Landau Singularities from Whitney Stratifications

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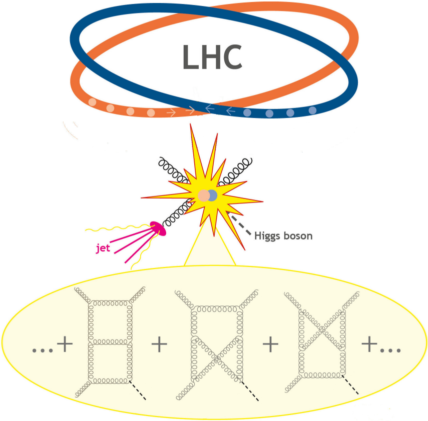
2408.14544, with Martin Helmer and Felix Tellander

Feynman graphs/integrals

Building blocks of theoretical predictions for collider physics.



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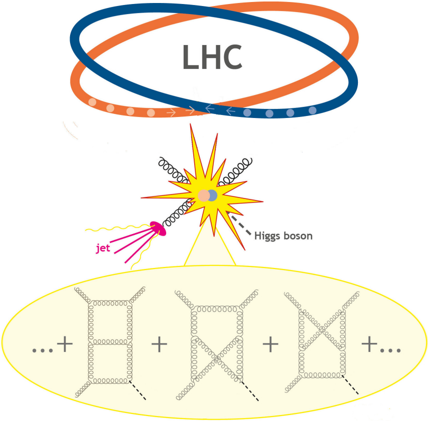

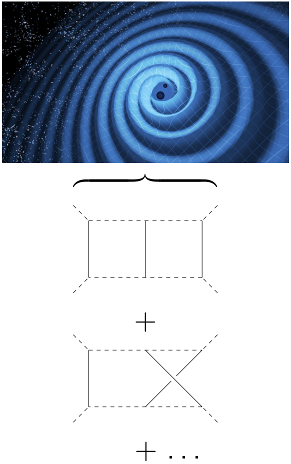

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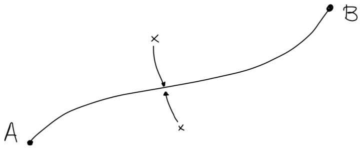

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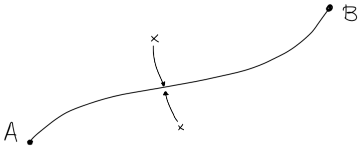

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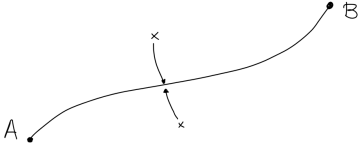

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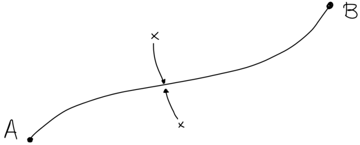

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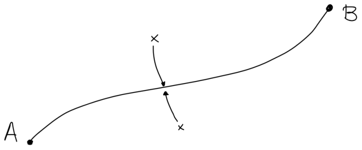

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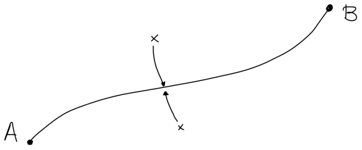

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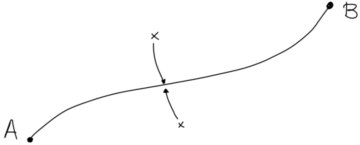

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B. Numerically: Sector decomposition [Binoth,Heinrich'00]

How to compute kinematic/Landau singularities in practice?

1. `HyperInt` [Panzer'14]
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This Work

- Algorithm for complete+non-redundant set of Landau singularities
- Based on Whitney stratifications. ∃ software implementation
- Bonus: Also yields singularities in kinematic limits (e.g. $p_i^2 \to 0$) of original integral

Outline

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Starting point: Lee-Pomeransky representation

For Feynman integral with n edges and L loops,

$$\mathcal{I} = \int_{\mathbb{R}_+^n} \left(\prod_{i=1}^{n} \frac{x_i^{\nu_i}\, dx_i}{x_i \Gamma(\nu_i)} \right) \frac{1}{\mathcal{G}^{D/2}}, \qquad \mathcal{G} = \mathcal{U} + \mathcal{F},$$

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- Depends on p_v^2, m_e^2 for each external leg v, internal edge e.

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- $\nu_0 = (L+1)D/2 - \sum_{i=1}^n \nu_i$, ensures projective invariance
- $\mathbb{P}_+^n = \{(x_0,\ldots,x_n)\sim(\lambda x_0,\ldots,\lambda x_n)\in\mathbb{R}^{n+1}, \lambda\neq 0 |\, x_i \geq 0,\ i=0,\ldots,n\}$ positive real projective space

The Landau equations

In homogenized Lee-Pomeransky representation, Landau equations describing the entrapment of the integration contour become:

$$\mathcal{G}_h = 0 \text{ and } x_i \frac{\partial \mathcal{G}_h}{\partial x_i} = 0 \;\forall\; i = 0, 1, \ldots, n.$$

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- Solutions with different $x_i \to 0$ scalings, $\nexists$ systematics of finding them

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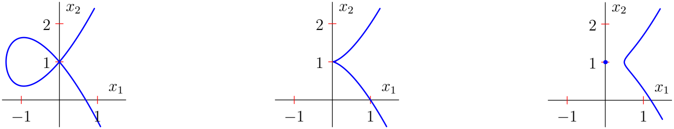

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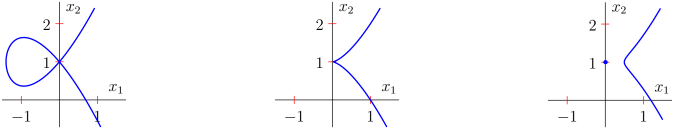

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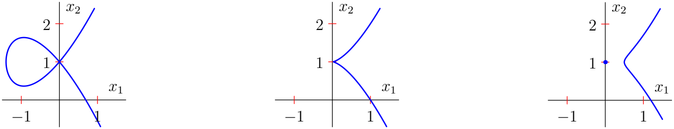

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Such topology changes algorithmically captured by Whitney stratifications.

Outline

Whitney stratification of a variety

- *Stratification of a variety X of dim. d*: Decomposition $X_0 \subset \ldots \subset X_d := X$ so that $X_i \backslash X_{i-1}$ are (open) smooth manifolds $\forall i$

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Definition

A pair of strata, M, N, whose closures obey $\overline{M} \subset \overline{N}$, satisfy Whitney's condition B at a point $x \in M$ with respect to N if for every sequence $\{p_\ell\} \subset M$ and $\{q_\ell\} \subset N$ limiting to x, the limit of secant lines between p_ℓ, q_ℓ is contained in the limit of tangent planes to N at q_ℓ.

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Example: The Whitney cusp $X = \mathbf{V}(x^2 + z^3 - y^2 z^2)^* \subset \mathbb{R}^3$

*Notation: $\mathbf{V}(g_1, \ldots, g_r) := \{\vec{x} \mid g_1(\vec{x}) = \cdots = g_r(\vec{x}) = 0\}$ for polynomials g_i in $\vec{x}$.

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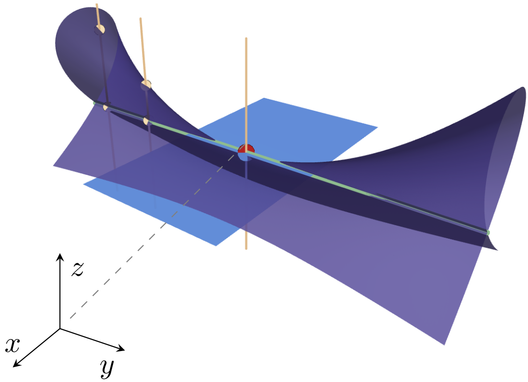

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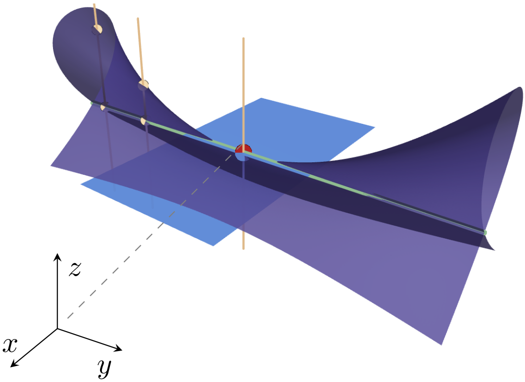

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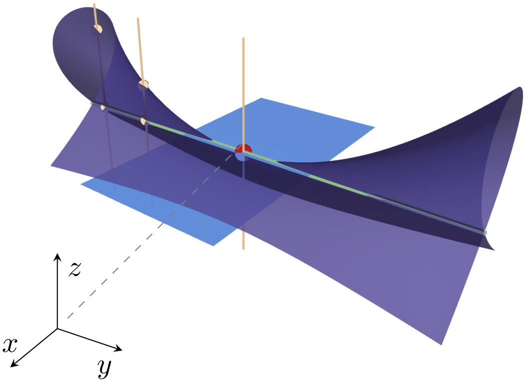

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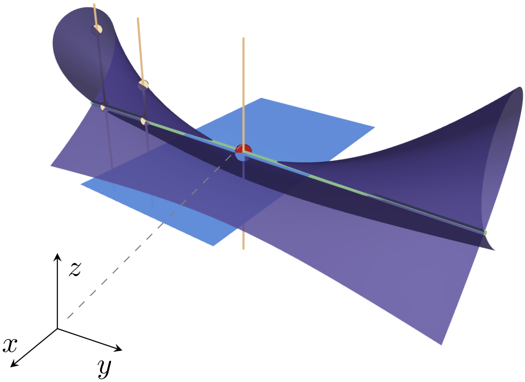

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↳ place in separate stratum.

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If X, Y varieties and $X_\bullet$, $Y_\bullet$ their stratifications, this is defined as the pair $(X_\bullet, Y_\bullet)$ with one additional condition:

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$\exists$ **unique** *minimal Whitney stratification* $\forall$ complex algebraic variety [Teissier'81]

$\Rightarrow$ $\exists$ unique minimal Whitney stratification of a map [Helmer,Nanda'23]

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$\Downarrow$

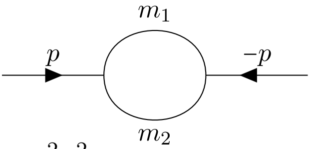
Fully algorithmic calculation of Landau singularities!

Example Usage of WhitneyStratifications

```
Macaulay2, version 1.24.05-2141-gcfa0a8fb4d-dirty (vanilla)
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems

i1 : needsPackage "WhitneyStratifications"

o1 = WhitneyStratifications

o1 : Package

i2 : R=QQ[z,x1,x2];

i3 : Gh=(x2-1)^2-(x1-z)*x1^2;

i4 : X=ideal(Gh);

o4 : Ideal of R

i5 : params={z};

i6 : S=QQ[params];

i7 : ms=mapStratify(params,X,ideal(0_S));

i8 : peek ms_1

o8 = MutableHashTable • 0 ⇒ {ideal z}
                      • 1 ⇒ {ideal 0}
```

Outline

Generic 1-loop bubble



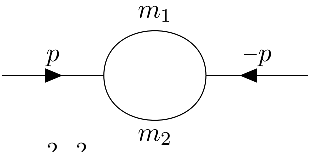
$(x_0, x_1, x_2) \in \mathbb{P}^2$,

$(m_1^2, m_2^2, p^2) \in \mathbb{C}^3 =: Y$,

$\mathcal{G}_h = x_0(x_1 + x_2) + (m_1^2 + m_2^2 - p^2)x_1x_2 + m_1^2x_1^2 + m_2^2x_2^2$,

$X = \mathbf{V}(x_0x_1x_2\mathcal{G}_h) \subset \mathbb{P}^2 \times \mathbb{C}^3$

Generic 1-loop bubble



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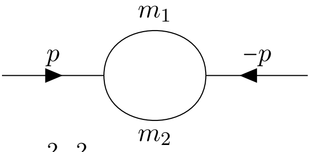
Minimal stratification of projection $\pi : X \to Y$ gives $Y_\bullet$ with $Y_3 = Y$ and

$$\begin{aligned} Y_2 =& \mathbf{V}(m_1^2) \cup \mathbf{V}(m_2^2) \cup \mathbf{V}(p^2) \\ & \cup \mathbf{V}(p^4 + m_1^4 + m_2^4 - 2p^2 m_1^2 - 2p^2 m_2^2 - 2m_1^2 m_2^2), \\ Y_1 =& \mathbf{V}(p^2,\, m_1^2 - m_2^2) \cup \mathbf{V}(m_2^2 - p^2,\, m_1^2) \cup \mathbf{V}(m_2^2,\, m_1^2 - p^2) \\ & \cup \mathbf{V}(p^2,\, m_1^2) \cup \mathbf{V}(p^2,\, m_2^2) \cup \mathbf{V}(m_2^2,\, m_1^2), \\ Y_0 =& \mathbf{V}(p^2,\, m_1^2,\, m_2^2). \end{aligned}$$

Generic 1-loop bubble



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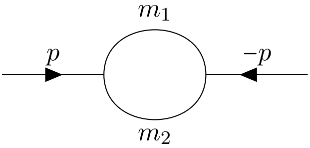
Y_0 =& \mathbf{V}(p^2,\, m_1^2,\, m_2^2).
\end{aligned}
$$

Landau variety

Generic 1-loop bubble II

Bonus: Landau variety of kinematic limits of integral contained in lower-dim. strata!



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$$
\begin{aligned}
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& \cup \mathbf{V}(p^4 + m_1^4 + m_2^4 - 2p^2 m_1^2 - 2p^2 m_2^2 - 2m_1^2 m_2^2), \\
Y_1 =& \mathbf{V}(p^2, m_1^2 - m_2^2) \cup \mathbf{V}(m_2^2 - p^2, m_1^2) \cup \mathbf{V}(m_2^2, m_1^2 - p^2) \\
& \cup \mathbf{V}(p^2, m_1^2) \cup \mathbf{V}(p^2, m_2^2) \cup \mathbf{V}(m_2^2, m_1^2), \\
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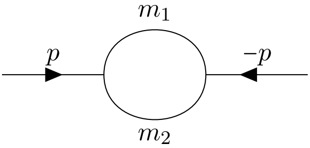
\end{aligned}
$$

Generic 1-loop bubble II

Bonus: Landau variety of kinematic limits of integral contained in lower-dim. strata!



E.g. in $m_1^2 \to 0$, Landau variety $\mathbf{V}(m_2^2 - p^2) \cup \mathbf{V}(p^2) \cup \mathbf{V}(m_2^2)$.

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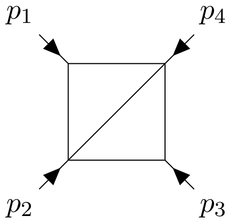
& \cup \boxed{\mathbf{V}(p^2, m_1^2)} \cup \mathbf{V}(p^2, m_2^2) \cup \boxed{\mathbf{V}(m_2^2, m_1^2)}, \\
Y_0 =& \mathbf{V}(p^2, m_1^2, m_2^2).
\end{aligned}$$

Two-mass hard slashed box

$p_3^2 \neq 0,\ p_4^2 \neq 0$, everything else massless.



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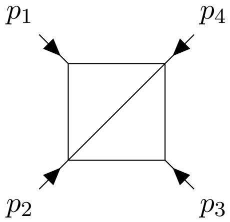

$$(x_0, \ldots, x_5) \in \mathbb{P}^5 \,,$$
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$$X = \mathbf{V}(x_0 \cdots x_5 \mathcal{G}_h) \subset \mathbb{P}^5 \times \mathbb{C}^4$$

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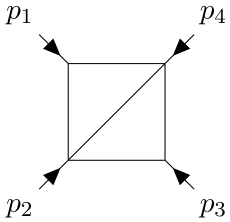

Minimal stratification of projection $\pi : X \to Y$ gives Landau variety,

$$\begin{aligned} Y_3 =& \mathbf{V}(p_3^2) \cup \mathbf{V}(s) \cup \mathbf{V}(st + t^2 - tp_3^2 - tp_4^2 + p_3^2 p_4^2) \\ & \cup \mathbf{V}(t - p_4^2) \cup \mathbf{V}(s^2 - 2sp_3^2 + p_3^4 - 2sp_4^2 - 2p_3^2 p_4^2 + p_4^4) \\ & \cup \mathbf{V}(t - p_3^2) \cup \mathbf{V}(t) \cup \mathbf{V}(p_4^2) \cup \mathbf{V}(p_4^2 - s - t). \end{aligned}$$

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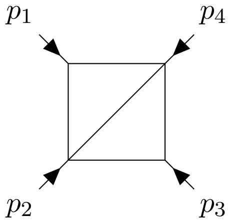

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- missed by PLD

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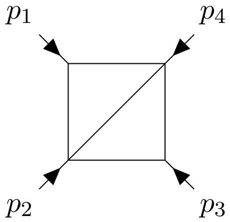

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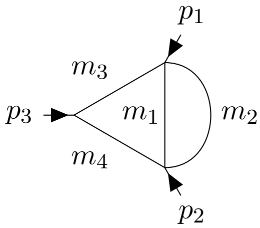
- missed by `PLD`
- agrees with `HyperInt` + known analytic expression for integral ✓

[Henn,Melnikov,Smirnov'14]

Parachute

Example with initially overlooked singularity, [Berghoff,Panzer'22]



$$p_3^2(m_4^2-p_2^2)(m_3^2-p_1^2)+(m_3^2-m_4^2-p_1^2+p_2^2)(m_3^2p_2^2-m_4^2p_1^2)$$

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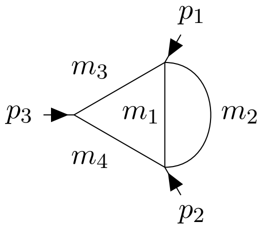

Currently too taxing for `WhitneyStratifications`: 7 kin. parameters.

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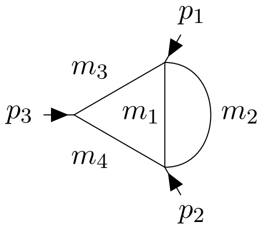

$$p_3^2(m_4^2-p_2^2)(m_3^2-p_1^2)+(m_3^2-m_4^2-p_1^2+p_2^2)(m_3^2p_2^2-m_4^2p_1^2)$$

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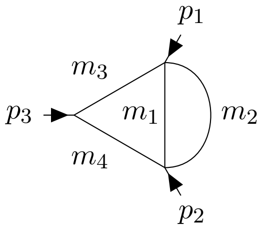

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Parachute

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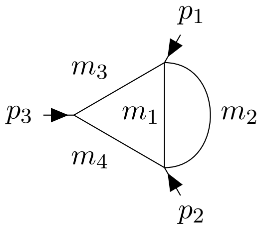

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- Limit of above singularity! Also checked with direct integration ✓

Parachute

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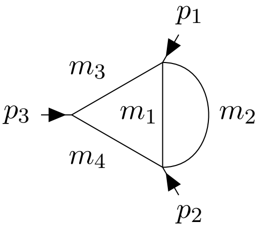

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Conclusions and Outlook

Algorithm for complete+non-redundant set of singularities $\forall$ integral!

- From Whitney stratification of projection map on its integrand.
- $\exists$ software package. Applied to nontrivial examples at two loops.
- Also yields singularities in limits (e.g. $p_i^2 \to 0$) of original integral.

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Next Stage

1. Efficieny improvements?
2. Values of integr. variables x_i on singularities?
   Needed for method of regions, num. integration.
3. Application to intersection theory for Feynman integrals?
   [Mastrolia,Mizera'19]
4. Extend to predict alphabet and prove cluster-algebraic structures?
   [Chicherin,Henn,GP'20] [Aliaj,GP'24]

Symanzik Polynomials

$$\mathcal{U} = \sum_{\substack{T \text{ a spanning} \\ \text{tree}^{\dagger} \text{ of } G}} \prod_{e \notin T} x_e ,$$

$$\mathcal{F} = \mathcal{U} \sum_{e \in E} m_e^2 x_e - \sum_{\substack{F \text{ a spanning} \\ 2\text{–forest}^{\ddagger} \text{ of } G}} p(F)^2 \prod_{e \notin F} x_e ,$$

† Connected subgraph of G containing all vertices but no loops.

‡ Defined similarly, but with 2 connected components.