Two-loop QCD corrections to multiscale amplitudes: Progress towards ttj, $W\gamma\gamma$ and Hbb final states

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based on work with:

Becchetti, Brancaccio, Giraudo, Hartanto, Wu, Zhang, Zoia

APCTP-HOCTools meeting, Athens
3rd October 2024

RICERCA IN ITALIA
FRAMEWORK PER L'ATTRAZIONE E IL RAFFORZAMENTO DELLE ECCELLENZE PER LA RICERCA IN ITALIA

motivation:

where is the precision frontier?

part I:

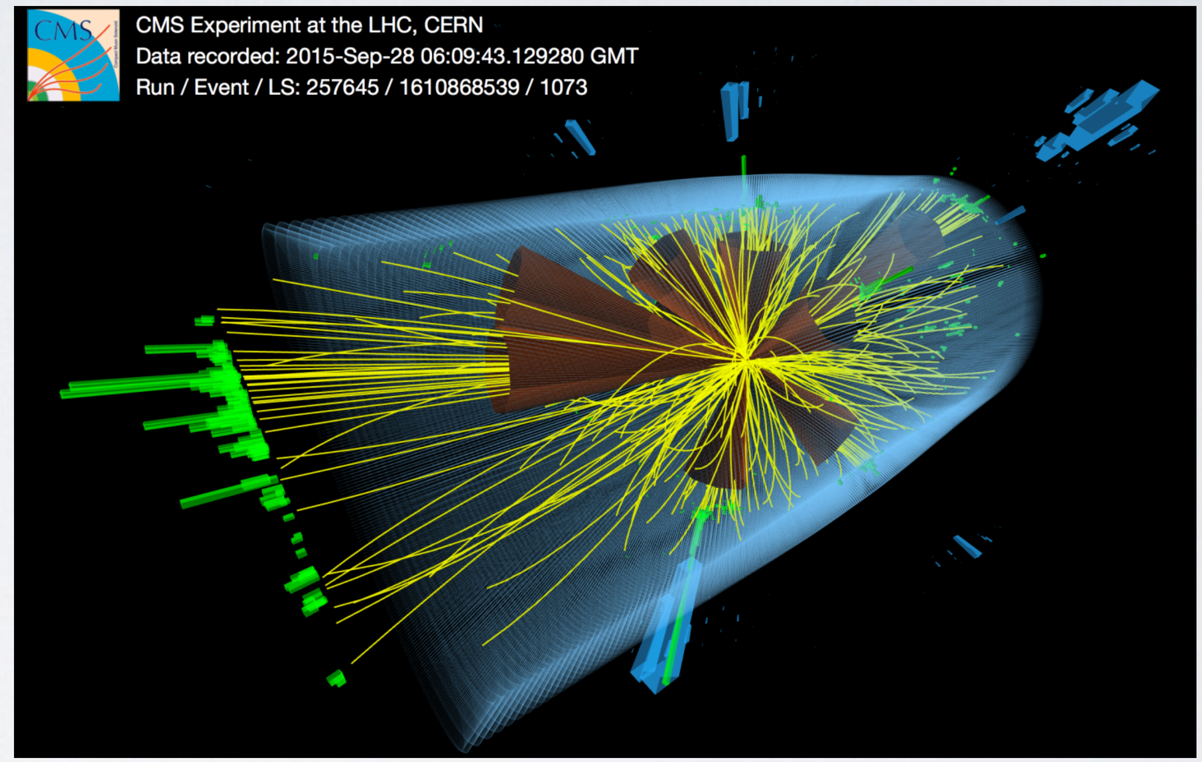
two-loop amplitudes - challenges and solutions

part II:

new results for $W\gamma\gamma$, Hbb and ttj

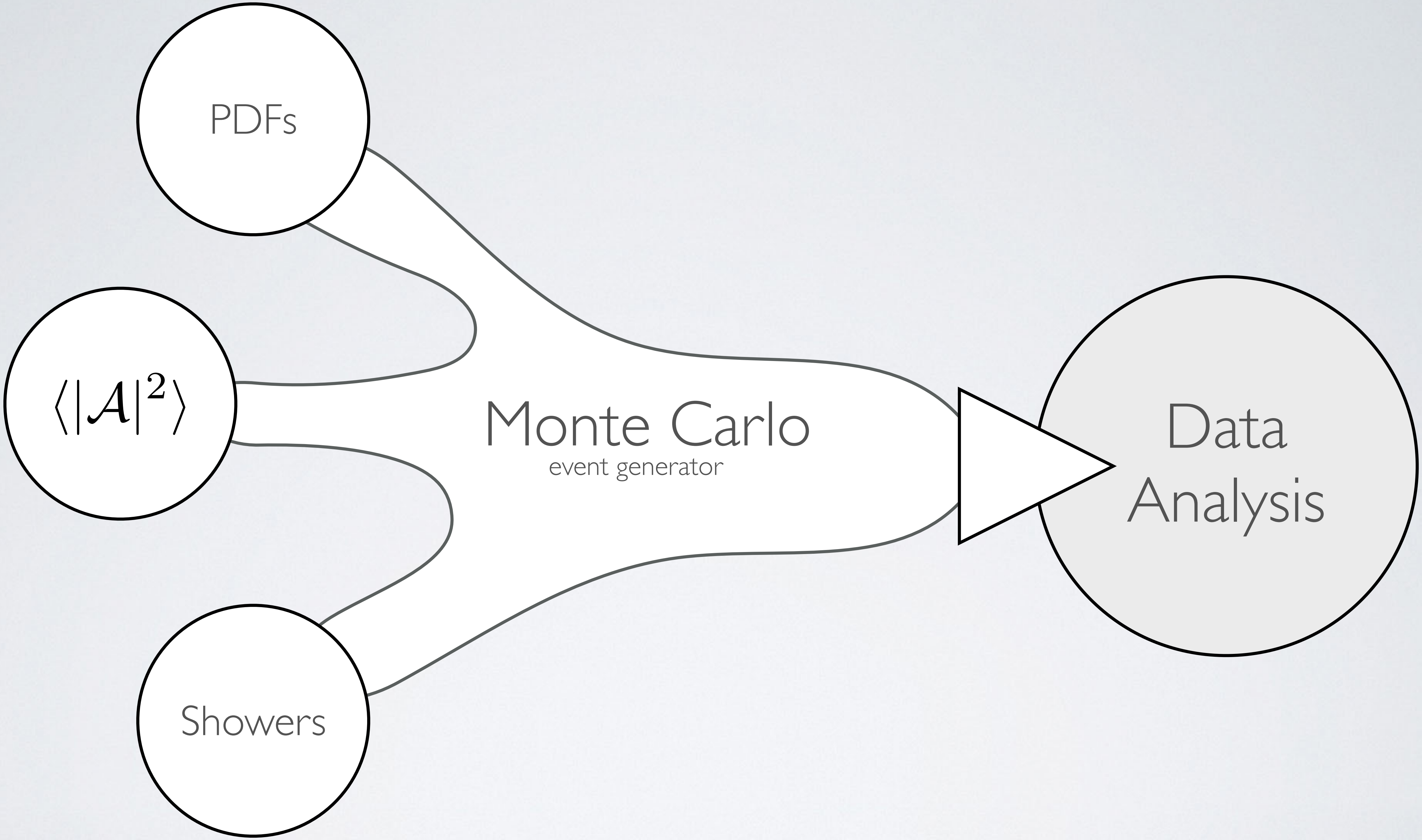
precision at the LHC

$$\mathcal{L} \longleftrightarrow$$

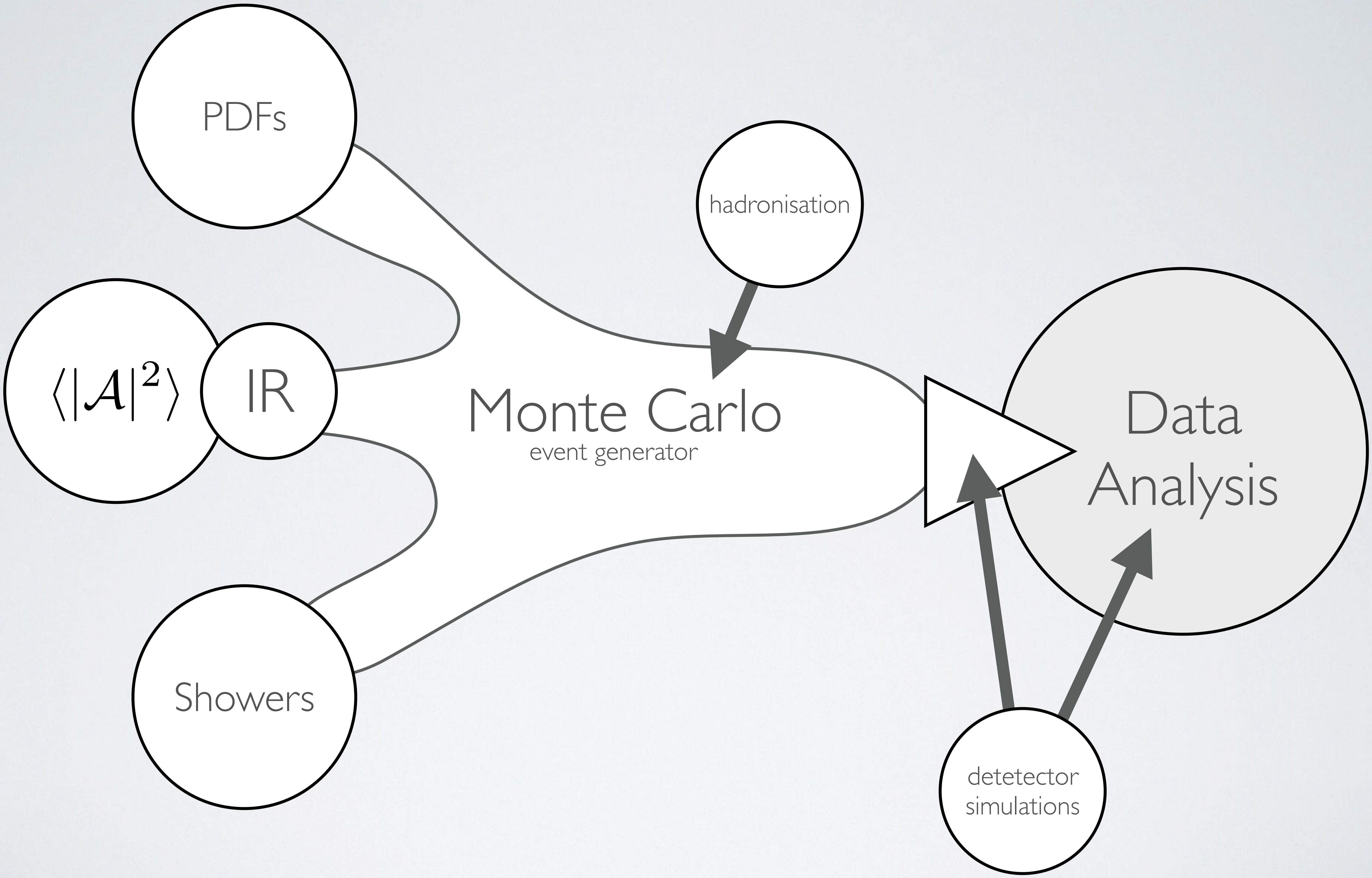


hadron colliders are not the obvious choice for precision studies of fundamental interactions but it's what we have to work with…
[LHC (-2025), HL-LHC (2030-2041)]

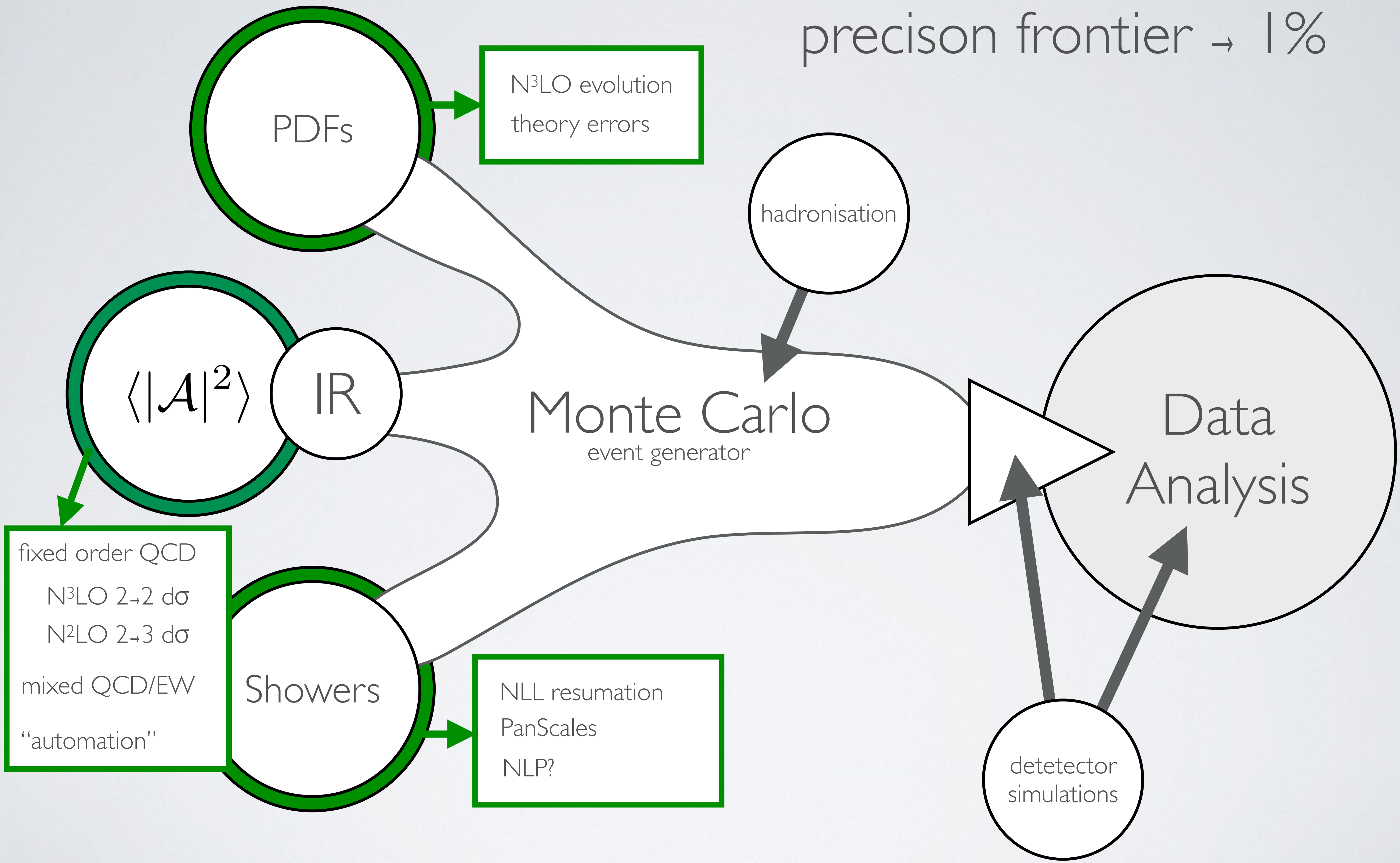
from theory to experiment



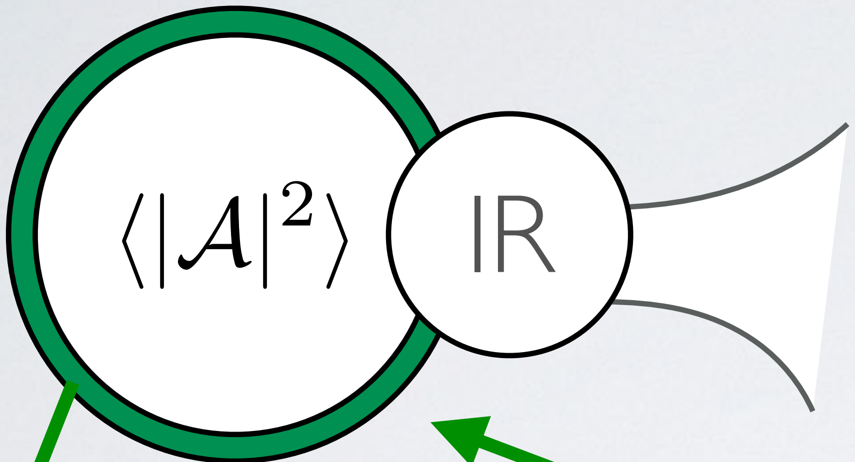
from theory to experiment



from theory to experiment: precison frontier → 1%



from theory to experiment: precison frontier → 1%



fixed order QCD

N^{3}LO 2→2 dσ

N^{2}LO 2→3 dσ

mixed QCD/EW

"automation"

new technology required!

reduction techniques

strenuous computer alegbra

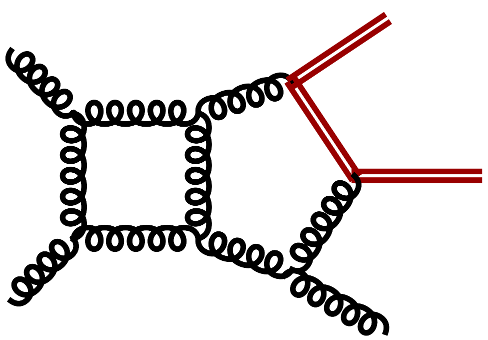
challenging integration

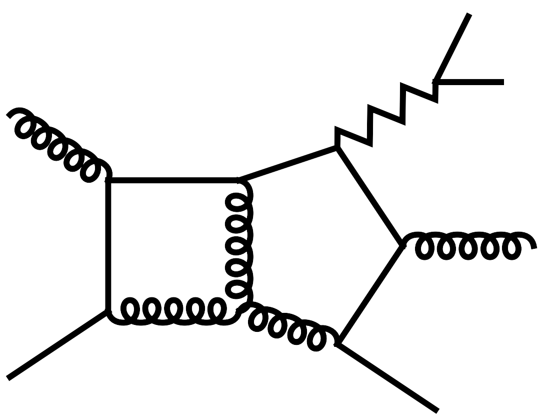
$$d\sigma = d\sigma^{\rm LO} + \alpha_s d\sigma^{\rm NLO} + \alpha_s^2 d\sigma^{\rm NNLO}$$

~10-30 %

~1-10 %

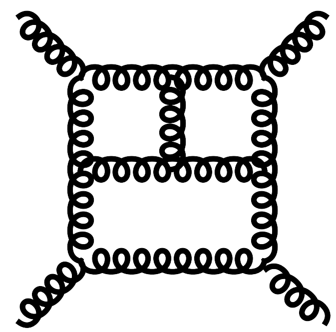
multiplicity frontier

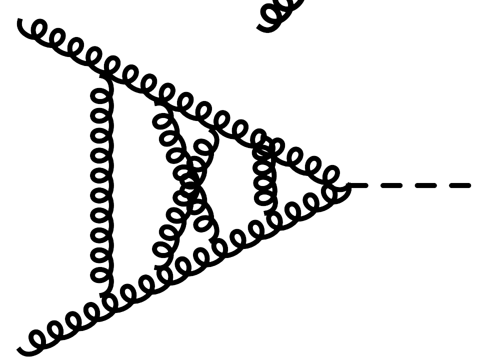




N^2LO $2\to3$ ($pp\to3j$ $pp\to W2j$, $pp\to ttj$,…)

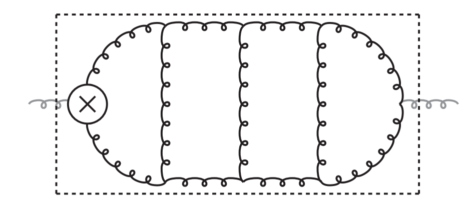
loop frontier

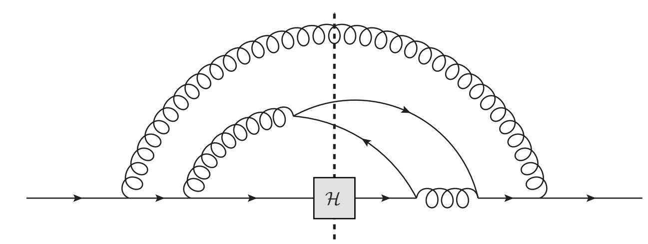




N^3LO $2\to2$, N^4LO $2\to1$ ($gg\to H$)

IR frontier





N^3LO splitting functions, analytic resummation, SCET, beam functions, IR subtraction etc.

the precision wishlist

latest update LH2021 Huss, Huston, Jones, Pellen [2207.02122]

process	known	desired
$pp \to H$	$\mathrm{N^3LO_{HTL}}$ $\mathrm{NNLO^{(t)}_{QCD}}$ $\mathrm{N^{(1,1)}LO^{(HTL)}_{QCD\otimes EW}}$	$\mathrm{N^4LO_{HTL}}$ (incl.) $\mathrm{NNLO^{(b,c)}_{QCD}}$
$pp \to H + j$	$\mathrm{NNLO_{HTL}}$ $\mathrm{NLO_{QCD}}$ $\mathrm{N^{(1,1)}LO_{QCD\otimes EW}}$	$\mathrm{NNLO_{HTL} \otimes NLO_{QCD} + NLO_{EW}}$
$pp \to H + 2j$	$\mathrm{NLO_{HTL} \otimes LO_{QCD}}$ $\mathrm{N^3LO^{(VBF^*)}_{QCD}}$ (incl.) $\mathrm{NNLO^{(VBF^*)}_{QCD}}$ $\mathrm{NLO^{(VBF)}_{EW}}$	$\mathrm{NNLO_{HTL} \otimes NLO_{QCD} + NLO_{EW}}$ $\mathrm{N^3LO^{(VBF^*)}_{QCD}}$ $\mathrm{NNLO^{(VBF)}_{QCD}}$

process	known	desired
$pp \to 2\,\text{jets}$	$\mathrm{NNLO_{QCD}}$ $\mathrm{NLO_{QCD} + NLO_{EW}}$	$\mathrm{N^3LO_{QCD} + NLO_{EW}}$
$pp \to 3\,\text{jets}$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$	

Table 2: Precision wish list: jet final states.

$pp \to V + 2j$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (QCD component) $\mathrm{NLO_{QCD} + NLO_{EW}}$ (EW component)	$\mathrm{NNLO_{QCD}}$
$pp \to V + b\bar{b}$	$\mathrm{NLO_{QCD}}$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$
$pp \to VV' + 1j$	$\mathrm{NLO_{QCD} + NLO_{EW}}$	$\mathrm{NNLO_{QCD}}$

$pp \to \gamma\gamma + j$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$ $+\mathrm{NLO_{QCD}}$ (gg channel)	
$pp \to \gamma\gamma\gamma$	$\mathrm{NNLO_{QCD}}$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$

$pp \to t\bar{t} + j$	$\mathrm{NLO_{QCD}}$ (off-shell effects) $\mathrm{NLO_{EW}}$ (w/o decays)	$\mathrm{NNLO_{QCD} + NLO_{EW}}$ (w/ decays)
$pp \to t\bar{t} + 2j$	$\mathrm{NLO_{QCD}}$ (w/o decays)	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (w/ decays)
$pp \to t\bar{t} + V'$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (w/o decays)	$\mathrm{NNLO_{QCD} + NLO_{EW}}$ (w/ decays)
$pp \to t\bar{t} + \gamma$	$\mathrm{NLO_{QCD}}$ (off-shell effects)	
$pp \to t\bar{t} + Z$	$\mathrm{NLO_{QCD}}$ (off-shell effects)	
$pp \to t\bar{t} + W$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (off-shell effects)	

$pp \to t\bar{t}t\bar{t}$	Full $\mathrm{NLO_{QCD} + NLO_{EW}}$ (w/o decays)	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (off-shell effects) $\mathrm{NNLO_{QCD}}$

the precision wishlist

latest update LH2021 Huss, Huston, Jones, Pellen [2207.02122]

process	known	desired
$pp \to H$	$\mathrm{N^3LO_{HTL}}$ $\mathrm{NNLO^{(t)}_{QCD}}$ $\mathrm{N^{(1,1)}LO^{(HTL)}_{QCD\otimes EW}}$	$\mathrm{N^4LO_{HTL}}$ (incl.) $\mathrm{NNLO^{(b,c)}_{QCD}}$
$pp \to H + j$	$\mathrm{NNLO_{HTL}}$ NL… $\mathrm{N^{(1}}$…	
$pp \to H + 2j$	NL… $\mathrm{N^3}$… NN… NL…	

process	known	desired
$pp \to 2\,\mathrm{jets}$	$\mathrm{NNLO_{QCD}}$ $\mathrm{NLO_{QCD} + NLO_{EW}}$	$\mathrm{N^3LO_{QCD} + NLO_{EW}}$
$pp \to 3\,\mathrm{jets}$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$	

Table 2: Precision wish list: jet final states.

fully differential predicitons essential LHC analyses

$\to V + 2j$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (QCD component) $\mathrm{NLO_{QCD} + NLO_{EW}}$ (EW component)	$\mathrm{NNLO_{QCD}}$
$\to V + b\bar{b}$	$\mathrm{NLO_{QCD}}$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$
$\to VV' + 1j$	$\mathrm{NLO_{QCD} + NLO_{EW}}$	$\mathrm{NNLO_{QCD}}$

see Bayu's talk

$pp \to \gamma\gamma + j$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$ $+\mathrm{NLO_{QCD}}$ (gg channel)	
$pp \to \gamma\gamma\gamma$	$\mathrm{NNLO_{QCD}}$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$

$pp \to t\bar{t} + j$	$\mathrm{NLO_{QCD}}$ (off-shell effects) $\mathrm{NLO_{EW}}$ (w/o decays)	$\mathrm{NNLO_{QCD} + NLO_{EW}}$ (w/ decays)
$pp \to t\bar{t} + 2j$	$\mathrm{NLO_{QCD}}$ (w/o decays)	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (w/ decays)
$pp \to t\bar{t} + V'$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (w/o decays)	$\mathrm{NNLO_{QCD} + NLO_{EW}}$ (w/ decays)
$pp \to t\bar{t} + \gamma$	$\mathrm{NLO_{QCD}}$ (off-shell effects)	
$pp \to t\bar{t} + Z$	$\mathrm{NLO_{QCD}}$ (off-shell effects)	
$pp \to t\bar{t} + W$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (off-shell effects)	

$pp \to t\bar{t}t\bar{t}$	Full $\mathrm{NLO_{QCD} + NLO_{EW}}$ (w/o decays)	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (off-shell effects) $\mathrm{NNLO_{QCD}}$

part I:
two-loop amplitudes - challenges and solutions

bare amplitudes

$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon, \{p\}) \mathcal{F}_i(\epsilon, \{p\})$$

rational functions

integrals/special functions

finite remainders

$$F^{(L)} = A^{(L),4-2\epsilon} - \sum_{k=1}^{L} I^{(k),4-2\epsilon} A^{(L-k),4-2\epsilon}$$

universal IR/UV poles

[Catani (1998)][Becher, Neubert (2009)]
[Magnea, Gardi (2009)]

'standard' approach

Feynman diagrams

computer algebra (FORM)

Tensor integrals

integration-by-parts identities

Master integrals

integration via DEQs, Mellin-Barnes,
direct integration (Hyperint),…

ε-expansion to MPLs

(multiple polylogarithms)

very large intermediate expressions

poor scaling with loops/legs

new computational toolbox

numerical unitarity
all-in-one cuts to master integrals

on-shell methods
hidden simplicity and underlying geometry

momentum twistors
rational kinematics

integrand reduction
algebraic reduction

syzygy relations
optimising systems of IBP identities

finite fields
exact numerics - truncated over e.g. prime numbers

recursion relations
reusing common blocks to evaluate diagrams efficiently

automated numerical algorithms @ **one-loop**
now standard in MC event generators
[Integrals: QCDloop, OneLoop…]
[Amplitudes: OpenLoops, HelacNLO,…]

finite field arithmetic

not a new idea - used in many computer algebra systems

solving IBP systems: e.g. FINRED [von Manteuffel],
KIRA+FIREFLY [Maierhoefer, Usovitsch, Uwer, Klappert, Lange]

framework for amplitude computations: FINITEFLOW [Peraro (2019)]

```
(* take some, reasonably large, prime number *)
FFPrimeNo[1]
(* all quantities evaluated modulo a prime number *)
Mod[-3,FFPrimeNo[1]]
Mod[87+FFPrimeNo[1],FFPrimeNo[1]]
Solve[b*3==87,Modulus→FFPrimeNo[1]][[1]]
(* already implemented in Mathematica *)
Mod[87/3+FFPrimeNo[1],FFPrimeNo[1]]
```

```
9 223 372 036 854 775 643
9 223 372 036 854 775 640
87
{b → 29}
29
```

NB: multiplicative inverse

extremely efficient solutions to large linear algebra systems

$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon, \{p\}) \mathcal{F}_i(\epsilon, \{p\})$$

rational functions

multiple numerical (mod prime) evaluations can used to reconstruct complete analytic information

Newton (polynomial) and Thiele (rational) interpolation

```
(* implement the Newton interpolation algorithm *)
NewtonReconstruct[z_, zvalues_List, fvalues_List, primeno_]:=Module[{res,maxdegree,aa,eqs,sol},
maxdegree = Length[zvalues]-1;
res = Sum[aa[r]*Product[(z-zvalues[[i+1]]),{i,0,r-1}],{r,0,maxdegree}];
eqs = Equal@@@Transpose[{res /. ({Rule[z,#]}&/@zvalues),fvalues}];
sol = Solve[eqs,Table[aa[i],{i,0,Length[fvalues]-1}],{Modulus->primeno}];
Return[res/. sol[[1]]];
]
```

```
fff[z_]:=15/2*z+119/6*z^2;
values = {19,44,78};
FFRatMod[fff/@values,FFPrimeNo[0]]
test = NewtonReconstruct[z,values,%,FFPrimeNo[0]]
Collect[%,z,FFRatRec[#,FFPrimeNo[0]]&]
```

{6 148 914 691 236 524 491, 6 148 914 691 236 555 916, 121 251}

6 148 914 691 236 524 491 + 1257 (-19 + z) + 1 537 228 672 809 129 317 (-44 + z) (-19 + z)

$$\frac{15\,z}{2} + \frac{119\,z^2}{6}$$

Rational external kinematics: e.g. Momentum Twistors (Hodges)

Trivial parallelisation of sample points

finite fields for amplitudes

useful features:

- reconstruct exact results using chinese remainder theorem
- extremely efficient solutions to large linear systems
- reconstruct rational functions using Newton/Thiele interpolation
- modular approach in FiniteFlow allows us to link different algorithms and avoid large intermediate steps

QGRAF + FORM/MATHEMATICA + rational phase-space (Momentum Twistors) —pre-processing→ colour ordered helicity amplitudes

$$M^{(2)}(\{p\},\epsilon) = \sum_i c_i(\{p\},\epsilon)\mathcal{F}_i(\{p\},\epsilon)$$

↓ IBPs

$$M^{(2)}(\{p\},\epsilon) = \sum_i d_i(\{p\},\epsilon)\mathrm{MI}_i(\{p\},\epsilon)$$

↓ IR/UV sub + expansion to function basis

$$F^{(2)}(\{p\}) = \sum_i e_i(\{p\})\mathrm{mon}_i(f_j^{(w)})$$

↓ linear relations, univariate apart, analytic reconstruction

complete reduction setup implemented in FINITEFLOW

IBPs generated with help from LITERED/FINITEFLOW

analytic reconstruction over finite fields

new developments have allowed major
breakthrougths in amplitude computations

$pp \to \gamma\gamma\gamma$, 3j, Hbb, W+2j, W$\gamma$j , $\gamma\gamma$+j, γ+2j

Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi, Abreu, De Laurentis, Dormans, Febres Cordero, Ita, Kraus, Klinkert, Page, Pascual, Ruf, Sotnikov, SB, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Krys, Marcoli, Moodie, Peraro, Zoia,…

refined sampling get reduce the number of points
needed to fully reconstruct analytic behaviour

Q-linear relations, factor matching, partial fractioning,…

further reading: de Laurentis, Page [2203.04269], Liu [2306.12262], Chawdhry [2312.03672]

$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon, \{p\}) \boxed{\mathcal{F}_i(\epsilon, \{p\})}$$

integrals/special functions

many ways to evaluate Feynman integrals:

Sector decomposition (numerical)	[Binoth, Heinrich….]
Differential equations	[Kotikov, Gehrmann, Remiddi, Henn…]
Mellin-Barnes	[Smirnov, Tausk, Czakon, Kosower, …]
Direct parametric integration	[Panzer, Borinsky…]

`secdec,Fiesta` `mbtools, ambre` `Hyperint` `feyntrop`

what type of functions? MPLs, elliptic or beyond

canonical form differential equations (DEQs)

[Henn (2013)]

$$\partial_x M_i = \epsilon A_{ij}(x) M_j$$

$d = 4 - 2\epsilon$

M_i integral basis usually called 'master integrals' (MIs)

A_{ij} matrix depends on kinematic invariants

if $dA = \sum_i d\log(W_i)$ it is (relatively) easy to define a special function basis from iterated integrals

e.g. 1M pentagon functions: Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia [2306.15431]

W_i alphabet

automated approaches to find canonical bases (Fuchsia, epsilon, initial, dlog…) often not sufficient to handle complicated kinematics

Analytic forms for DEQ matrices, A can be obtained via IBP reduction over finite fields

optimized IBP systems

Even with finite field technology, IBP systems are computationally intensive using the Laporta algorithm

$$I^{(2)}_{n\geq 5} = \frac{2^{D-6}}{\pi^5\Gamma(D-5)J}\int\prod_{i=1}^{11}\mathrm{d}z_i\, F(z)^{\frac{D-7}{2}}\,\frac{P(z)}{z_1\cdots z_k}$$

$$bF - \sum_{i=1}^{k-c} b_{r_i} z_{r_i}\frac{\partial F}{\partial z_{r_i}} + \sum_{j=k-c+1}^{m-c} a_{r_j}\frac{\partial F}{\partial z_{r_j}} = 0$$

syzygy relations can be used to generate compact systems free of dotted propagators

Gluza, Kadja, Kosower (2011), Ita (2015), Larsen, Zhang (2015),…

automated tool: NeatIBP [Wu et al. 2305.08783]

even with all the new ideas, putting everything together is a real challenge… but no longer a proof of concept

beyond leading colour, the number of **permutations** of each family can be very large

internal masses are important for phenomenology but present serious technical problems

memory and **CPU** usuage can still be large - need to build stable code framework

part II:
new results for Wγγ, Hbb and ttj

pp→Wγγ finite remainders

SB, Hartanto, Wu, Zhang, Zoia [2409.08146]

pp→Wγγ finite remainders

$$0 \to \bar{u}(p_1,h_1) + d(p_2,h_2) + \gamma(p_3,h_3) + \gamma(p_4,h_4) + \nu_\ell(p_5,h_5) + \ell^+(p_6,h_6)$$

$$M_6^{(L)}(1_{\bar{u}},2_d,3_\gamma,4_\gamma,5_\nu,6_{\bar{\ell}}) = e^2 g_W^2 \left[(4\pi)^\epsilon e^{-\epsilon\gamma_E}\frac{\alpha_s}{4\pi}\right]^L \delta_{i_2}^{\;\bar{i}_1}\, A_6^{(L)}(1_{\bar{u}},2_d,3_\gamma,4_\gamma,5_\nu,6_{\bar{\ell}})$$

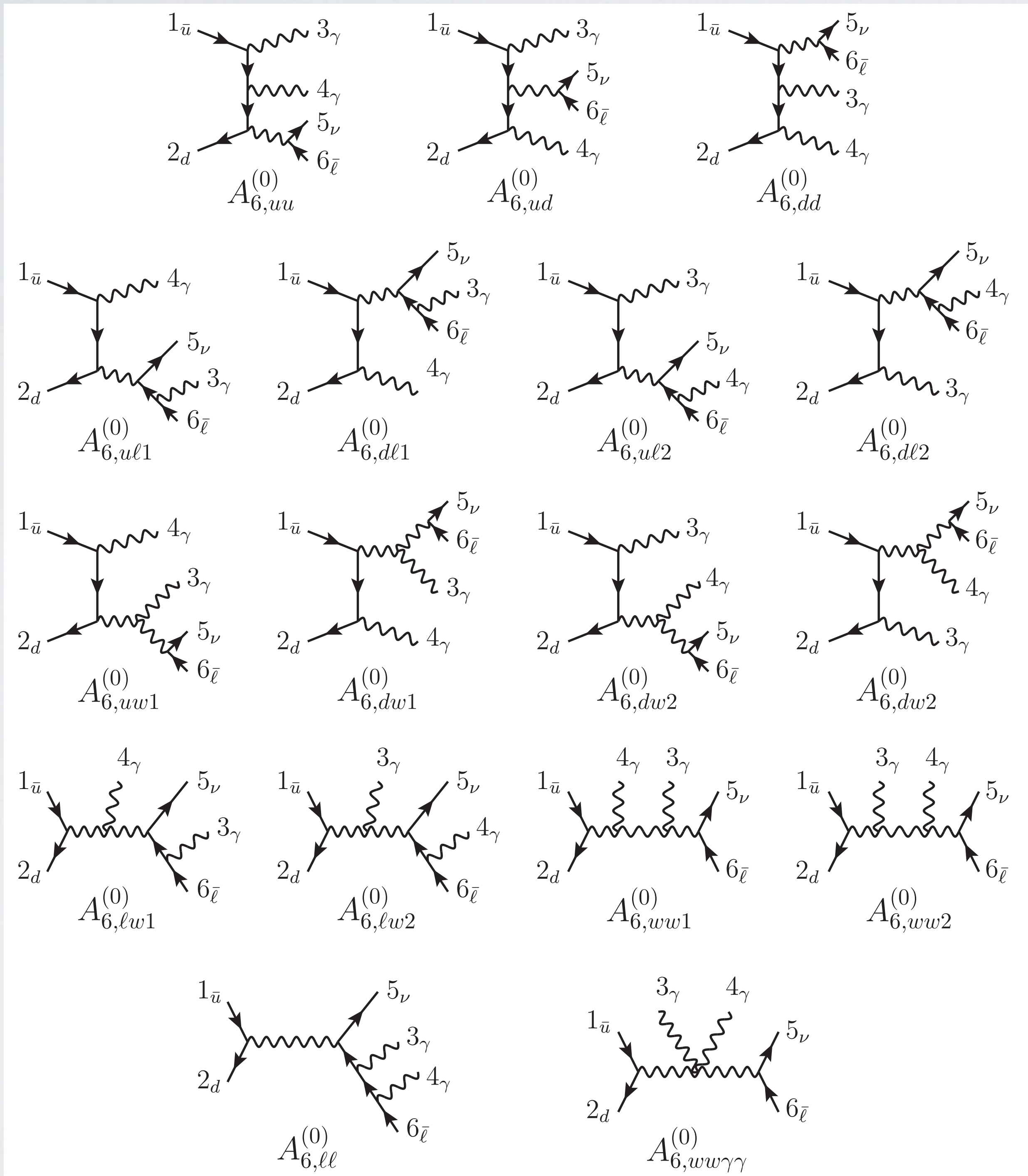
$$\begin{aligned}
A_6^{(L)} =& \left[Q_u^2 A_{6,uu}^{(L)} + Q_u Q_d A_{6,ud}^{(L)} + Q_d^2 A_{6,dd}^{(L)} + \left(\sum_{q=1}^{n_f} Q_q^2\right) A_{6,q}^{(L)}\right] P(s_{56}) \\
&+ \left[Q_u Q_\ell A_{6,u\ell 1}^{(L)} + Q_d Q_\ell A_{6,d\ell 1}^{(L)}\right] P(s_{356}) + \left[Q_u Q_\ell A_{6,u\ell 2}^{(L)} + Q_d Q_\ell A_{6,d\ell 2}^{(L)}\right] P(s_{456}) \\
&+ \left[Q_u Q_w A_{6,uw1}^{(L)} + Q_d Q_w A_{6,dw1}^{(L)}\right] P(s_{356}) P(s_{56}) \\
&+ \left[Q_u Q_w A_{6,uw2}^{(L)} + Q_d Q_w A_{6,dw2}^{(L)}\right] P(s_{456}) P(s_{56}) \\
&+ Q_\ell Q_w A_{6,\ell w1}^{(L)} P(s_{356}) P(s_{3456}) + Q_\ell Q_w A_{6,\ell w2}^{(L)} P(s_{456}) P(s_{3456}) \\
&+ Q_w^2 A_{6,ww1}^{(L)} P(s_{56}) P(s_{356}) P(s_{3456}) + Q_w^2 A_{6,ww2}^{(L)} P(s_{56}) P(s_{456}) P(s_{3456}) \\
&+ Q_\ell^2 A_{6,\ell\ell}^{(L)} P(s_{3456}) + A_{6,ww\gamma\gamma}^{(L)} P(s_{56}) P(s_{3456}) \,,
\end{aligned}$$

$$P(s) = \frac{1}{s - \mu_W^2}$$

$$\begin{aligned}
A_{6,i}^{(1)} &= \left(N_c - \frac{1}{N_c}\right) A_{6,i}^{(1),N_c} \,, \\
A_{6,i}^{(2)} &= N_c^2 A_{6,i}^{(2),N_c^2} - \left(A_{6,i}^{(2),N_c^2} + A_{6,i}^{(2),1/N_c^2}\right) + \frac{1}{N_c^2} A_{6,i}^{(2),1/N_c^2} + \left(N_c - \frac{1}{N_c}\right) n_f A_{6,i}^{(2),N_c n_f} \\
A_{6,q}^{(2)} &= \left(N_c - \frac{1}{N_c}\right) A_{6,q}^{(2),N_c} \,.
\end{aligned}$$

colour structure is relatively simple, flavour structure more involved

pp→Wγγ finite remainders



tree-level diagrams representing the various sub-amplitude contributions

pp→Wγγ finite remainders

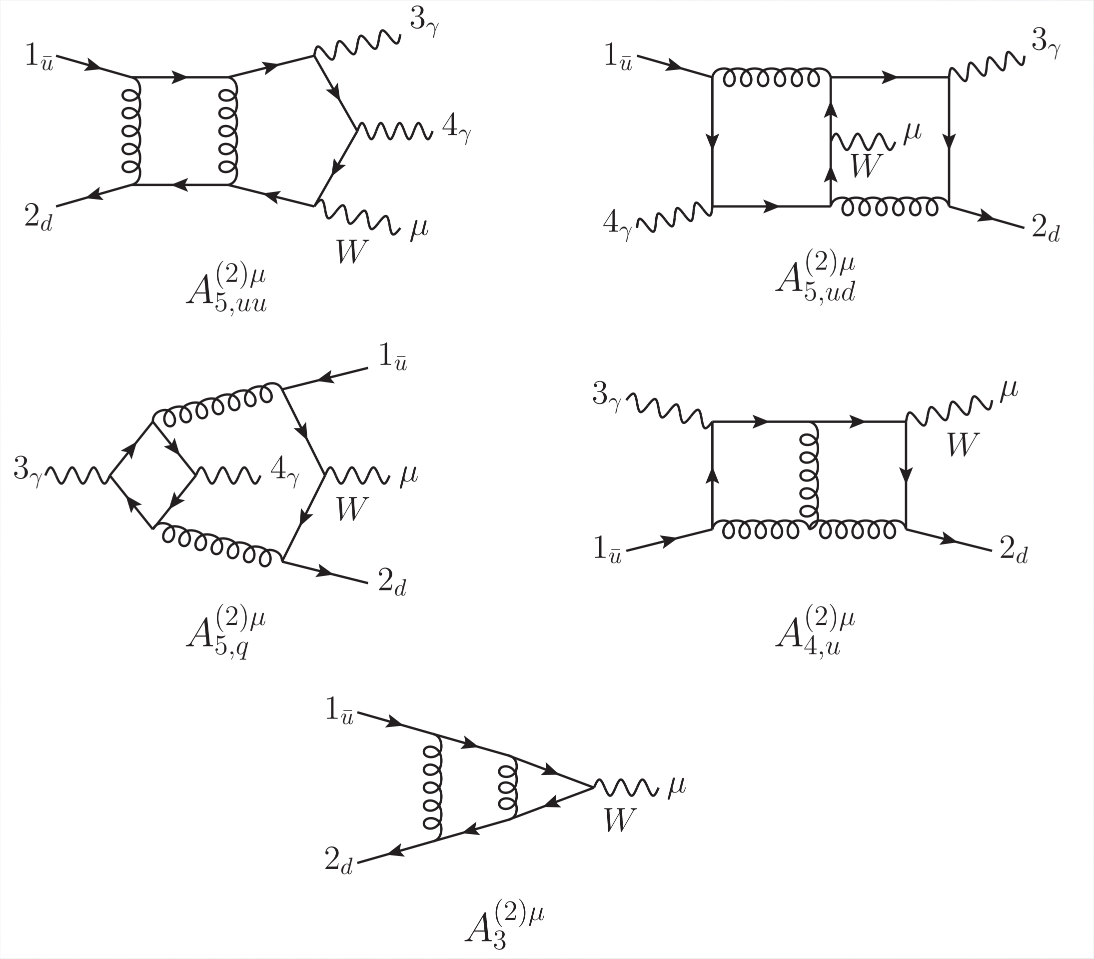
# of external particles	independent contracted helicity amplitudes
5	$\tilde{A}^{(L)+-++}_{5,uu;i}, \tilde{A}^{(L)+-+-}_{5,uu;i}, \tilde{A}^{(L)+--+}_{5,uu;i}, \tilde{A}^{(L)+---}_{5,uu;i},$ $\tilde{A}^{(L)+-++}_{5,ud;i}, \tilde{A}^{(L)+-+-}_{5,ud;i},$ $\tilde{A}^{(L)+-++}_{5,q;i}, \tilde{A}^{(L)+-+-}_{5,q;i}$
4	$\tilde{A}^{(L)+-+}_{4,u;i}$
3	$\tilde{A}^{(L)+-}_{3;i}$

$$F^{(L)}_{6,i} = \lim_{\epsilon\to 0}\left[A^{(L)}_{6,i} - \mathcal{P}^{(L)}_{6,i}\right]$$

$$\mathcal{P}^{(1)}_{6,i} = 2\,I_1(\epsilon)A^{(0)}_{6,i}\,,$$

$$\mathcal{P}^{(2)}_{6,i} = 4\,I_2(\epsilon)A^{(0)}_{6,i} + 2\left(I_1(\epsilon) + \frac{\beta_0}{\epsilon}\right)A^{(1)}_{6,i}$$

different multiplicites must be combined to get (QED) gauge invariant result



all non-planar integral families appear with different permutations - pushes reduction technology to the limit

pp→Wγγ finite remainders

	original	stage 1	stage 2	stage 3	stage 4	# points (# primes)	analytic expression
$\tilde{A}^{(2),N_c^2}_{5,uu;1}$	159/155	159/155	159/0	33/31	33/0	27728 (2)	✓
$\tilde{A}^{(2),N_c^2}_{5,uu;2}$	147/143	147/143	147/0	33/31	33/0	37132 (2)	✓
$\tilde{A}^{(2),N_c^2}_{5,uu;3}$	157/153	157/153	157/0	31/29	31/0	31610 (2)	✓
$\tilde{A}^{(2),N_c^2}_{5,uu;4}$	143/139	143/139	143/0	35/33	35/0	38710 (2)	✓
$\tilde{A}^{(2),1/N_c^2}_{5,uu;1}$	223/219	223/219	223/0	50/48	50/0	134551 (?)	✗
$\tilde{A}^{(2),1/N_c^2}_{5,uu;2}$	208/204	208/204	208/0	41/42	41/0	81973 (?)	✗
$\tilde{A}^{(2),1/N_c^2}_{5,uu;3}$	219/215	219/215	219/0	49/46	49/0	130146 (?)	✗
$\tilde{A}^{(2),1/N_c^2}_{5,uu;4}$	202/199	202/199	202/0	48/49	48/0	143320 (?)	✗
$\tilde{A}^{(2),N_c^2}_{5,ud;1}$	163/160	163/160	163/0	33/32	33/0	30371 (2)	✓
$\tilde{A}^{(2),N_c^2}_{5,ud;2}$	167/165	167/165	167/0	35/34	34/0	37506 (2)	✓
$\tilde{A}^{(2),N_c^2}_{5,ud;3}$	150/147	150/147	150/0	33/29	31/0	29698 (2)	✓
$\tilde{A}^{(2),N_c^2}_{5,ud;4}$	152/150	152/150	152/0	35/32	34/0	36726 (2)	✓
$\tilde{A}^{(2),1/N_c^2}_{5,ud;1}$	219/217	217/215	217/0	55/53	55/0	173066 (?)	✗
$\tilde{A}^{(2),1/N_c^2}_{5,ud;2}$	228/225	228/225	228/0	51/49	51/0	172337 (?)	✗
$\tilde{A}^{(2),1/N_c^2}_{5,ud;3}$	218/213	216/211	216/0	47/45	47/0	118142 (?)	✗
$\tilde{A}^{(2),1/N_c^2}_{5,ud;4}$	208/205	206/203	206/0	50/51	50/0	153605 (?)	✗
$\tilde{A}^{(2),N_c}_{5,q;1}$	136/135	119/118	119/0	34/32	34/0	26059 (2)	✓
$\tilde{A}^{(2),N_c}_{5,q;2}$	137/137	122/122	122/0	51/52	51/0	194872 (2)	✓
$\tilde{A}^{(2),N_c}_{5,q;3}$	148/147	130/129	130/0	43/44	43/0	108803 (2)	✓
$\tilde{A}^{(2),N_c}_{5,q;4}$	136/135	130/129	130/0	47/48	47/0	147167 (3)	✓

univariate partial fraction in momentum twistor variables peforms quite well - some sub-leading colour would take a lot of time to reconstruct

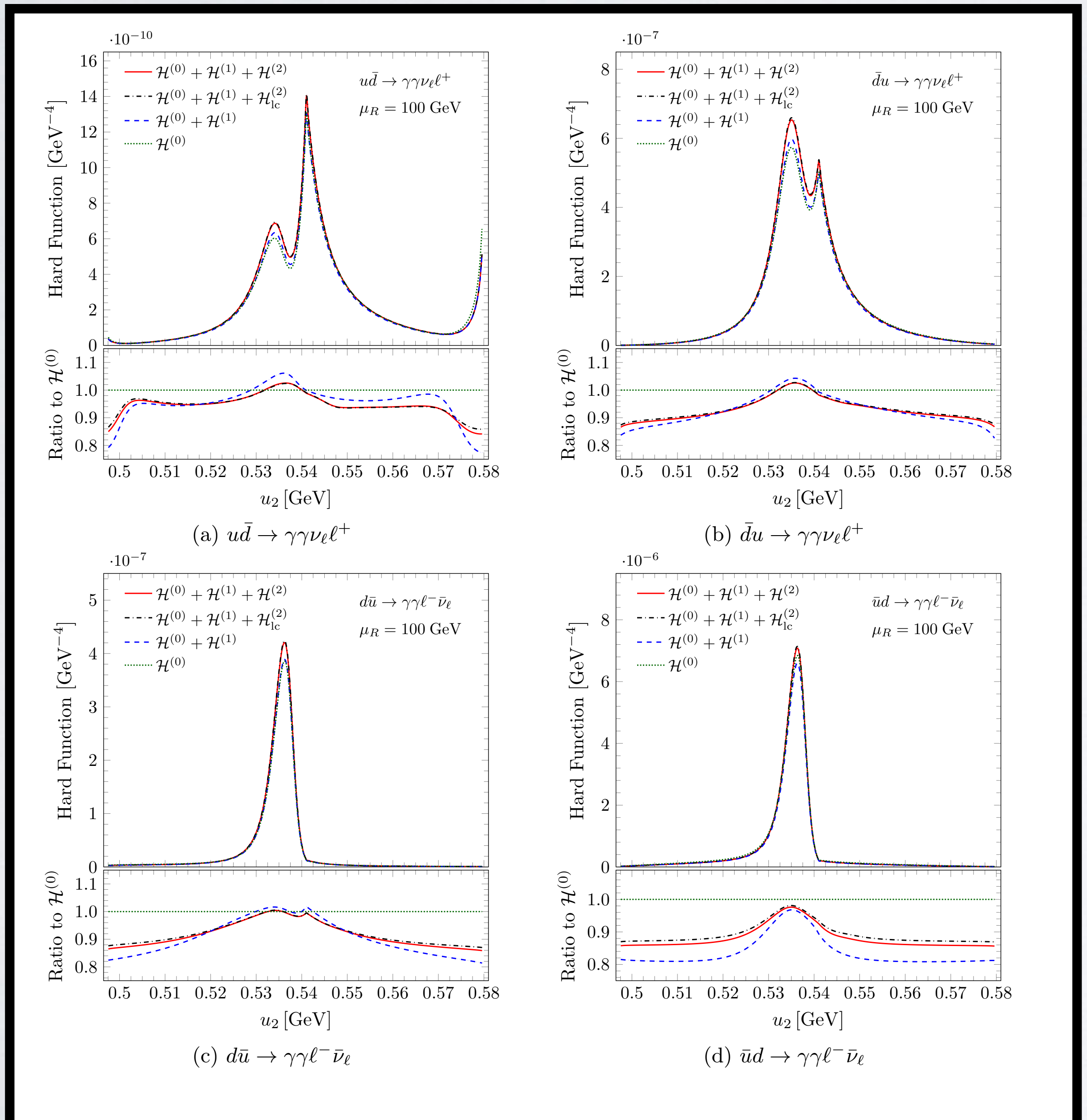
	original	stage 1	stage 2
$T_j^\dagger \cdot \tilde{A}^{(2),1/N_c^2}_{5,uu,1}$	100/97	100/97	100/0
$T_j^\dagger \cdot \tilde{A}^{(2),1/N_c^2}_{5,uu,2}$	99/96	99/96	99/0
$T_j^\dagger \cdot \tilde{A}^{(2),1/N_c^2}_{5,uu,3}$	101/97	101/97	101/0
$T_j^\dagger \cdot \tilde{A}^{(2),1/N_c^2}_{5,uu,4}$	101/97	101/97	101/0
$T_j^\dagger \cdot \tilde{A}^{(2),1/N_c^2}_{5,ud,1}$	97/93	96/92	96/0
$T_j^\dagger \cdot \tilde{A}^{(2),1/N_c^2}_{5,ud,2}$	97/93	97/93	97/0
$T_j^\dagger \cdot \tilde{A}^{(2),1/N_c^2}_{5,ud,3}$	97/93	96/92	96/0
$T_j^\dagger \cdot \tilde{A}^{(2),1/N_c^2}_{5,ud,4}$	97/93	96/92	96/0

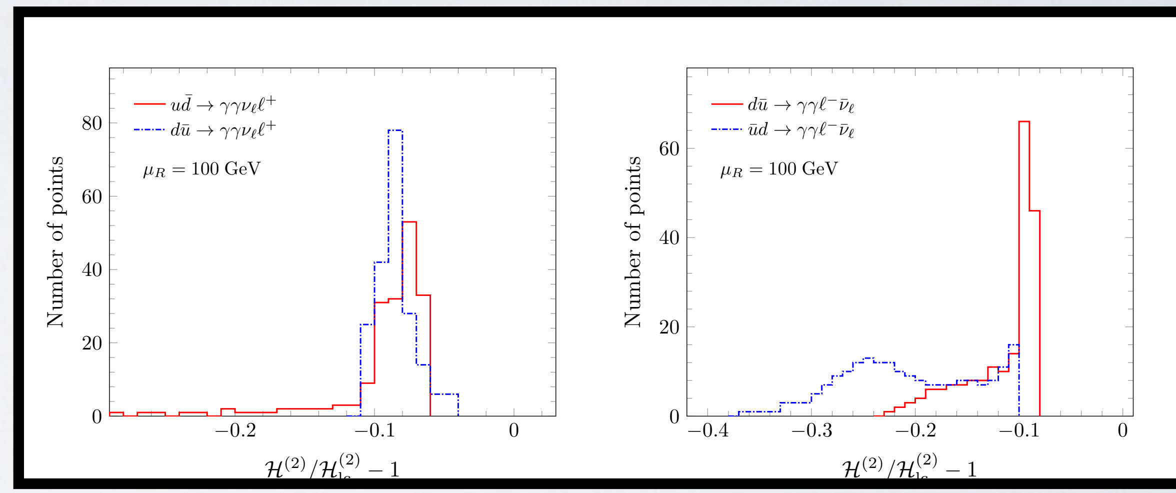
projector method used to evaluate sub-leading colour on rationalized values of s_{ij}

since sub-leading colour is supressed - we can manage with few phase space points

pp→Wγγ finite remainders

smooth evaluation across phasespace slices





subleading colour effects peaked around 10% of VV as expected

pp→Hbb finite remainders

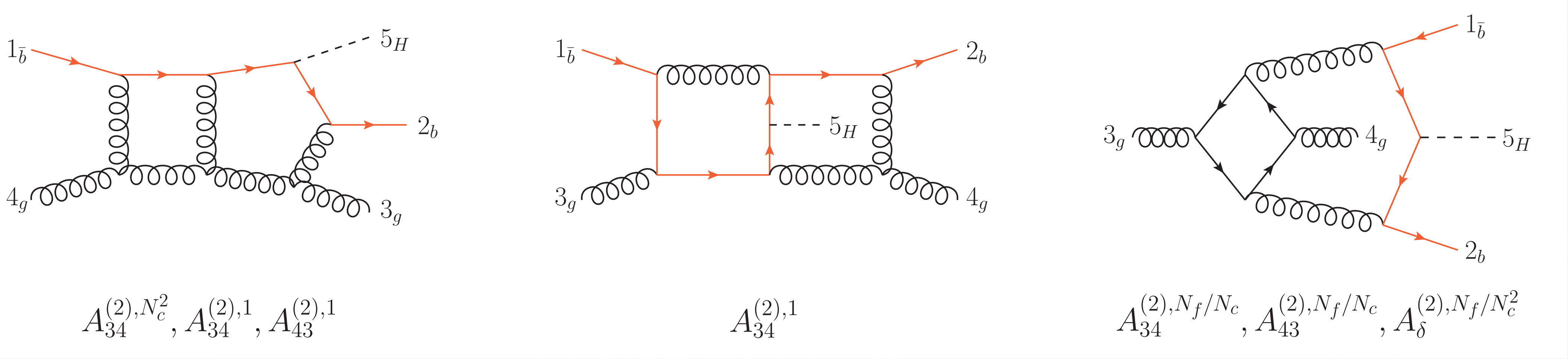
Leading colour: SB, Hartanto, Krys, Zoia [2107.14733]

Full colour: SB, Hartanto, Wu, Zhang, Zoia [2410.xxxxx]

pp→Hbb finite remainders

massless b, non-zero Yukawa

$$0 \to \bar{b}(p_1) + b(p_2) + g(p_3) + g(p_4) + H(p_5)$$
$$0 \to \bar{b}(p_1) + b(p_2) + \bar{q}(p_3) + q(p_4) + H(p_5)$$
$$0 \to \bar{b}(p_1) + b(p_2) + \bar{b}(p_3) + b(p_4) + H(p_5)$$



through 'massification' methods can be used to approximate ttH in certain phase-space regions
Catani et al. [2210.07846] Buonacore et al. [2306.16311]

pp→Hbb finite remainders

fully analytic expressions obtained: reconstruction summary

	$A_{34}^{(2),N_c^2}$	$A_{34}^{(2),1}$		$A_{\delta}^{(2),N_c}$		$A_{\delta}^{(2),1/N_c}$	
helicity	$\vec{h}_g$	$\vec{h}'_g$		$\vec{h}_g$		$\vec{h}_g$	
$x_1 = 1$	$133/132$	$176/175$	$176/175$	$265/269$	$265/269$	$214/207$	$214/207$
ansatz for linear relations	-	-	$A_{34}^{(2),1/N_c^2}$	-	$A_{34}^{(2),1}$ $A_{43}^{(2),1}$ $A_{34}^{(2),1/N_c^2}$ $A_{43}^{(2),1/N_c^2}$ $A_{\delta}^{(2),1/N_c}$	-	$A_{34}^{(2),1}$ $A_{43}^{(2),1}$ $A_{34}^{(2),1/N_c^2}$ $A_{43}^{(2),1/N_c^2}$
linear relations	$133/132$	$176/175$	$110/109$	$233/233$	$121/115$	$141/140$	$135/130$
number of coefficients	162	572	216	582	151	581	258
denominator matching	$133/0$	$176/0$	$110/0$	$233/0$	$119/0$	$141/0$	$135/0$
univariate partial fraction	$32/25$	$38/33$	$25/22$	$36/33$	$36/33$	$32/28$	$32/28$
factor matching	$28/0$	$38/0$	$24/0$	$36/0$	$36/0$	$32/0$	$32/0$
number of points	16711	47608	10382	38221	37002	21150	20337
evaluation time per point	t_0	$100t_0$	$67t_0$	$106t_0$	$57t_0$	$74t_0$	$62t_0$

towards pp→ttj helicity amplitudes

SB, Becchetti, Chaubey, Marzucca, Sarandrea [2201.12188]

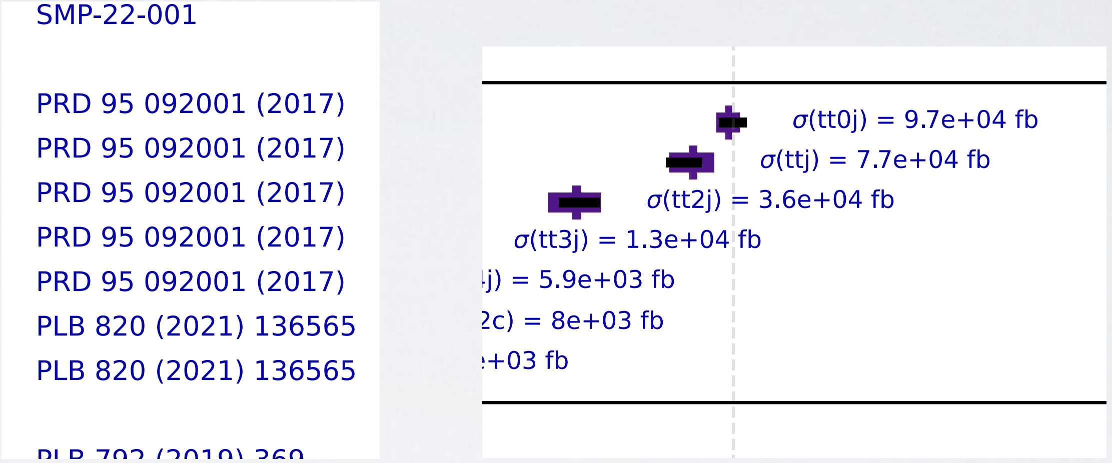
SB, Becchetti, Chaubey, Marzucca [2210.17477]

SB, Becchetti, Giraudo, Zoia [2404.12325]

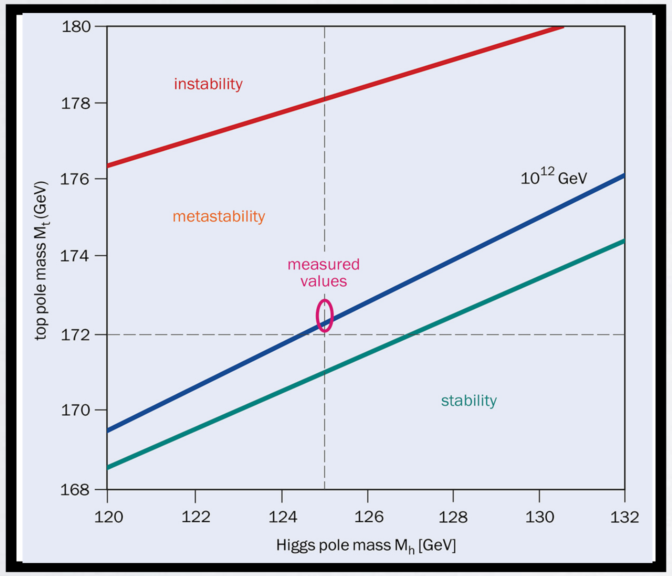
SB, Becchetti, Brancaccio, Zoia [241x.xxxx]

towards pp→ttj helicity amplitudes

around 50% of **top quark pair** events at LHC are associated **with an additional jet**



CMS XS+jets SM summary



normalised tt+j XS very sensitive to **top mass**

[see Alioli et al. 2202.07975 for latest phenomenological studies]

top quark helicity amplitudes

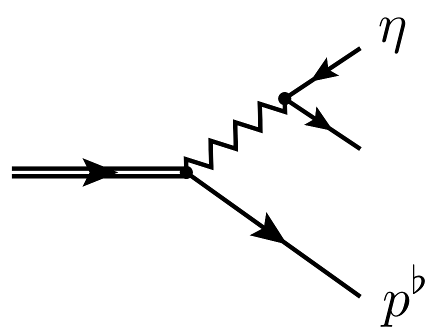
colour decomposition

$$\mathcal{A} = \sum_k C_k A_k$$

consider **leading colour** only for now

$$A_k = \sum_l c_{kl} I_l$$

helicity amplitudes encode spin correlations in the narrow width approximation



$$\overline{u_\pm}(p,m;n) = \frac{\langle \eta \mp | (\not{p} + m)}{\langle \eta \mp | p^\flat \pm \rangle}$$

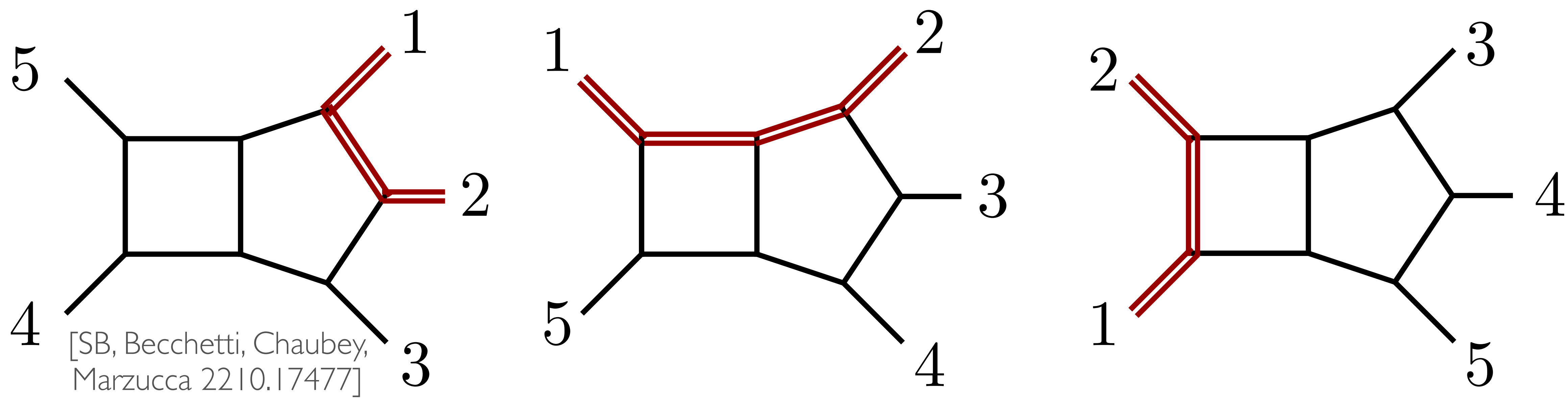
e.g. Melnikov, Schulze '09

in this case the master integral basis was unknown…

integral basis for leading colour ttj

SB, Becchetti, Giraudo, Zoia [2404.12325]

$$4_p 5_p \to 1_t 2_t 3_j$$



[SB, Becchetti, Chaubey, Marzucca 2210.17477]

88 MIs	121 MIs	109 MIs
dlog ✓	dlog ✗	dlog ✓
74 letters		79 letters

high precision boundary values with Auxiliary Mass Flow (AMFLOW Liu '22)

integral basis for leading colour ttj

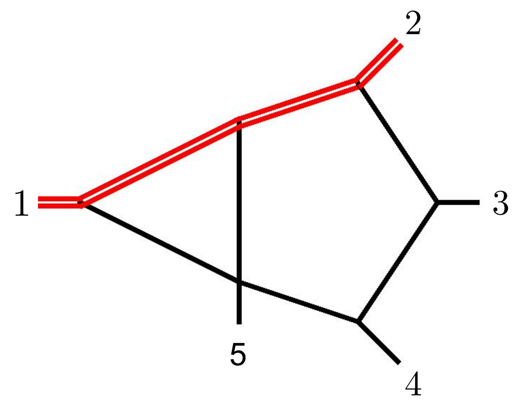
SB, Becchetti, Giraudo, Zoia [2404.12325]

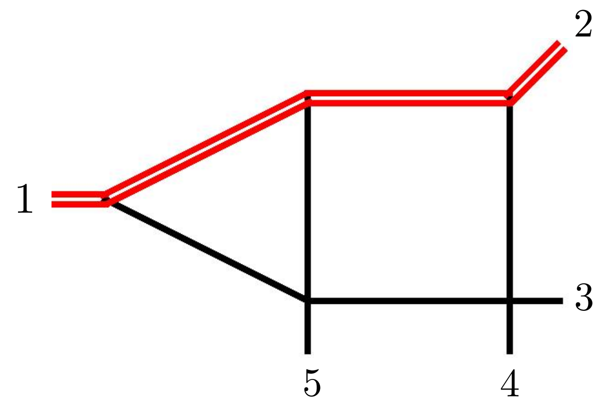
dlog candidates for most integrals follow patterns seen in previous examples:

- local and extra-dim. numerator insertions
- square roots (e.g. Gram determinants)
  8 square roots in total
- algebraic letters of the form

$$\frac{a+\sqrt{b}}{a-\sqrt{b}}$$

problem sectors



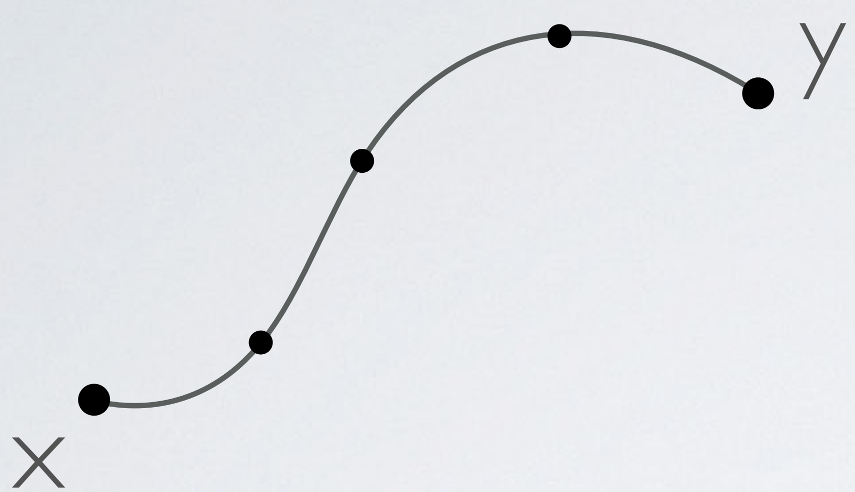


nested square root required to rotate into ε factorised form

Picard-Fuchs analysis confirms it to contain an elliptic curve

technology for analytic representation of elliptic integrals not yet ready for example of this complexity: resort to numerical evaluation of the DEQs

solving differential equations numerically



transport point x to point y using
generalised series expansion
of the differential equations

[Moriello 1907.13234]

[DiffExp Hidding 2006.05510]

B topology basis chosen such that k_{max}=2

$$\mathrm{d}A^{(B)}(\vec{x},\varepsilon) = \sum_{k=0}^{k_{\max}} \sum_i \varepsilon^k \, \omega_i(\vec{x}) \, c_{k,i}^{(B)}$$

ω are linearly independent one-forms (135 of which 72 are dlog)

compact analytic representation (for all topologies)

Path split into segments: evaluation time per segment ~30s

Efficient phase-space integration possible if the number of segments between points is minimized
(e.g. [Becchetti et al. 2010.09451])

defining a function basis

while the DEQ is not in ε-factorised form - we can do suprisingly well with the expansion around d=4

- most of the DEQ is in dlog form
- elliptic sectors only appear at order 4 in ε [check with BCs]

$$d\vec{g} = (dA^{(0)}(x) + \epsilon dA^{(1)}(x) + \epsilon dA^{(2)}(x))\vec{g} \qquad \vec{g} = \sum_{k=0} \epsilon^k \vec{g}^{(k)}$$

$$d\vec{g}^{(w)} = dA^{(0)}\vec{g}^{(w)} + dA^{(1)}\vec{g}^{(w-1)} + dA^{(2)}\vec{g}^{(w-2)}$$

- $dA^{(0)}$ and $dA^{(2)}$ mostly zero's and up to ε^3 everything is dlog.
- elliptic sectors do not decouple at ε^4 so simply keep MI component as basis function (6x2 perms = 12 functions)

result is an (overcomplete) basis that can be evaluated via generalized series expansions

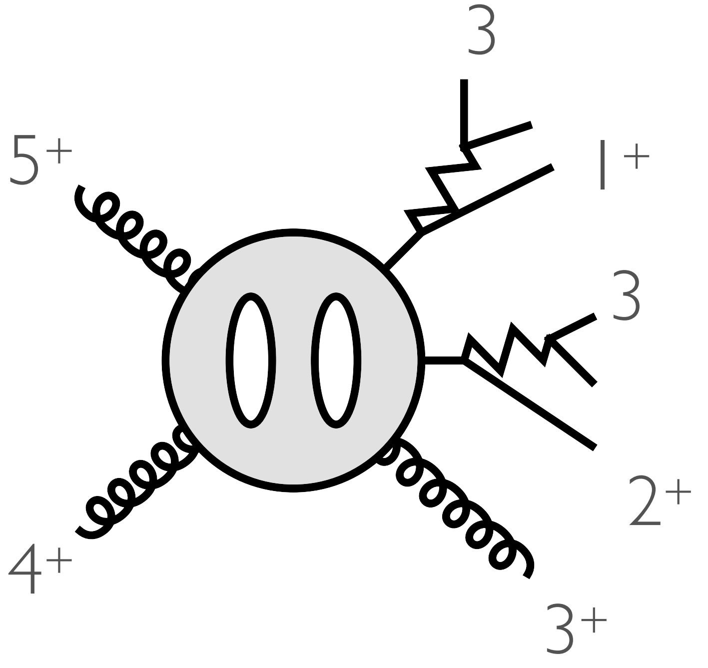
first amplitude level results

- first (numerical) evaluations in the physical remainder for the finite remainder of one helicity amplitude
- mass renormalization counter-terms to restore gauge invariance
- analytic cancellation of IR and UV poles

produced by Colomba Brancaccio for TOP2024

Phase-Space points	$A^{2L}_{LC}(+++++; n_t n_{\bar{t}})[\mathrm{GeV}^{-2}]$
$d_{12} \to 0.1074, d_{23} \to 0.2719, d_{34} \to -0.1563,$ $d_{45} \to 0.5001, d_{15} \to -0.03196, mt^2 \to 0.02502$	$19.028262 - 3.1078961\, i$
$d_{12} \to 0.3915, d_{23} \to 0.06997, d_{34} \to -0.06034,$ $d_{45} \to 0.5002, d_{15} \to -0.1293, mt^2 \to 0.02499$	$0.07061470 - 0.00649655\, i$
$d_{12} \to 0.2167, d_{23} \to 0.02186, d_{34} \to -0.01149,$ $d_{45} \to 0.5007, d_{15} \to -0.04709, mt^2 \to 0.02502$	$-29.219122 - 27.542150\, i$
$d_{12} \to 0.2986, d_{23} \to 0.1599, d_{34} \to -0.05978,$ $d_{45} \to 0.4998, d_{15} \to -0.2899, mt^2 \to 0.02500$	$-0.97280521 + 0.86357506\, i$
$d_{12} \to 0.2882, d_{23} \to 0.04770, d_{34} \to -0.1080,$ $d_{45} \to 0.5000, d_{15} \to -0.1583, mt^2 \to 0.02502$	$-0.40407926 - 0.53165671\, i$

with $d_{ij} = p_i \cdot p_j$, normalised here w.r.t $2\, p_4 \cdot p_5$



first test of full set of elements - not quite ready for pheno though

Conclusions

2L QCD corrections to 2→3 (massless) processes no longer present the same bottleneck they did 5 years ago thanks to **analytic reconstruction** techniques and well behaved **special function bases**

new results for Hbb and Wγγ - first non-planar 5pt with an off-shell leg

some progress for ttj - more work to guarentee fast and stable evaluations for pheno. studies