheta_2^4)$$

$\Gamma_0(2)$

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{R}(\tau) \cdot D_\tau^k \right] f^{R}(\tau) = 0$$

(i) Insert $\{\alpha_i^{NS}, \alpha_i^R\}$ to fix parameters in an MLDE with a given ℓ

(ii) Solve the MLDE

(iii) check if the solutions satisfy the physical conditions

ANY HELP FROM

1. integrality
2. positivity
3. unique vacuum
4. (unitarity)

?

INTERALITY CONJECTURE

(N : even integer)

$$\left\{ \text{v.v.m.f. of } \Gamma_\theta(2)\ (\Gamma_0'(2)) \Big/ \text{integrality} \right\} = \left\{ \text{v.v.m.f. of } \underbrace{\Gamma_\theta(2)/\Gamma(N)}_{\text{FINITE GROUP!}}\ \left(\Gamma_0(2)/\Gamma(N)\right) \right\}$$

$$\Gamma(N) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}),\quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

INTERALITY CONJECTURE

$$\underbrace{\left\{\begin{array}{l} \text{v.v.m.f. of} \\ \Gamma_\theta(2)/\Gamma(N) \quad \left(\Gamma_0(2)/\Gamma(N)\right) \end{array}\right\}}_{\text{FINITE GROUP!}} \quad (N:\ \text{even integer})$$

$\longleftarrow$ representation theory of $\Gamma_\theta(2)/\Gamma(N)$

(i) Constrain the possible values of $\{\alpha_i^{NS}, \alpha_i^{R}\}$

(ii) Improve the hol. modular bootstrap method

INTERALITY CONJECTURE

$(N:$ even integer$)$

$$\left\{ \text{v.v.m.f. of } \underbrace{\Gamma_\theta(2)/\Gamma(N)} \right\}$$

2-Sylow subgroup of $SL(2,\mathbb{Z}_N)$

① irrep. of $SL(2,\mathbb{Z}_N)$: classified

② $\exists$ algorithm to obtain the character table of $\Gamma_\theta(2)/\Gamma(N)$

$$\chi^{NS}(\tau) \simeq q^{\alpha^{NS}} \cdot \left(a_0 + a_{\frac{1}{2}} q^{\frac{1}{2}} + a_1 q^1 + \cdots \right)$$

(i) physical meaning of N

integrality conjecture

$$T^N = 1 \in \Gamma_\theta(2)/\Gamma(N) \longrightarrow \chi_i^{NS}(\tau+N) = e^{2\pi i N \alpha_i^{NS}} \chi_i^{NS}(\tau) = \chi_i^{NS}(\tau)$$

$$\therefore \quad \alpha_i^{NS} = \frac{m_i}{N} \mod \mathbb{Z}$$

$$m_i = 0, 1, 2, \cdots, (N-1)$$

INTERALITY CONJECTURE

(N : even integer)

$$\underbrace{\left\{ \text{v.v.m.f. of } \Gamma_\theta(2)/\Gamma(N) \right\}}_{}$$

2-Sylow subgroup of $SL(2,\mathbb{Z}_N)$

① irrep. of $SL(2,\mathbb{Z}_N)$: classified

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(i) physical meaning of N $\quad \alpha_i^{NS} = \frac{m_i}{N} \mod \mathbb{Z}$

$m_i = 0, 1, 2, \cdots, (N-1)$

(ii) for each d, $\exists$ finitely many possible values for N

d	N	Number of irrep.
1	$\{2,4,6,8,12,16,24,48\}$	48 irreps.
2	$\{4,8,12,16,20,24,32,40,48,60,80,96,120,240\}$	300 irreps.
3	$\{6,10,12,14,20,24,28,30,40,42,48,56,60,80,84,$ $112,120,168,240,336\}$	208 irreps.
4	$\{8,10,12,16,18,20,24,28,30,32,36,40,48,56,60,$ $64,72,80,84,96,112,120,144,160,168,192,240,336,480\}$	1206 irreps.

INTERALITY CONJECTURE

(N : even integer)

$$\underbrace{\left\{ \begin{array}{l} \text{v.v.m.f. of} \\ \Gamma_\theta(2)/\Gamma(N) \end{array} \right\}}$$

2 - Sylow subgroup of $SL(2,\mathbb{Z}_N)$

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(iii) character table determines $\{2\alpha_i^{NS}, \alpha_i^{R}\} \mod \mathbb{Z}$

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$$\underbrace{\left\{ \text{v.v.m.f. of } \Gamma_\theta(2)/\Gamma(N) \right\}}$$

2 - Sylow subgroup of $SL(2, \mathbb{Z}_N)$

① irrep. of $SL(2, \mathbb{Z}_N)$: classified

② $\exists$ algorithm to obtain the character table of $\Gamma_\theta(2)/\Gamma(N)$

(iii) character table determines $\{2\alpha_i^{NS}, \alpha_i^R\}$ mod $\mathbb{Z}$ ✓

① $\{2\alpha_i^{NS}\} \longleftrightarrow$ eigenvalues of $\rho_{NS}(T^2)$ $\qquad T^2 \in \Gamma_\theta(2)/\Gamma(N)$

$$\rho_{NS}(T^2) \doteq \begin{bmatrix} e^{4\pi i \alpha_0^{NS}} & & & \\ & e^{4\pi i \alpha_1^{NS}} & & \\ & & \ddots & \\ & & & e^{4\pi i \alpha_{d-1}^{NS}} \end{bmatrix}$$

$$\chi_i^{NS}(\tau+2) = \rho_{NS}(T^2)_{ij}\, \chi_j^{NS}(\tau)$$

$$\doteq e^{4\pi i \alpha_i^{NS}} \cdot \chi_i^{NS}(\tau)$$

$$\chi^{NS}(\tau) \equiv q^{\alpha^{NS}} \cdot \left(a_0 + a_{\frac{1}{2}} q^{\frac{1}{2}} + a_1 q^1 + \cdots \right)$$

(N : even integer)

$$\left\{ \begin{array}{c} \text{v.v.m.f. of} \\ \underbrace{\Gamma_\theta(2) / \Gamma(N)} \end{array} \right\}$$

2-Sylow subgroup of $SL(2, \mathbb{Z}_N)$

① irrep. of $SL(2, \mathbb{Z}_N)$: classified

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① $\{2\alpha_i^{NS}\} \longleftrightarrow$ eigenvalues of $\rho_{NS}(T^2)$

$$\rho_{NS}(T^2) \doteq \begin{bmatrix} e^{4\pi i \alpha_0^{NS}} & & & \\ & e^{4\pi i \alpha_1^{NS}} & & \\ & & \ddots & \\ & & & e^{4\pi i \alpha_{d-1}^{NS}} \end{bmatrix}$$

② character of $(T^2)^\ell$ can be expressed in terms of the eigenvalues

$$ch_\rho\left[(T^2)^\ell\right] = tr\left[\rho_{NS}^\ell(T^2)\right] \qquad \left(\ell = 1, 2, \dots, \frac{N}{2}\right)$$

(N : even integer)

$$\underbrace{\left\{ \text{v.v.m.f. of } \ \Gamma_\theta(2) / \Gamma(N) \right\}}$$

2 - Sylow subgroup of $SL(2,\mathbb{Z}_N)$

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① $\{2\alpha_i^{NS}\} \longleftrightarrow$ eigenvalues of $\rho_{NS}(T^2)$

$$\rho_{NS}(T^2) \doteq \begin{bmatrix} e^{4\pi i \alpha_0^{NS}} & & & \\ & e^{4\pi i \alpha_1^{NS}} & & \\ & & \ddots & \\ & & & e^{4\pi i \alpha_{d-1}^{NS}} \end{bmatrix}$$

② $ch_\rho[(T^2)^\ell] = tr[\rho_{NS}^\ell(T^2)]$ $\quad (\ell = 1, 2, \cdots, \frac{N}{2})$

③ each character can be read off from the character table once the conjugacy class to which $(T^2)^\ell$ belongs

$\longrightarrow$ $\{2\alpha_i^{NS}\}$ mod $\mathbb{Z}$ are determined

(N : even integer)

$$\underbrace{\left\{ \text{v.v.m.f. of } \Gamma_0(2)/\Gamma(N) \right\}}_{\text{2-Sylow subgroup of } SL(2,\mathbb{Z}_N)}$$

① irrep. of $SL(2,\mathbb{Z}_N)$: classified

② $^{\exists}$ algorithm to obtain the character table of $\Gamma_0(2)/\Gamma(N)$

(iii) character table determines $\{2\alpha_i^{NS}, \alpha_i^{R}\}$ mod $\mathbb{Z}$

e.g. $N=4$ & 2-dim'l rep. ρ_q

$T^4 = 1_2$ $\qquad$ T^2

	1a	4a	4b	4c	4d	2a	2b	2c	2d	2e
ch_1	1	1	1	1	1	1	1	1	1	1
ch_2	1	−1	1	−1	1	−1	1	1	−1	1
ch_3	1	−1	−1	−1	−1	1	1	1	1	1
ch_4	1	1	−1	1	−1	−1	1	1	−1	1
ch_5	1	i	$-i$	$-i$	i	1	1	−1	−1	−1
ch_6	1	$-i$	$-i$	i	i	−1	1	−1	1	−1
ch_7	1	$-i$	i	i	$-i$	1	1	−1	−1	−1
ch_8	1	i	i	$-i$	$-i$	−1	1	−1	1	−1
ch_9	2	0	0	0	0	0	−2	2	0	−2
ch_{10}	2	0	0	0	0	0	−2	−2	0	2

$$\rho_q(T^2) \doteq \begin{bmatrix} e^{4\pi i \alpha_0^{NS}} & \\ & e^{4\pi i \alpha_1^{NS}} \end{bmatrix} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \longrightarrow (2\alpha_0^{NS}, 2\alpha_1^{NS}) = (0, 1/2)$$

mod integer

(N : even integer)

$$\underbrace{\left\{ \begin{array}{c} \text{v.v.m.f. of} \\ \Gamma_\theta(2)/\Gamma(N) \end{array} \right\}}_{\text{2-Sylow subgroup of } SL(2,\mathbb{Z}_N)}$$

① irrep. of $SL(2,\mathbb{Z}_N)$: classified

② $\exists$ algorithm to obtain the character table of $\Gamma_\theta(2)/\Gamma(N)$

(iii) character table determines $\{2\alpha_i^{NS}, \alpha_i^{R}\}$ mod $\mathbb{Z}$ ✓

① the character table of $\Gamma_\theta(2)/\Gamma(N)$ knows the Ramond spectrum as well.

$$\rho_{NS}[TS\cdot T\,(TS)^{-1}] = \rho_R[T] \doteq \begin{bmatrix} e^{2\pi i\alpha_0^R} & & & \\ & e^{2\pi i\alpha_1^R} & & \\ & & \ddots & \\ & & & e^{2\pi i\alpha_{d-1}^R} \end{bmatrix}$$

$\because\ |i\rangle_{NS} = (TS)_{ij}\,|j\rangle_R$

$$\mathrm{NS}\ \underset{\mathrm{NS}}{\boxed{\mathrm{NS}}} \xleftrightarrow{T} \mathrm{R}\ \underset{\mathrm{NS}}{\boxed{\widetilde{\mathrm{NS}}}} \xleftrightarrow{S} \mathrm{NS}\ \underset{\mathrm{R}}{\boxed{\mathrm{R}}}$$

(N : even integer)

$$\left\{ \text{v.v.m.f. of } \underbrace{\Gamma_\theta(2) / \Gamma(N)} \right\}$$

2 - Sylow subgroup of $SL(2, \mathbb{Z}_N)$

① irrep. of $SL(2, \mathbb{Z}_N)$: classified

② $\exists$ algorithm to obtain the character table of $\Gamma_\theta(2) / \Gamma(N)$

(iii) character table determines $\{2\alpha_i^{NS}, \alpha_i^R\}$ mod $\mathbb{Z}$

① $$\rho_{NS}\left[TS \cdot T \cdot (TS)^{-1}\right] = \rho_R\left[T\right] \doteq \begin{bmatrix} e^{2\pi i \alpha_0^R} & & & \\ & e^{2\pi i \alpha_1^R} & & \\ & & \ddots & \\ & & & e^{2\pi i \alpha_{d-1}^R} \end{bmatrix}$$

② the characters of $TS \cdot T^\ell \cdot (TS)^{-1}$ determine the eigenvalues

$(\ell = 1, 2, \cdots, N)$

(N : even integer)

$$\left\{ \text{v.v.m.f. of } \; \underbrace{\Gamma_0(2) / \Gamma(N)}_{} \right\}$$

2 - Sylow subgroup of $SL(2, \mathbb{Z}_N)$

① irrep. of $SL(2, \mathbb{Z}_N)$: classified

② $^{\exists}$ algorithm to obtain the character table of $\Gamma_0(2) / \Gamma(N)$

(iii) character table determines $\{ 2\alpha_i^{NS}, \alpha_i^{R} \}$ mod $\mathbb{Z}$

e.g. $N = 4$ & 2-dim'l rep. ρ_q

$g = TS \cdot T \cdot (TS)^{-1}$

	g^4	g		g^3						g^2
	1a	4a	4b	4c	4d	2a	2b	2c	2d	2e
ch_1	1	1	1	1	1	1	1	1	1	1
ch_2	1	-1	1	-1	1	-1	1	1	-1	1
ch_3	1	-1	-1	-1	-1	1	1	1	1	1
ch_4	1	1	-1	1	-1	-1	1	1	-1	1
ch_5	1	i	$-i$	$-i$	i	1	1	-1	-1	-1
ch_6	1	$-i$	$-i$	i	i	-1	1	-1	1	-1
ch_7	1	$-i$	i	i	$-i$	1	1	-1	-1	-1
ch_8	1	i	i	$-i$	$-i$	-1	1	-1	1	-1
ch_9	2	0	0	0	0	0	-2	2	0	-2
ch_{10}	2	0	0	0	0	0	-2	-2	0	2

$$\rho_q(g) \doteq \begin{bmatrix} e^{2\pi i \alpha_0^R} & \\ & e^{2\pi i \alpha_1^R} \end{bmatrix} = \begin{bmatrix} i & \\ & -i \end{bmatrix} \longrightarrow (\alpha_0^R, \alpha_1^R) = \left(\frac{1}{2}, \frac{3}{2} \right)$$

mod integer

(N : even integer)

$$\underbrace{\left\{ \text{v.v.m.f. of } \Gamma_0(2)/\Gamma(N) \right\}}_{}$$

2 - Sylow subgroup of $SL(2, \mathbb{Z}_N)$

① irrep. of $SL(2, \mathbb{Z}_N)$: classified

② $\exists$ algorithm to obtain the character table of $\Gamma_0(2)/\Gamma(N)$

$\{2\alpha_i^{NS}, \alpha_i^{R}\}$ candidates

$d = 2$ exponents: $\{2\alpha^{\mathrm{NS}}, \alpha^{\mathrm{R}}\} + \frac{k}{24}\{-\mathbf{1}_2, \mathbf{1}_2\}$ for $k \in \mathbb{Z}$

No.	Representative	No.	Representative
1	$\{(0, \frac{1}{8}), (\frac{7}{16}, \frac{15}{16})\}$	8	$\{(0, \frac{1}{3}), (\frac{1}{4}, \frac{11}{12})\}$
2	$\{(0, \frac{1}{4}), (\frac{3}{8}, \frac{7}{8})\}$	9	$\{(0, \frac{1}{3}), (\frac{5}{12}, \frac{3}{4})\}$
3	$\{(0, \frac{3}{8}), (\frac{5}{16}, \frac{13}{16})\}$	10	$\{(\frac{1}{120}, \frac{49}{120}), (\frac{17}{120}, \frac{113}{120})\}$
4	$\{(0, \frac{1}{2}), (\frac{1}{8}, \frac{7}{8})\}$	11	$\{(\frac{1}{120}, \frac{49}{120}), (\frac{53}{120}, \frac{77}{120})\}$
5	$\{(0, \frac{1}{2}), (\frac{1}{4}, \frac{3}{4})\}$	12	$\{(\frac{1}{60}, \frac{49}{60}), (\frac{1}{30}, \frac{19}{30})\}$
6	$\{(\frac{1}{48}, \frac{25}{48}), (\frac{1}{24}, \frac{11}{12})\}$	13	$\{(\frac{1}{60}, \frac{49}{60}), (\frac{2}{15}, \frac{8}{15})\}$
7	$\{(\frac{1}{48}, \frac{25}{48}), (\frac{1}{6}, \frac{19}{24})\}$		

$d = 3$ exponents: $\{2\alpha^{\mathrm{NS}}, \alpha^{\mathrm{R}}\} + \frac{k}{24}\{-\mathbf{1}_3, \mathbf{1}_3\}$ for $k \in \mathbb{Z}$

No.	Representative	No.	Representative
1	$\{(0, \frac{1}{5}, \frac{4}{5}), (0, \frac{2}{5}, \frac{3}{5})\}$	6	$\{(0, \frac{1}{3}, \frac{2}{3}), (0, \frac{1}{3}, \frac{2}{3})\}$
2	$\{(0, \frac{1}{5}, \frac{4}{5}), (\frac{1}{10}, \frac{1}{2}, \frac{9}{10})\}$	7	$\{(\frac{1}{168}, \frac{25}{168}, \frac{121}{168}), (\frac{11}{168}, \frac{107}{168}, \frac{155}{168})\}$
3	$\{(0, \frac{2}{5}, \frac{3}{5}), (0, \frac{1}{5}, \frac{4}{5})\}$	8	$\{(\frac{1}{168}, \frac{25}{168}, \frac{121}{168}), (\frac{23}{168}, \frac{71}{168}, \frac{95}{168})\}$
4	$\{(0, \frac{2}{5}, \frac{3}{5}), (\frac{3}{10}, \frac{1}{2}, \frac{7}{10})\}$	9	$\{(\frac{1}{56}, \frac{9}{56}, \frac{25}{56}), (\frac{11}{56}, \frac{43}{56}, \frac{51}{56})\}$
5	$\{(0, \frac{1}{3}, \frac{2}{3}), (\frac{1}{6}, \frac{1}{2}, \frac{5}{6})\}$	10	$\{(\frac{1}{56}, \frac{9}{56}, \frac{25}{56}), (\frac{15}{56}, \frac{23}{56}, \frac{39}{56})\}$

$\{2\alpha_i^{NS}, \alpha_i^{R}\}$ candidates

(N : even integer)

$$\underbrace{\left\{ \text{v.v.m.f. of } \Gamma_0(2)/\Gamma(N) \right\}}$$

2-Sylow subgroup of $SL(2, \mathbb{Z}_N)$

① irrep. of $SL(2, \mathbb{Z}_N)$: classified

② $\exists$ algorithm to obtain the character table of $\Gamma_0(2)/\Gamma(N)$

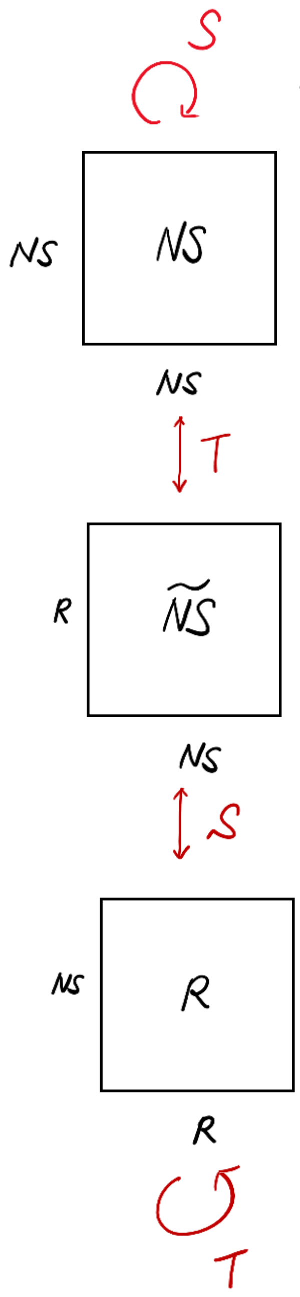
$d = 4$ exponents: $\{2\alpha^{\mathrm{NS}}, \alpha^{\mathrm{R}}\} + \frac{k}{24}\{-\mathbf{1}_4, \mathbf{1}_4\}$ for $k \in \mathbb{Z}$

No.	Representative	No.	Representative
1	$\{(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}),(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})\}$	29	$\{(\frac{1}{120},\frac{3}{40},\frac{49}{120},\frac{27}{40}),(\frac{23}{120},\frac{47}{120},\frac{21}{40},\frac{29}{40})\}$
2	$\{(0,\frac{1}{24},\frac{3}{8},\frac{2}{3}),(\frac{1}{16},\frac{19}{48},\frac{9}{16},\frac{43}{48})\}$	30	$\{(\frac{1}{120},\frac{2}{15},\frac{49}{120},\frac{8}{15}),(\frac{19}{240},\frac{91}{240},\frac{139}{240},\frac{211}{240})\}$
3	$\{(0,\frac{1}{16},\frac{1}{4},\frac{9}{16}),(\frac{5}{32},\frac{13}{32},\frac{21}{32},\frac{29}{32})\}$	31	$\{(\frac{1}{120},\frac{19}{120},\frac{49}{120},\frac{91}{120}),(\frac{1}{15},\frac{4}{15},\frac{17}{30},\frac{23}{30})\}$
4	$\{(0,\frac{1}{16},\frac{9}{16},\frac{3}{4}),(\frac{1}{32},\frac{9}{32},\frac{17}{32},\frac{25}{32})\}$	32	$\{(\frac{1}{120},\frac{49}{120},\frac{61}{120},\frac{109}{120}),(\frac{1}{60},\frac{1}{15},\frac{4}{15},\frac{49}{60})\}$
5	$\{(0,\frac{1}{12},\frac{1}{3},\frac{3}{4}),(\frac{1}{24},\frac{3}{8},\frac{13}{24},\frac{7}{8})\}$	33	$\{(\frac{1}{120},\frac{49}{120},\frac{61}{120},\frac{109}{120}),(\frac{23}{120},\frac{47}{120},\frac{83}{120},\frac{107}{120})\}$
6	$\{(0,\frac{1}{9},\frac{4}{9},\frac{7}{9}),(0,\frac{2}{9},\frac{5}{9},\frac{8}{9})\}$	34	$\{(\frac{1}{120},\frac{49}{120},\frac{73}{120},\frac{97}{120}),(\frac{11}{120},\frac{59}{120},\frac{83}{120},\frac{107}{120})\}$
7	$\{(0,\frac{1}{9},\frac{4}{9},\frac{7}{9}),(\frac{1}{18},\frac{7}{18},\frac{1}{2},\frac{13}{18})\}$	35	$\{(\frac{1}{120},\frac{49}{120},\frac{73}{120},\frac{97}{120}),(\frac{17}{120},\frac{41}{120},\frac{89}{120},\frac{113}{120})\}$
8	$\{(0,\frac{1}{9},\frac{4}{9},\frac{7}{9}),(\frac{5}{36},\frac{1}{4},\frac{17}{36},\frac{29}{36})\}$	36	$\{(\frac{1}{120},\frac{49}{120},\frac{73}{120},\frac{97}{120}),(\frac{29}{120},\frac{53}{120},\frac{77}{120},\frac{101}{120})\}$
9	$\{(0,\frac{1}{9},\frac{4}{9},\frac{7}{9}),(\frac{11}{36},\frac{23}{36},\frac{3}{4},\frac{35}{36})\}$	37	$\{(\frac{1}{96},\frac{25}{96},\frac{49}{96},\frac{73}{96}),(\frac{1}{48},\frac{1}{12},\frac{25}{48},\frac{5}{6})\}$
10	$\{(0,\frac{1}{8},\frac{1}{3},\frac{11}{24}),(\frac{3}{16},\frac{17}{48},\frac{11}{16},\frac{41}{48})\}$	38	$\{(\frac{1}{96},\frac{25}{96},\frac{49}{96},\frac{73}{96}),(\frac{7}{48},\frac{5}{24},\frac{11}{24},\frac{31}{48})\}$
11	$\{(0,\frac{1}{8},\frac{1}{2},\frac{5}{8}),(\frac{1}{16},\frac{5}{16},\frac{9}{16},\frac{13}{16})\}$	39	$\{(\frac{1}{80},\frac{9}{80},\frac{41}{80},\frac{49}{80}),(\frac{1}{40},\frac{9}{40},\frac{13}{20},\frac{17}{20})\}$
12	$\{(0,\frac{1}{7},\frac{2}{7},\frac{4}{7}),(\frac{1}{28},\frac{9}{28},\frac{3}{4},\frac{25}{28})\}$	40	$\{(\frac{1}{80},\frac{9}{80},\frac{41}{80},\frac{49}{80}),(\frac{1}{10},\frac{31}{40},\frac{9}{10},\frac{39}{40})\}$
13	$\{(0,\frac{1}{7},\frac{2}{7},\frac{4}{7}),(\frac{1}{4},\frac{11}{28},\frac{15}{28},\frac{23}{28})\}$	41	$\{(\frac{1}{60},\frac{1}{15},\frac{4}{15},\frac{49}{60}),(\frac{1}{120},\frac{49}{120},\frac{61}{120},\frac{109}{120})\}$
14	$\{(0,\frac{1}{6},\frac{1}{2},\frac{2}{3}),(0,\frac{1}{3},\frac{1}{2},\frac{5}{6})\}$	42	$\{(\frac{1}{60},\frac{17}{120},\frac{49}{60},\frac{113}{120}),(\frac{17}{240},\frac{113}{240},\frac{137}{240},\frac{233}{240})\}$
15	$\{(0,\frac{1}{6},\frac{1}{2},\frac{2}{3}),(\frac{1}{8},\frac{3}{8},\frac{11}{24},\frac{17}{24})\}$	43	$\{(\frac{1}{60},\frac{3}{20},\frac{7}{20},\frac{49}{60}),(\frac{1}{20},\frac{23}{60},\frac{9}{20},\frac{47}{60})\}$
16	$\{(0,\frac{3}{16},\frac{1}{4},\frac{11}{16}),(\frac{3}{32},\frac{11}{32},\frac{19}{32},\frac{27}{32})\}$	44	$\{(\frac{1}{60},\frac{3}{20},\frac{7}{20},\frac{49}{60}),(\frac{17}{60},\frac{11}{20},\frac{53}{60},\frac{19}{20})\}$
17	$\{(0,\frac{3}{16},\frac{11}{16},\frac{3}{4}),(\frac{7}{32},\frac{15}{32},\frac{23}{32},\frac{31}{32})\}$	45	$\{(\frac{1}{60},\frac{23}{120},\frac{47}{120},\frac{49}{60}),(\frac{83}{240},\frac{107}{240},\frac{203}{240},\frac{227}{240})\}$
18	$\{(0,\frac{2}{9},\frac{5}{9},\frac{8}{9}),(0,\frac{1}{9},\frac{4}{9},\frac{7}{9})\}$	46	$\{(\frac{1}{60},\frac{19}{60},\frac{31}{60},\frac{49}{60}),(\frac{1}{120},\frac{31}{120},\frac{49}{120},\frac{79}{120})\}$
19	$\{(0,\frac{2}{9},\frac{5}{9},\frac{8}{9}),(\frac{1}{36},\frac{1}{4},\frac{13}{36},\frac{25}{36})\}$	47	$\{(\frac{1}{60},\frac{19}{60},\frac{31}{60},\frac{49}{60}),(\frac{17}{60},\frac{23}{60},\frac{47}{60},\frac{53}{60})\}$
20	$\{(0,\frac{2}{9},\frac{5}{9},\frac{8}{9}),(\frac{7}{36},\frac{19}{36},\frac{3}{4},\frac{31}{36})\}$	48	$\{(\frac{1}{48},\frac{7}{48},\frac{25}{48},\frac{31}{48}),(\frac{5}{48},\frac{11}{48},\frac{29}{48},\frac{35}{48})\}$
21	$\{(0,\frac{2}{9},\frac{5}{9},\frac{8}{9}),(\frac{5}{18},\frac{1}{2},\frac{11}{18},\frac{17}{18})\}$	49	$\{(\frac{1}{48},\frac{7}{48},\frac{25}{48},\frac{31}{48}),(\frac{17}{48},\frac{23}{48},\frac{41}{48},\frac{47}{48})\}$
22	$\{(0,\frac{1}{4},\frac{1}{2},\frac{3}{4}),(0,\frac{1}{4},\frac{1}{2},\frac{3}{4})\}$	50	$\{(\frac{1}{48},\frac{3}{16},\frac{25}{48},\frac{11}{16}),(0,\frac{1}{8},\frac{2}{3},\frac{19}{24})\}$
23	$\{(0,\frac{3}{7},\frac{5}{7},\frac{6}{7}),(\frac{3}{28},\frac{1}{4},\frac{19}{28},\frac{27}{28})\}$	51	$\{(\frac{1}{48},\frac{3}{16},\frac{25}{48},\frac{11}{16}),(\frac{1}{24},\frac{3}{8},\frac{5}{12},\frac{3}{4})\}$
24	$\{(0,\frac{3}{7},\frac{5}{7},\frac{6}{7}),(\frac{5}{28},\frac{13}{28},\frac{17}{28},\frac{3}{4})\}$	52	$\{(\frac{1}{48},\frac{13}{48},\frac{25}{48},\frac{37}{48}),(\frac{1}{24},\frac{1}{6},\frac{13}{24},\frac{2}{3})\}$
25	$\{(\frac{1}{240},\frac{49}{240},\frac{121}{240},\frac{169}{240}),(\frac{1}{120},\frac{17}{60},\frac{49}{120},\frac{53}{60})\}$	53	$\{(\frac{1}{48},\frac{13}{48},\frac{25}{48},\frac{37}{48}),(\frac{5}{48},\frac{17}{48},\frac{29}{48},\frac{41}{48})\}$
26	$\{(\frac{1}{240},\frac{49}{240},\frac{121}{240},\frac{169}{240}),(\frac{1}{30},\frac{31}{120},\frac{19}{30},\frac{79}{120})\}$	54	$\{(\frac{1}{48},\frac{13}{48},\frac{25}{48},\frac{37}{48}),(\frac{11}{48},\frac{23}{48},\frac{35}{48},\frac{47}{48})\}$
27	$\{(\frac{1}{120},\frac{1}{30},\frac{49}{120},\frac{19}{30}),(\frac{31}{240},\frac{79}{240},\frac{151}{240},\frac{199}{240})\}$	55	$\{(\frac{1}{32},\frac{9}{32},\frac{17}{32},\frac{25}{32}),(0,\frac{1}{16},\frac{9}{16},\frac{3}{4})\}$
28	$\{(\frac{1}{120},\frac{3}{40},\frac{49}{120},\frac{27}{40}),(\frac{1}{40},\frac{9}{40},\frac{83}{120},\frac{107}{120})\}$	56	$\{(\frac{1}{32},\frac{9}{32},\frac{17}{32},\frac{25}{32}),(\frac{1}{8},\frac{3}{16},\frac{3}{8},\frac{11}{16})\}$

MODULAR SYMMETRY

MODULAR LINEAR DIFFERENTIAL EQUATION

EHANCED HOLOMORPHIC MODULAR BOOTSTRAP



$\Gamma_\theta(2)$

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{NS}(\tau) \cdot D_\tau^k \right] f^{NS}(\tau) = 0$$

$\updownarrow T \quad (\vartheta_2^4, \vartheta_3^4, \vartheta_4^4) \to (-\vartheta_2^4, \vartheta_4^4, \vartheta_3^4)$

$\Gamma^0(2)$

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{\widetilde{NS}}(\tau) \cdot D_\tau^k \right] f^{\widetilde{NS}}(\tau) = 0$$

$\updownarrow S \quad (\vartheta_2^4, \vartheta_3^4, \vartheta_4^4) \to -\tau^2 \cdot (\vartheta_4^4, \vartheta_3^4, \vartheta_2^4)$

$\Gamma_0(2)$

$$\left[D_\tau^d + \sum_{k=0}^{d-1} \phi_k^{R}(\tau) \cdot D_\tau^k \right] f^{R}(\tau) = 0$$

(i) Insert $\{2\alpha_i^{NS}, \alpha_i^R\}$ to fix parameters in an MLDE with a given ℓ

We now have the data as the consequence of integrality

(ii) Solve the MLDE

(iii) check if the solutions satisfy the physical conditions

1. integrality
2. positivity
3. unique vacuum
4. (unitarity)

CLASSIFICATION OF FERMIONIC $RCFT_2$

Rank two, unitary, $\ell=0$, $\{c, h_1^{\mathrm{NS}}, h_0^{\mathrm{R}}, h_1^{\mathrm{R}}\}_{\mathrm{type}}$		
$\{\frac{7}{10}, \frac{1}{10}, \frac{3}{80}, \frac{7}{16}\}_B$	$\{11, \frac{5}{6}, \frac{11}{24}, \frac{9}{8}\}_S$	$\{21, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}\}_B$
$\{\frac{3}{4}, \frac{1}{4}, \frac{1}{32}, \frac{5}{32}\}_S$	$\{\frac{133}{10}, \frac{9}{10}, \frac{57}{80}, \frac{21}{16}\}_B$	$\{\frac{85}{4}, \frac{5}{4}, \frac{51}{32}, \frac{55}{32}\}_B$
$\{1, \frac{1}{6}, \frac{1}{24}, \frac{3}{8}\}_S$	$\{\frac{91}{5}, \frac{11}{10}, \frac{49}{40}, \frac{13}{8}\}_B$	$\{22, \frac{4}{3}, \frac{3}{2}, \frac{11}{6}\}_B$
$\{\frac{9}{4}, \frac{1}{4}, \frac{3}{32}, \frac{15}{32}\}_S$	$\{\frac{39}{2}, \frac{7}{6}, \frac{65}{48}, \frac{27}{16}\}_B$	$\{\frac{114}{5}, \frac{7}{5}, \frac{3}{2}, \frac{19}{10}\}_B$
$\{\frac{39}{4}, \frac{3}{4}, \frac{13}{32}, \frac{33}{32}\}_S$	$\{\frac{102}{5}, \frac{6}{5}, \frac{3}{2}, \frac{17}{10}\}_B$	

All of them are supersymmetric

No solution found for ($d=2$, $\ell=1$)

Rank three, $\ell=0$, $\{c, h_1^{\mathrm{NS}}, h_2^{\mathrm{NS}}, h_0^{\mathrm{R}}, h_1^{\mathrm{R}}, h_2^{\mathrm{R}}\}_{\mathrm{type}}$		
$\{\frac{11}{14}, \frac{1}{14}, \frac{3}{14}, \frac{3}{112}, \frac{5}{16}, \frac{99}{112}\}_N$	$\{\frac{77}{5}, \frac{7}{10}, \frac{11}{10}, \frac{33}{40}, \frac{49}{40}, \frac{13}{8}\}_B$	$\{\frac{187}{7}, \frac{13}{14}, \frac{25}{14}, \frac{79}{56}, \frac{17}{8}, \frac{143}{56}\}_B$
$\{1, \frac{1}{10}, \frac{2}{5}, \frac{1}{40}, \frac{9}{40}, \frac{5}{8}\}_N$	$\{\frac{33}{2}, \frac{9}{10}, \frac{11}{10}, \frac{77}{80}, \frac{93}{80}, \frac{25}{16}\}_B$	$\{\frac{323}{10}, \frac{17}{10}, \frac{19}{10}, \frac{31}{16}, \frac{171}{80}, \frac{187}{80}\}_B$
$\{\frac{7}{5}, \frac{1}{10}, \frac{1}{5}, \frac{3}{40}, \frac{19}{40}, \frac{7}{8}\}_B$	$\{\frac{35}{2}, \frac{5}{6}, \frac{7}{6}, \frac{49}{48}, \frac{65}{48}, \frac{27}{16}\}_B$	$\{\frac{465}{14}, \frac{25}{14}, \frac{27}{14}, \frac{31}{16}, \frac{249}{112}, \frac{265}{112}\}_B$
$\{2, \frac{1}{6}, \frac{1}{3}, \frac{1}{12}, \frac{5}{12}, \frac{3}{4}\}_S$	$\{\frac{130}{7}, \frac{13}{14}, \frac{8}{7}, \frac{5}{4}, \frac{39}{28}, \frac{47}{28}\}_B$	$\{\frac{182}{5}, \frac{11}{10}, \frac{11}{5}, \frac{49}{20}, \frac{57}{20}, \frac{13}{4}\}_B$
$\{\frac{18}{5}, \frac{3}{10}, \frac{2}{5}, \frac{3}{20}, \frac{11}{20}, \frac{3}{4}\}_S$	$\{19, \frac{9}{10}, \frac{8}{5}, \frac{19}{40}, \frac{51}{40}, \frac{15}{8}\}_N$	$\{39, \frac{7}{6}, \frac{7}{3}, \frac{65}{24}, \frac{73}{24}, \frac{27}{8}\}_B$
$\{\frac{81}{10}, \frac{3}{10}, \frac{9}{10}, \frac{9}{80}, \frac{57}{80}, \frac{21}{16}\}_N$	$\{\frac{99}{5}, \frac{9}{10}, \frac{6}{5}, \frac{11}{8}, \frac{63}{40}, \frac{71}{40}\}_B$	$\{\frac{391}{10}, \frac{13}{10}, \frac{12}{5}, \frac{207}{80}, \frac{239}{80}, \frac{51}{16}\}_B$
$\{\frac{42}{5}, \frac{3}{5}, \frac{7}{10}, \frac{7}{20}, \frac{3}{4}, \frac{19}{20}\}_S$	$\{\frac{207}{10}, \frac{13}{10}, \frac{8}{5}, \frac{23}{80}, \frac{119}{80}, \frac{27}{16}\}_N$	$\{\frac{204}{5}, \frac{6}{5}, \frac{12}{5}, 3, \frac{16}{5}, \frac{17}{5}\}_B$
$\{9, \frac{3}{5}, \frac{9}{10}, \frac{9}{40}, \frac{5}{8}, \frac{41}{40}\}_N$	$\{21, \frac{7}{6}, \frac{4}{3}, \frac{9}{8}, \frac{35}{24}, \frac{43}{24}\}_B$	$\{42, \frac{11}{10}, \frac{12}{5}, \frac{63}{20}, \frac{67}{20}, \frac{15}{4}\}_B$
$\{10, \frac{2}{3}, \frac{5}{6}, \frac{5}{12}, \frac{3}{4}, \frac{13}{12}\}_S$	$\{22, \frac{5}{6}, \frac{5}{3}, \frac{11}{12}, \frac{19}{12}, \frac{9}{4}\}_S$	$\{\frac{297}{7}, \frac{19}{14}, \frac{18}{7}, \frac{165}{56}, \frac{181}{56}, \frac{27}{8}\}_B$
$\{\frac{143}{14}, \frac{11}{14}, \frac{13}{14}, \frac{39}{112}, \frac{55}{112}, \frac{17}{16}\}_N$	$\{22, \frac{11}{10}, \frac{7}{5}, \frac{5}{4}, \frac{33}{20}, \frac{37}{20}\}_B$	$\{44, \frac{4}{3}, \frac{8}{3}, 3, \frac{10}{3}, \frac{11}{3}\}_B$
$\{\frac{165}{14}, \frac{5}{7}, \frac{15}{14}, \frac{45}{112}, \frac{11}{16}, \frac{141}{112}\}_N$	$\{\frac{221}{10}, \frac{13}{10}, \frac{7}{5}, \frac{17}{16}, \frac{117}{80}, \frac{149}{80}\}_B$	$\{\frac{310}{7}, \frac{17}{14}, \frac{18}{7}, \frac{93}{28}, \frac{97}{28}, \frac{15}{4}\}_B$
$\{\frac{63}{5}, \frac{4}{5}, \frac{9}{10}, \frac{27}{40}, \frac{7}{8}, \frac{51}{40}\}_B$	$\{\frac{170}{7}, \frac{17}{14}, \frac{11}{7}, \frac{5}{4}, \frac{51}{28}, \frac{55}{28}\}_B$	$\{\frac{228}{5}, \frac{7}{5}, \frac{14}{5}, 3, \frac{17}{5}, \frac{19}{5}\}_B$
$\{\frac{66}{5}, \frac{4}{5}, \frac{11}{10}, \frac{11}{20}, \frac{3}{4}, \frac{27}{20}\}_S$	$\{\frac{171}{7}, \frac{19}{14}, \frac{11}{7}, \frac{9}{8}, \frac{95}{56}, \frac{111}{56}\}_B$	$\{70, \frac{4}{3}, \frac{25}{6}, \frac{55}{12}, \frac{21}{4}, \frac{83}{12}\}_B$
$\{\frac{195}{14}, \frac{1}{14}, \frac{19}{14}, \frac{75}{112}, \frac{13}{16}, \frac{267}{112}\}_B$	$\{\frac{247}{10}, \frac{13}{10}, \frac{8}{5}, \frac{19}{16}, \frac{143}{80}, \frac{159}{80}\}_B$	
$\{\frac{195}{14}, \frac{6}{7}, \frac{15}{14}, \frac{75}{112}, \frac{13}{16}, \frac{155}{112}\}_B$	$\{\frac{133}{5}, \frac{9}{10}, \frac{9}{5}, \frac{57}{40}, \frac{81}{40}, \frac{21}{8}\}_B$	

Rank three, $\ell=1$, $\{c, h_1^{\mathrm{NS}}, h_2^{\mathrm{NS}}, h_0^{\mathrm{R}}, h_1^{\mathrm{R}}, h_2^{\mathrm{R}}\}_{\mathrm{type}}$		
$\{\frac{13}{7}, \frac{1}{7}, \frac{3}{14}, \frac{5}{56}, \frac{3}{8}, \frac{29}{56}\}_B$	$\{\frac{62}{7}, \frac{9}{14}, \frac{5}{7}, \frac{1}{4}, \frac{11}{28}, \frac{27}{28}\}_N$	$\{\frac{139}{7}, \frac{3}{14}, \frac{8}{7}, \frac{75}{56}, \frac{99}{56}, \frac{21}{8}\}_B$
$\{2, \frac{1}{5}, \frac{3}{10}, \frac{1}{20}, \frac{1}{4}, \frac{9}{20}\}_N$	$\{10, \frac{7}{10}, \frac{4}{5}, \frac{1}{4}, \frac{9}{20}, \frac{21}{20}\}_N$	$\{20, \frac{4}{5}, \frac{6}{5}, \frac{13}{10}, \frac{3}{2}, \frac{17}{10}\}_B$
$\{\frac{22}{7}, \frac{2}{7}, \frac{5}{14}, \frac{3}{28}, \frac{1}{4}, \frac{15}{28}\}_N$	$\{\frac{149}{14}, \frac{5}{14}, \frac{11}{14}, \frac{53}{112}, \frac{117}{112}, \frac{19}{16}\}_B$	$\{\frac{146}{7}, \frac{8}{7}, \frac{17}{14}, \frac{25}{28}, \frac{41}{28}, \frac{7}{4}\}_B$
$\{\frac{89}{14}, \frac{3}{14}, \frac{9}{14}, \frac{9}{112}, \frac{73}{112}, \frac{15}{16}\}_N$	$\{\frac{85}{7}, \frac{11}{14}, \frac{6}{7}, \frac{3}{8}, \frac{37}{56}, \frac{69}{56}\}_N$	$\{\frac{148}{7}, \frac{6}{7}, \frac{9}{7}, \frac{19}{14}, \frac{3}{2}, \frac{25}{14}\}_B$
$\{7, \frac{3}{10}, \frac{7}{10}, \frac{3}{40}, \frac{27}{40}, \frac{7}{8}\}_N$	$\{\frac{132}{7}, \frac{5}{7}, \frac{8}{7}, \frac{17}{14}, \frac{3}{2}, \frac{23}{14}\}_B$	$\{\frac{45}{2}, \frac{6}{5}, \frac{13}{10}, \frac{81}{80}, \frac{129}{80}, \frac{29}{16}\}_B$

Rank four, $\ell=0$, $\{c, h_1^{\mathrm{NS}}, h_2^{\mathrm{NS}}, h_3^{\mathrm{NS}}, h_0^{\mathrm{R}}, h_1^{\mathrm{R}}, h_2^{\mathrm{R}}, h_3^{\mathrm{R}}\}$		
$\{\frac{4}{5}, \frac{1}{40}, \frac{1}{8}, \frac{2}{5}, \frac{1}{40}, \frac{1}{8}, \frac{21}{40}, \frac{13}{8}\}_N$	$\{\frac{63}{8}, \frac{3}{8}, \frac{5}{8}, \frac{3}{4}, \frac{21}{64}, \frac{49}{64}, \frac{69}{64}, \frac{81}{64}\}_S$	$\{\frac{207}{7}, \frac{9}{14}, \frac{10}{7}, \frac{33}{14}, \frac{69}{56}, \frac{101}{56}, \frac{141}{56}, \frac{27}{8}\}_S$
$\{\frac{5}{6}, \frac{1}{18}, \frac{1}{6}, \frac{1}{3}, \frac{1}{48}, \frac{35}{144}, \frac{11}{16}, \frac{65}{48}\}_N$	$\{\frac{35}{4}, \frac{7}{12}, \frac{3}{4}, \frac{5}{6}, \frac{35}{96}, \frac{21}{32}, \frac{95}{96}, \frac{33}{32}\}_S$	$\{\frac{247}{8}, \frac{7}{8}, \frac{5}{4}, \frac{17}{8}, \frac{113}{64}, \frac{133}{64}, \frac{181}{64}, \frac{209}{64}\}_B$
$\{\frac{7}{8}, \frac{1}{8}, \frac{1}{4}, \frac{7}{8}, \frac{1}{64}, \frac{5}{64}, \frac{21}{64}, \frac{33}{64}\}_N$	$\{\frac{99}{8}, \frac{3}{4}, \frac{7}{8}, \frac{9}{8}, \frac{33}{64}, \frac{45}{64}, \frac{77}{64}, \frac{81}{64}\}_S$	$\{\frac{247}{8}, \frac{13}{8}, \frac{7}{4}, \frac{15}{8}, \frac{113}{64}, \frac{117}{64}, \frac{133}{64}, \frac{145}{64}\}_B$
$\{1, \frac{1}{32}, \frac{1}{8}, \frac{9}{32}, \frac{1}{32}, \frac{9}{32}, \frac{25}{32}, \frac{49}{32}\}_N$	$\{\frac{68}{5}, \frac{17}{40}, \frac{4}{5}, \frac{9}{8}, \frac{17}{40}, \frac{9}{8}, \frac{13}{8}, \frac{77}{40}\}_N$	$\{\frac{65}{2}, \frac{13}{24}, \frac{7}{6}, \frac{15}{8}, \frac{39}{16}, \frac{125}{48}, \frac{149}{48}, \frac{63}{16}\}_B$
$\{1, \frac{1}{14}, \frac{2}{7}, \frac{9}{14}, \frac{1}{56}, \frac{9}{56}, \frac{25}{56}, \frac{7}{8}\}_N$	$\{15, \frac{15}{32}, \frac{7}{8}, \frac{39}{32}, \frac{15}{32}, \frac{39}{32}, \frac{55}{32}, \frac{63}{32}\}_N$	$\{\frac{663}{20}, \frac{17}{10}, \frac{7}{4}, \frac{39}{20}, \frac{65}{32}, \frac{69}{32}, \frac{357}{160}, \frac{377}{160}\}_B$
$\{\frac{5}{4}, \frac{1}{32}, \frac{5}{32}, \frac{1}{4}, \frac{1}{16}, \frac{5}{16}, \frac{9}{16}, \frac{29}{16}\}_B$	$\{\frac{88}{5}, \frac{4}{5}, \frac{11}{10}, \frac{6}{5}, \frac{11}{10}, \frac{13}{10}, \frac{3}{2}, \frac{17}{10}\}_B$	$\{\frac{133}{4}, \frac{19}{32}, \frac{5}{4}, \frac{63}{32}, \frac{19}{8}, \frac{21}{8}, \frac{25}{8}, \frac{31}{8}\}_B$
$\{\frac{5}{4}, \frac{1}{12}, \frac{1}{4}, \frac{5}{6}, \frac{5}{96}, \frac{3}{32}, \frac{41}{96}, \frac{23}{32}\}_S$	$\{\frac{56}{3}, \frac{7}{9}, \frac{7}{6}, \frac{4}{3}, \frac{7}{6}, \frac{23}{18}, \frac{3}{2}, \frac{11}{6}\}_B$	$\{\frac{264}{7}, \frac{13}{14}, \frac{25}{14}, \frac{18}{7}, \frac{51}{28}, \frac{75}{28}, \frac{13}{4}, \frac{99}{28}\}_B$
$\{\frac{10}{7}, \frac{1}{14}, \frac{1}{7}, \frac{5}{14}, \frac{1}{14}, \frac{5}{14}, \frac{9}{14}, \frac{3}{2}\}_B$	$\{\frac{77}{4}, \frac{11}{12}, \frac{7}{6}, \frac{5}{4}, \frac{121}{96}, \frac{133}{96}, \frac{51}{32}, \frac{55}{32}\}_B$	$\{\frac{196}{5}, \frac{63}{80}, \frac{11}{10}, \frac{35}{16}, \frac{49}{16}, \frac{253}{80}, \frac{293}{80}, \frac{73}{16}\}_B$
$\{\frac{3}{2}, \frac{1}{18}, \frac{2}{9}, \frac{8}{9}, \frac{11}{144}, \frac{3}{16}, \frac{59}{144}, \frac{107}{144}\}_B$	$\{\frac{429}{20}, \frac{11}{10}, \frac{5}{4}, \frac{27}{20}, \frac{39}{32}, \frac{231}{160}, \frac{291}{160}, \frac{59}{32}\}_B$	$\{\frac{399}{10}, \frac{9}{10}, \frac{9}{5}, \frac{27}{10}, \frac{171}{80}, \frac{219}{80}, \frac{267}{80}, \frac{63}{16}\}_B$
$\{\frac{21}{10}, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{9}{80}, \frac{41}{80}, \frac{73}{80}, \frac{21}{16}\}_B$	$\{\frac{70}{3}, \frac{7}{6}, \frac{14}{9}, \frac{11}{6}, \frac{7}{12}, \frac{5}{4}, \frac{65}{36}, \frac{23}{12}\}_N$	$\{\frac{169}{4}, \frac{13}{32}, \frac{5}{4}, \frac{81}{32}, \frac{45}{16}, \frac{53}{16}, \frac{65}{16}, \frac{89}{16}\}_B$
$\{\frac{12}{5}, \frac{3}{40}, \frac{1}{5}, \frac{3}{8}, \frac{3}{40}, \frac{3}{8}, \frac{7}{8}, \frac{63}{40}\}_N$	$\{\frac{203}{8}, \frac{7}{8}, \frac{13}{8}, \frac{9}{4}, \frac{29}{64}, \frac{81}{64}, \frac{125}{64}, \frac{161}{64}\}_N$	$\{\frac{273}{5}, \frac{11}{10}, \frac{11}{5}, \frac{33}{10}, \frac{147}{40}, \frac{163}{40}, \frac{179}{40}, \frac{39}{8}\}_B$
$\{\frac{21}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{4}, \frac{3}{64}, \frac{15}{64}, \frac{35}{64}, \frac{63}{64}\}_N$	$\{27, \frac{13}{14}, \frac{12}{7}, \frac{33}{14}, \frac{27}{56}, \frac{75}{56}, \frac{115}{56}, \frac{21}{8}\}_N$	
$\{\frac{45}{7}, \frac{5}{14}, \frac{4}{7}, \frac{9}{14}, \frac{15}{56}, \frac{39}{56}, \frac{55}{56}, \frac{9}{8}\}_S$	$\{\frac{225}{8}, \frac{5}{8}, \frac{11}{8}, \frac{9}{4}, \frac{75}{64}, \frac{111}{64}, \frac{155}{64}, \frac{207}{64}\}_S$	

SUSY UNITARITY CONDITION

$$h_i^R \geq c/24$$

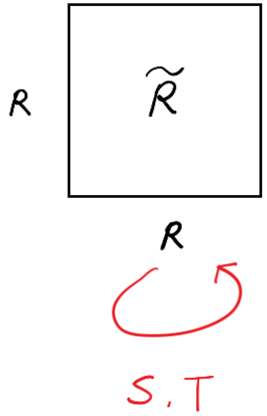
S : SUSY

B : ~~SUSY~~

N : non-SUSY

MODULAR SYMMETRY

PARTION FUNCTIONS



$SL(2,\mathbb{Z})$

$$Z_{\tilde{R}} = \mathrm{tr}_{\mathcal{H}_R}\left[(-1)^F \, q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - c/24}\right] = \sum_{i,j=0}^{d-1} N_{ij}\, \chi_i^{\tilde{R}}(\tau)\, \bar{\chi}_j^{\tilde{R}}(\bar{\tau})$$

integrability theorem

$$= \begin{cases} 0 & \text{for } h_i^R > c/24 \\ \text{Const.} \neq 0 & \text{for } h_0^R = c/24 \end{cases}$$

NO SUSY VACUA

$\exists$ SUSY VACUA