

Intersection Theory for Fundamental Interactions

Modern Calculus Concepts

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INPP Demokritos
APCTP meeting & HOCTOOLS workshop

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In collaboration with: **P. Benincasa, G. Brunello, S. Cacciatori, V. Chestnov, G. Crisanti, W. Flieger, M. Giroux, H. Frellesvig, M.K. Mandal, S. Matsubara-Heo, S. Mizera, S. Smith, F. Vazao, N. Takayama**



Differential Equations

Theoretical Physics goals: *modelling* Nature by *modelling* changes: Systems' Evolution

Describe how promptly a quantity changes with respect to the change in one or more other quantities

- **Differential Equations**

$$\partial_x^{(n)} f(x) + p_{n-1}(x) \partial_x^{(n-1)} f(x) + \dots + p_1(x) \partial_x^{(1)} f(x) + p_0(x) f(x) = 0$$

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- **Linear relations**

$$f_n(x) + a_{n-1} f_{n-1}(x) + \dots + a_1 f_1(x) + a_0 f_0(x) = 0$$

Differential Equations

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● Differential Equations

$$\partial_x^{(n)} f(x) + p_{n-1}(x) \partial_x^{(n-1)} f(x) + \dots + p_1(x) \partial_x^{(1)} f(x) + p_0(x) f(x) = 0$$

$$\partial_x^{(n)} f(x) = - p_{n-1}(x) \partial_x^{(n-1)} f(x) - \dots - p_1(x) \partial_x^{(1)} f(x) - p_0(x) f(x)$$

● Linear relations

$$f_n(x) + a_{n-1} f_{n-1}(x) + \dots + a_1 f_1(x) + a_0 f_0(x) = 0$$

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● Differential Equations

$$\partial_x^{(n)} f(x) + p_{n-1}(x) \partial_x^{(n-1)} f(x) + \dots + p_1(x) \partial_x^{(1)} f(x) + p_0(x) f(x) = 0$$

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● Linear relations

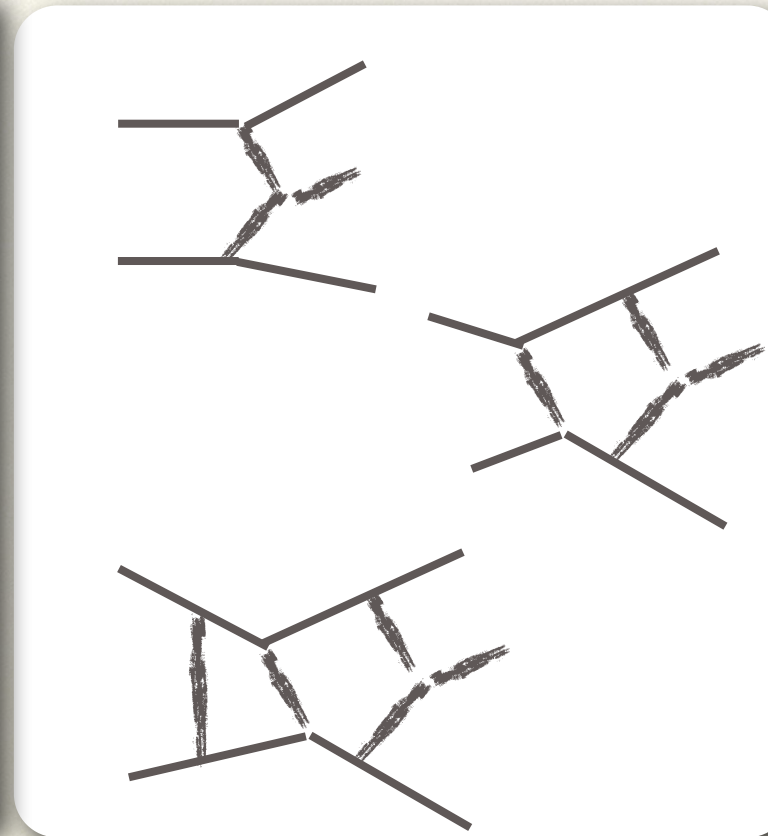
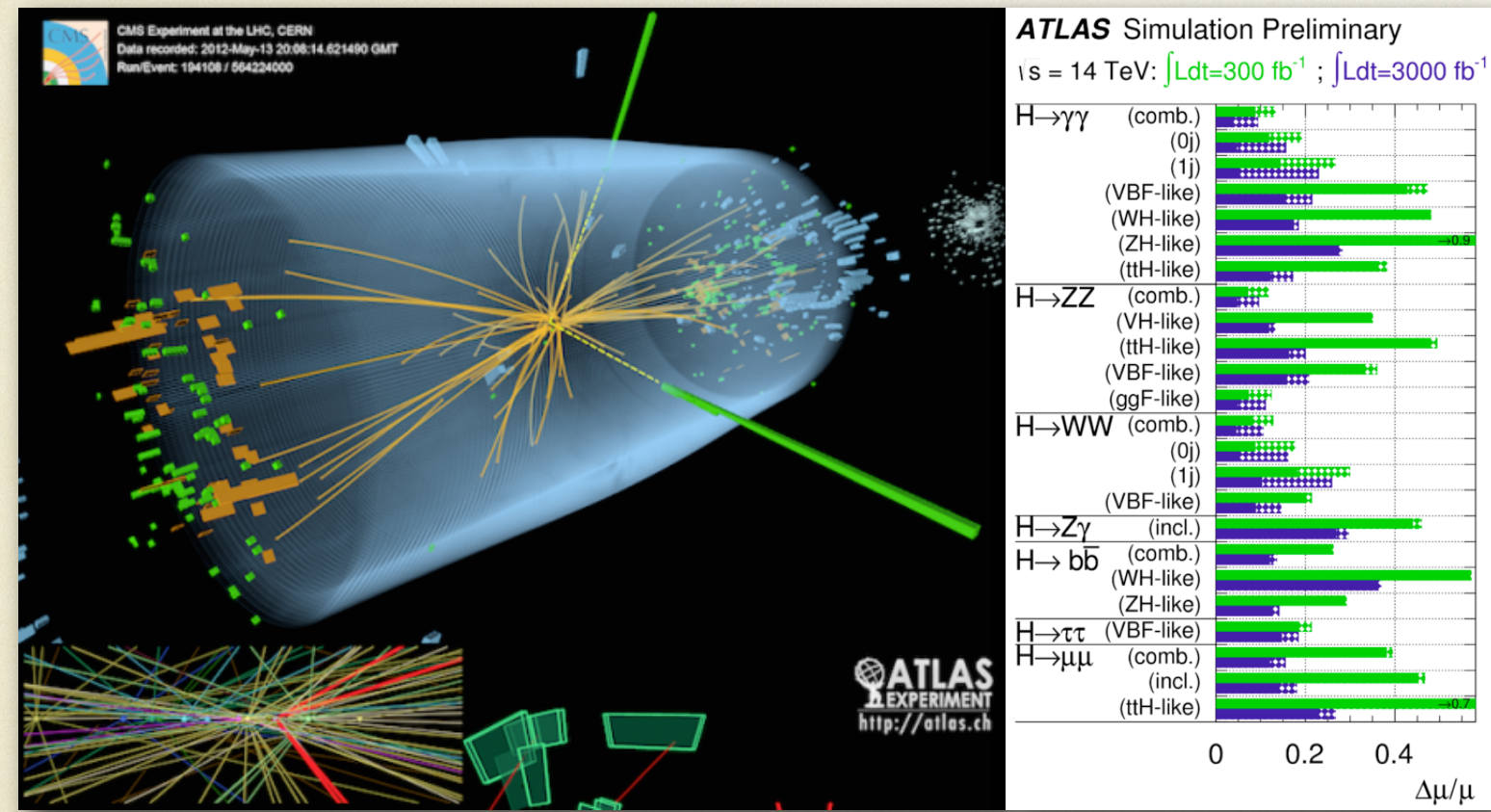
$$f_n(x) + a_{n-1} f_{n-1}(x) + \dots + a_1 f_1(x) + a_0 f_0(x) = 0$$

$$f_n(x) = -a_{n-1} f_{n-1}(x) - \dots - a_1 f_1(x) - a_0 f_0(x)$$

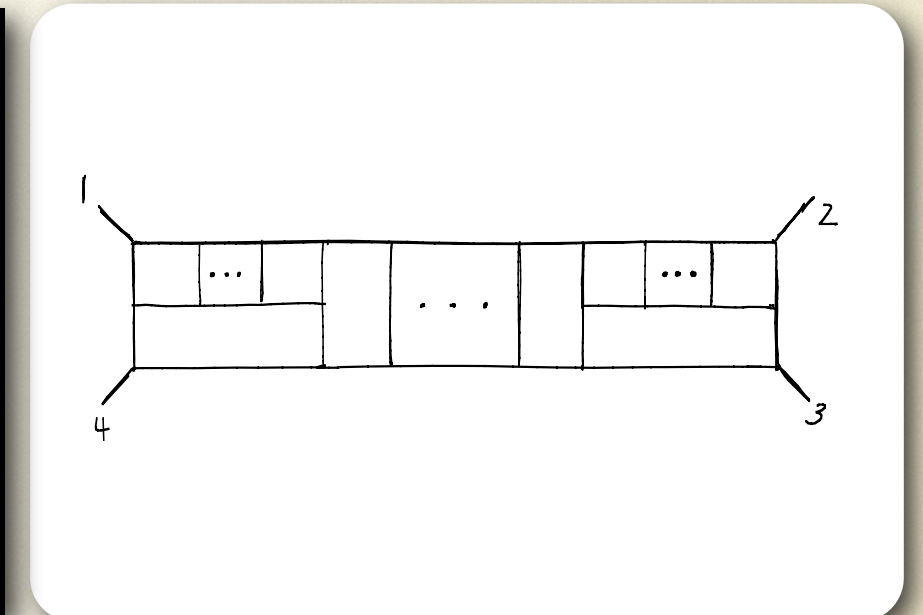
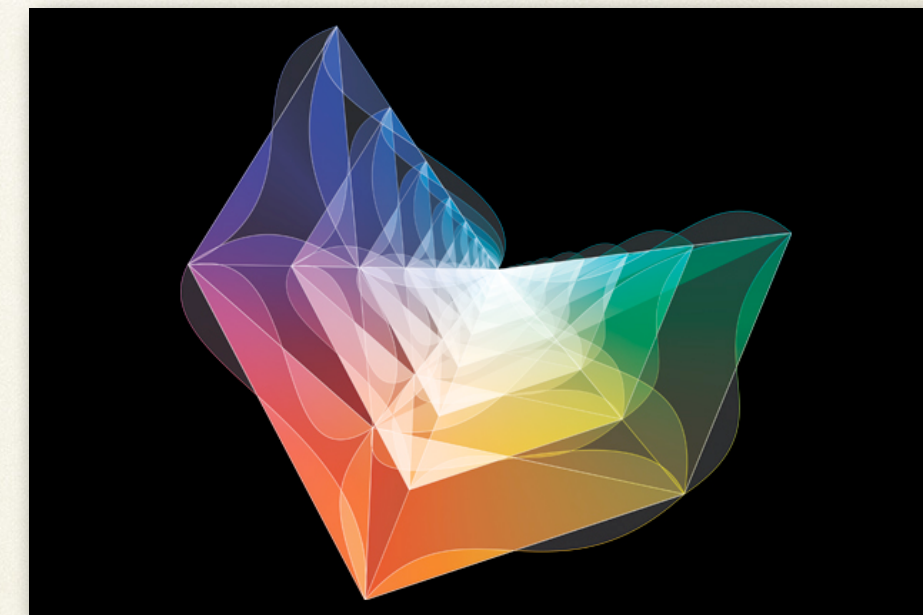
Decomposition formulas
in terms
of *n independent elements*

Impact of Scattering Amplitudes & Multiloop Calculus / Frontier of Theoretical Physics

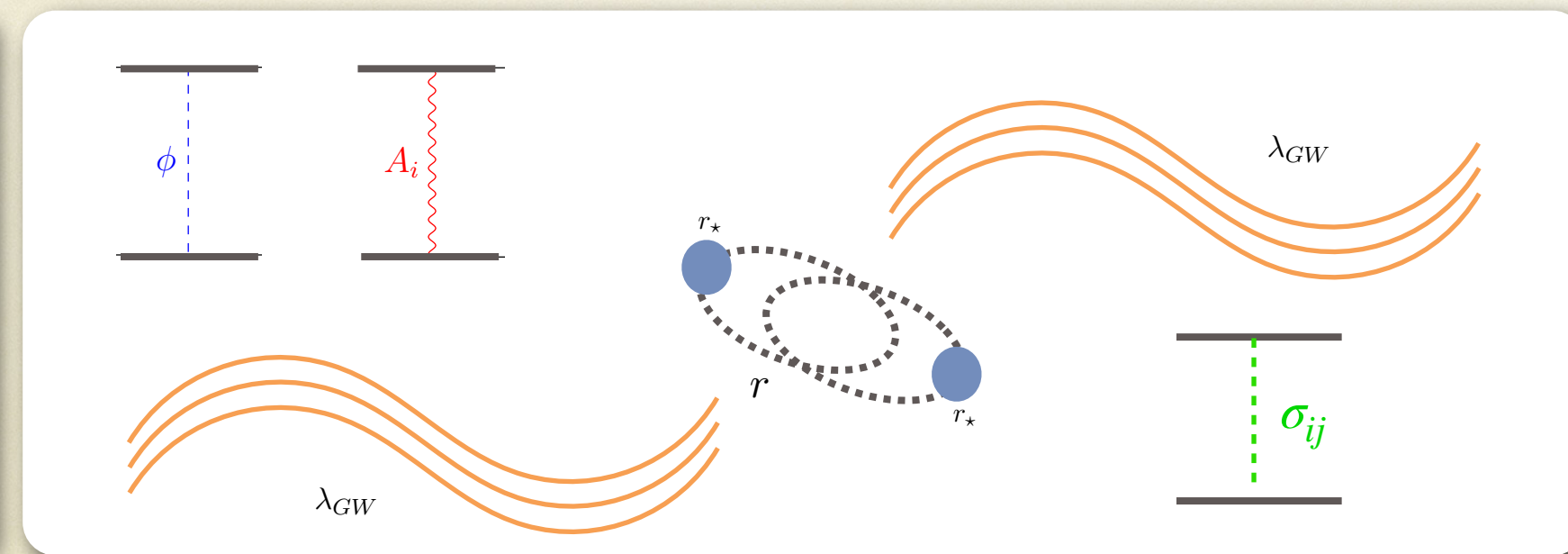
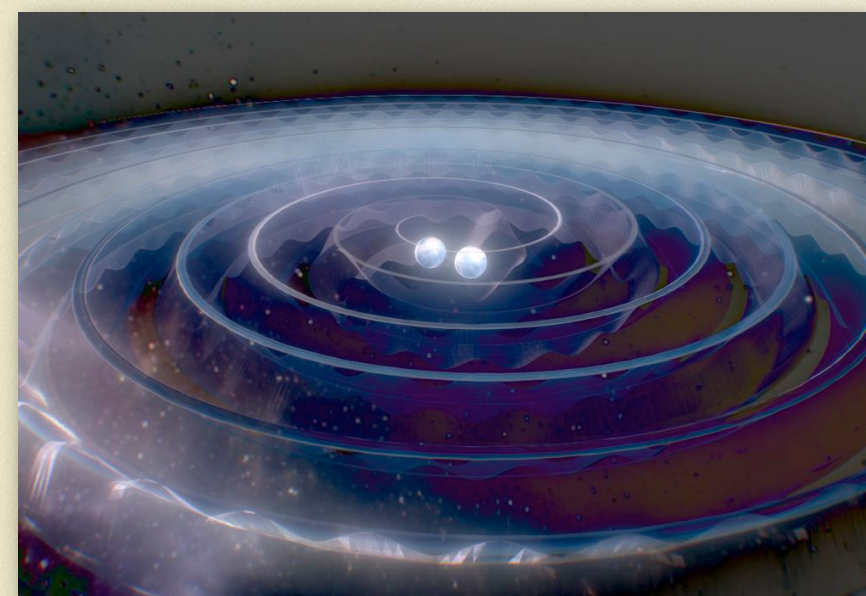
● Collider Phenomenology



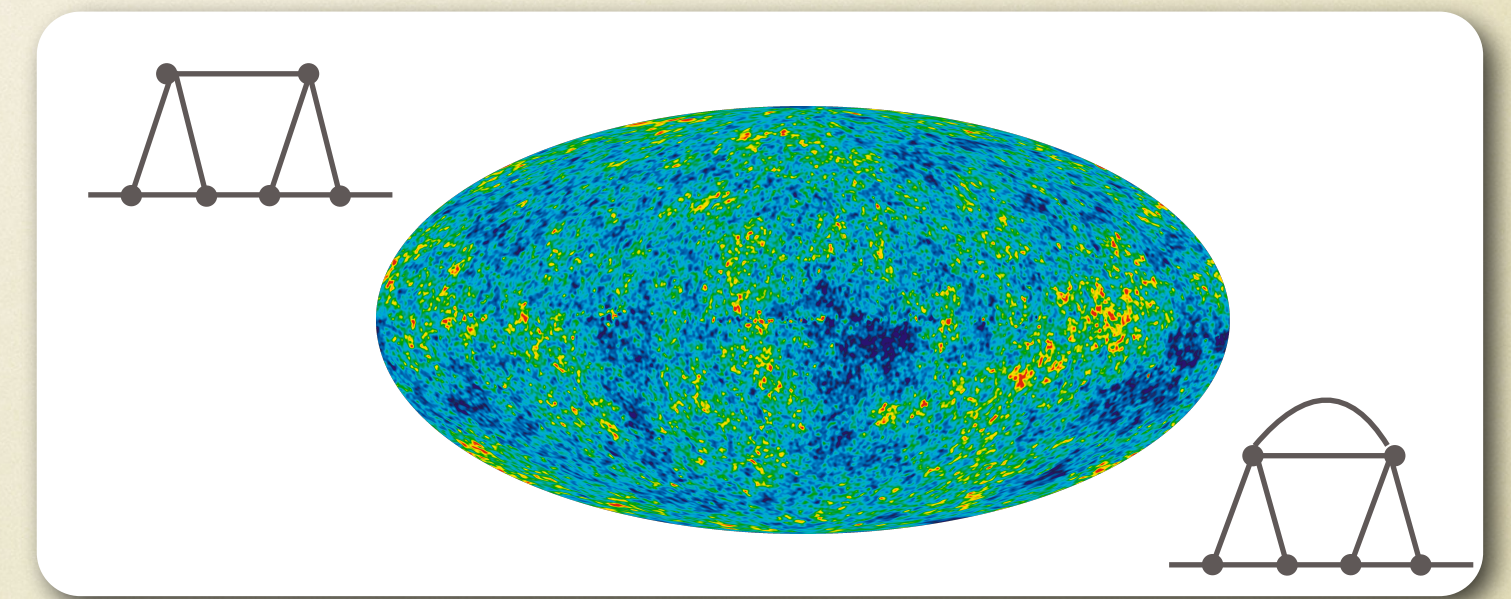
● Geometry of Quantum Field Theory



● EFT Classical General Relativity

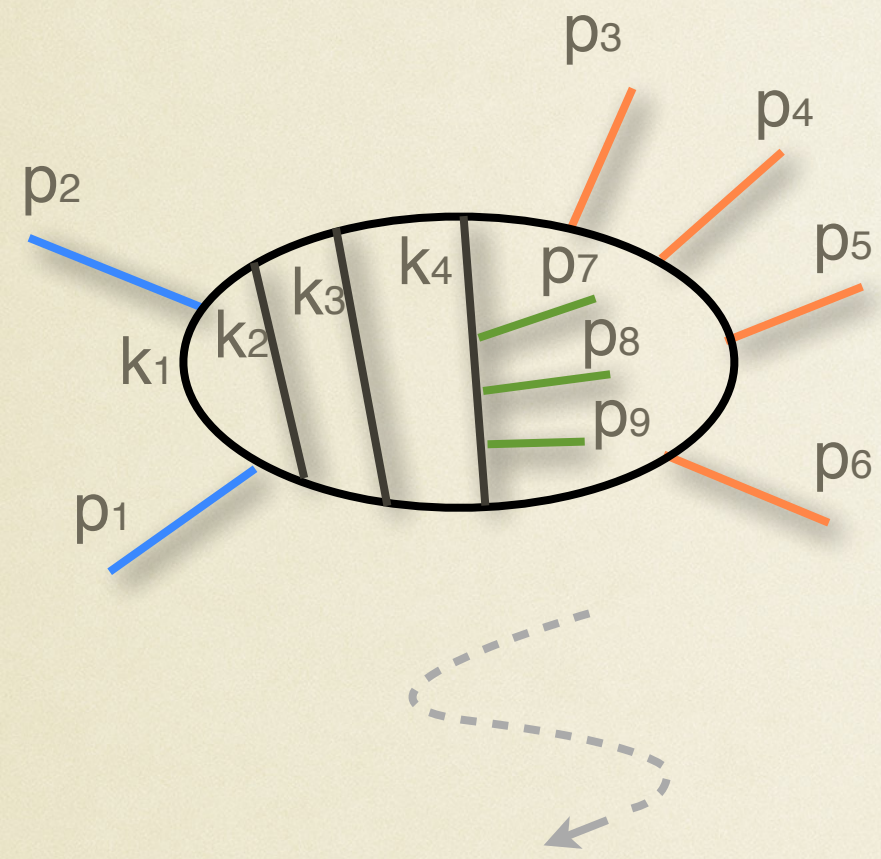


● Cosmology



Feynman Integrals

● Momentum-space Representation



N-denominator
generic Integral

$$= I_{a_1, \dots, a_N}^{[d]} = \int \prod_{i=1}^L d^d k_i \left(\prod_{n=1}^N \frac{1}{D_n^{a_n}} \right)$$

L loops, $E+1$ external momenta,

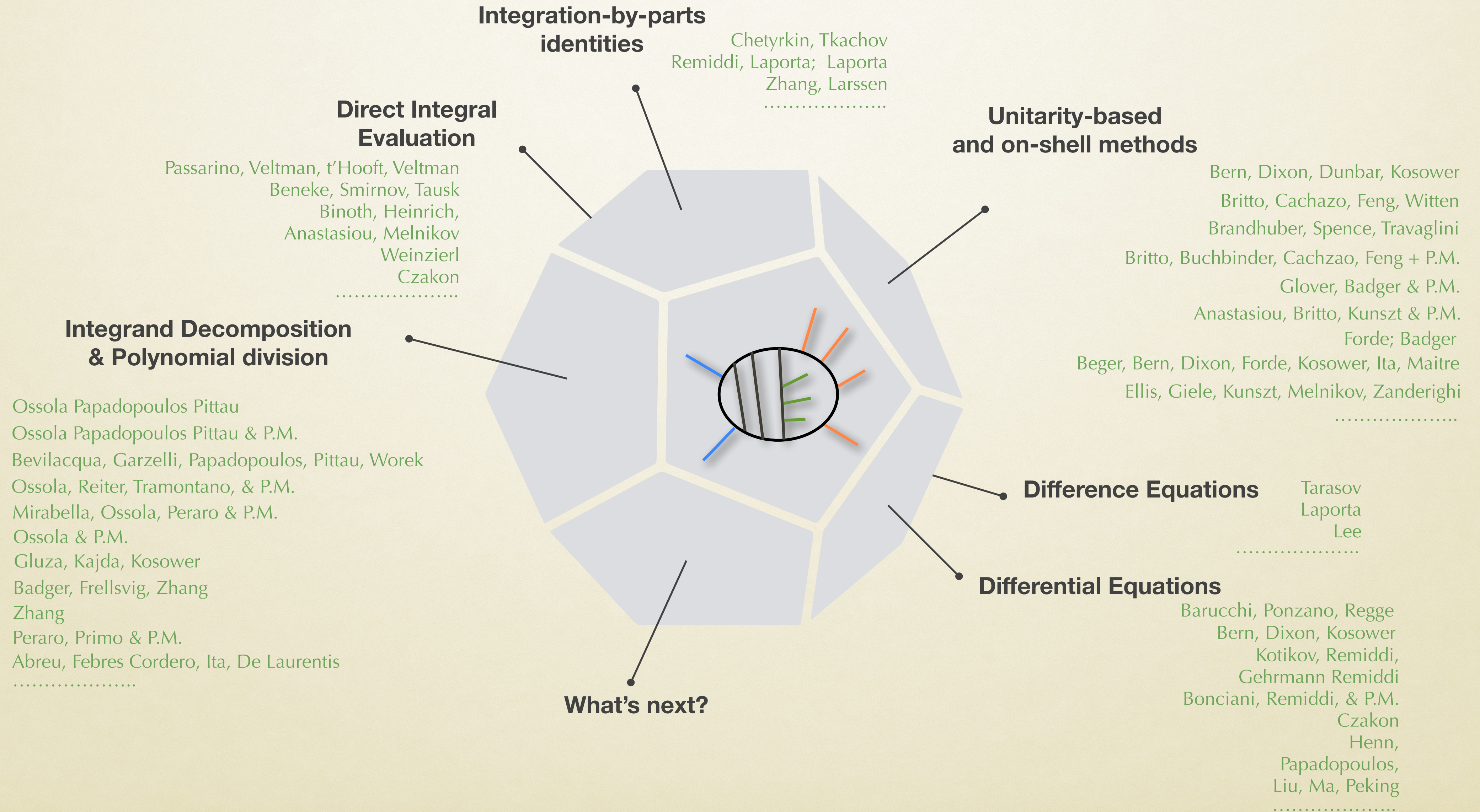
$N = LE + \frac{1}{2}L(L+1)$ (generalised) denominators

total number of *reducible* and *irreducible*
scalar products

't Hooft & Veltman

$$D_n = (p_1 \pm p_2 \pm \dots \pm k_1 \pm k_2 \pm \dots)^2 - m_n^2$$

Feynman Integrals / (a few) Evaluation Methods



Feynman Integrals

● Integration-by-parts Identities (IBPs)

Chetyrkin, Tkachov

Laporta, Remiddi

$$\int \prod_{i=1}^L d^d k_i \frac{\partial}{\partial k_j^\mu} \left(v_\mu \prod_{n=1}^N \frac{1}{D_n^{a_n}} \right) = 0$$

$$v_\mu = v_\mu(p_i, k_j)$$

arbitrary

● IBP equations

● Contiguity relations

$$\sum_i b_i I_{a_1, \dots, a_i \pm 1, \dots, a_N}^{[d]} = 0$$

⊕ **Generating an *overdimensioned (sparse) systems of linear equations***

⊕ **Solutions:**

✓ Gauss' Elimination

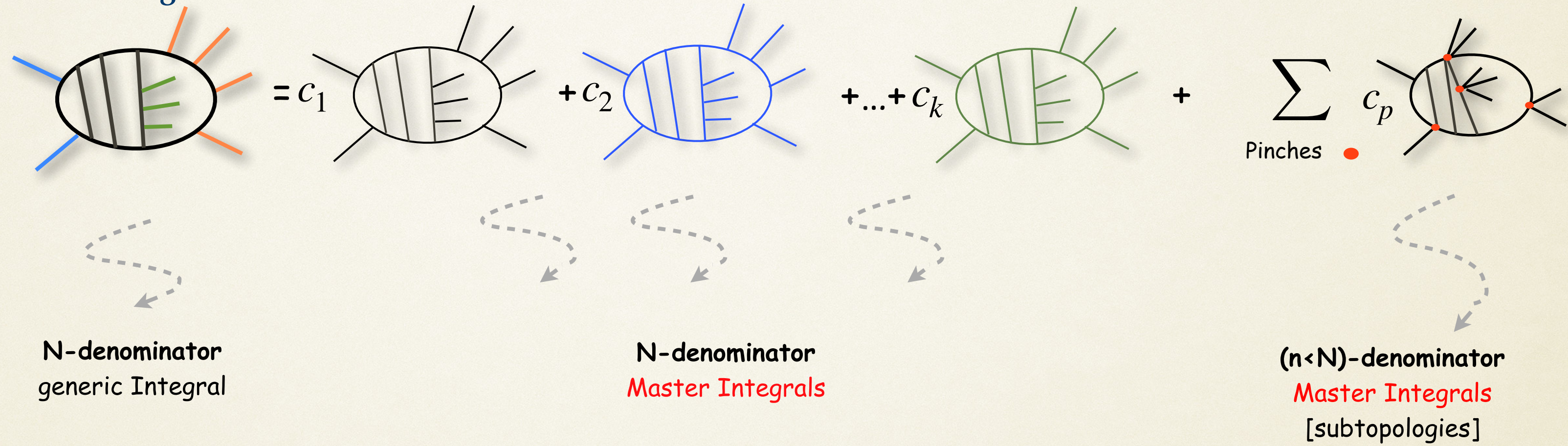
✓ Groebner Bases

✓ Syzygy Equations

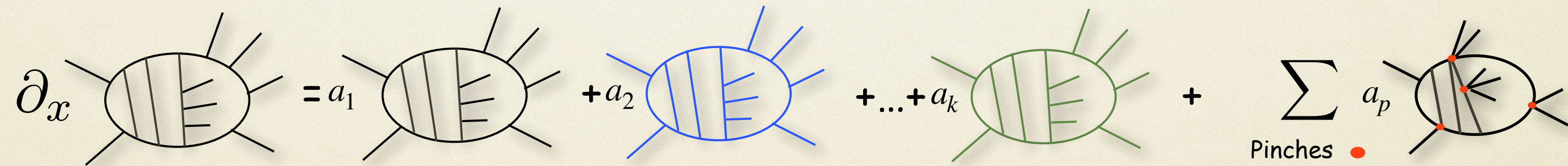
✓ **Finite Fields + Chinese Remainder Theorem + Rational Functions Reconstruction**

Linear relations for Feynman Integrals

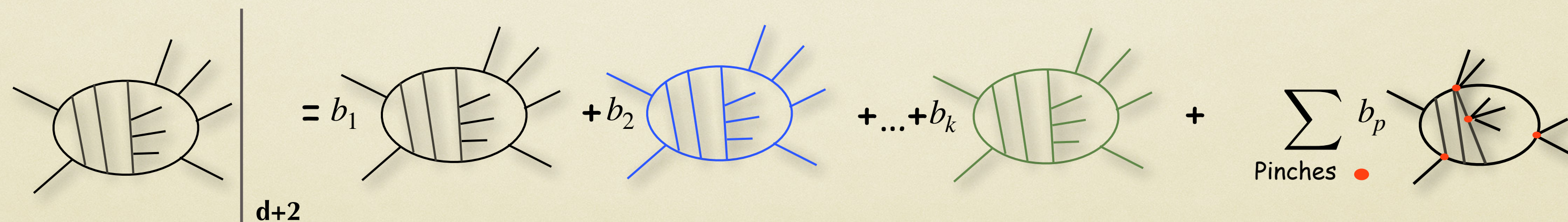
- Decomposition in terms of independent Master Integrals



- 1st order Differential Equations for MIs



- Dimension-Shift relations and Gram determinant relations



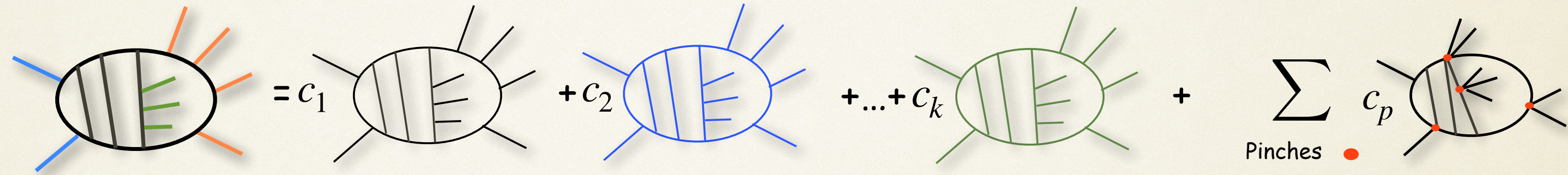
Linear relations for Feynman Integrals

- Relations among Integrals in dim. reg.

$$\text{Diagram} = c_1 \text{Diagram}_1 + c_2 \text{Diagram}_2 + \dots + c_k \text{Diagram}_k + \sum_{\text{Pinches}} c_p \text{Diagram}_p$$

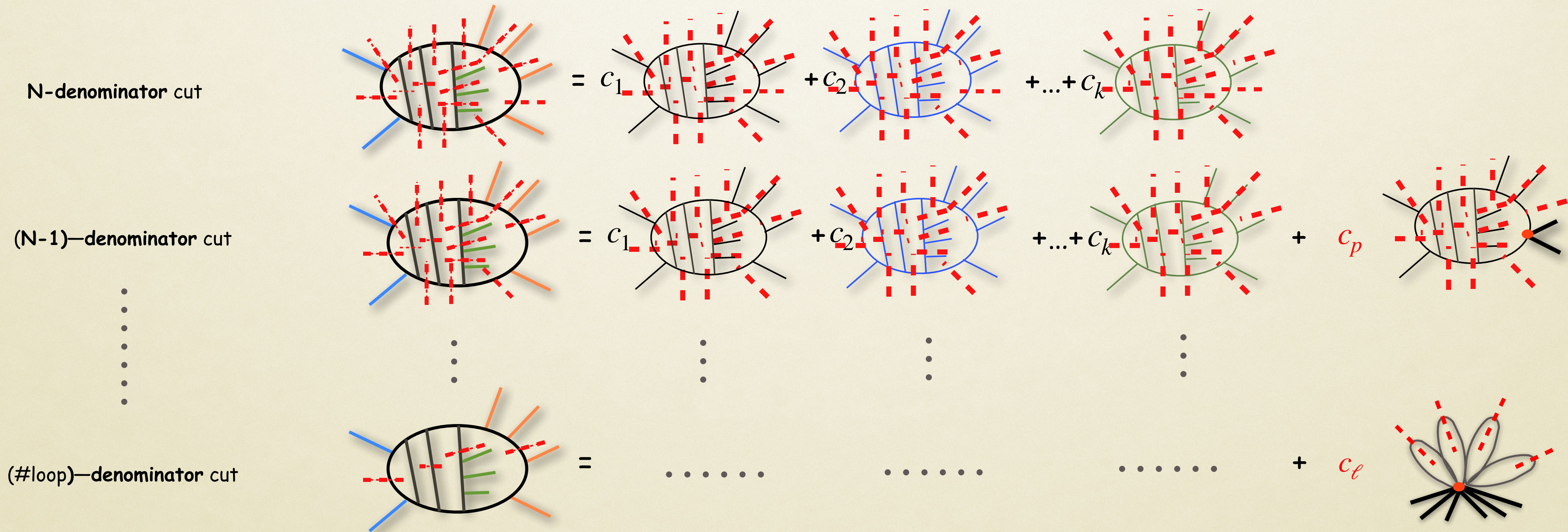
Linear relations for Feynman Integrals

- Relations among Integrals in dim. reg.



- Unitarity-based and on-shell methods

$$\frac{1}{D_i} \rightarrow \delta(D_i)$$



☑ cuts are like **pass-high** filters

⊕ (Block)-triangular system of linear equations: **integration required**

Linear relations for Feynman Integrands

- Relations among Integrals in dim. reg.

$$I = c_1 I_1 + c_2 I_2 + \dots + c_k I_k + \sum_{\text{Pinches}} c_p I_p$$

- OPP Integrand Decomposition

[integrand identity]

$$N(k_1, \dots, k_{\text{loop}}) I = \Delta_1 I_1 + \Delta_2 I_2 + \dots + \Delta_k I_k + \sum_{\text{Pinches}} \Delta_p I_p$$

- ✓ Δ_i are polynomials and $c_i \in \Delta_i$

- ✓ c_i determined by **polynomial fitting**

- ⊕ (Block)-triangular system of linear equations:

principle of polynomial identity: integration NOT required

- Cuts vs Residues vs Remainders

- ✓ Δ_i , therefore c_i , determined by **polynomial division** (Δ_i are the **remainders**)

Novel Perspective on (Feynman) Calculus

Outline

📌 Vector Space Structure of (Feynman, GKZ, Euler-Mellin, A-hypergeometric) twisted period Integrals

📌 De Rahm co-homology groups

📌 Space dimensions, Linear and Quadratic relations

📌 Intersection Numbers

📌 Computational Methods for n-forms:

📌 iterative method / fibration-based approach

📌 polynomial division and relative cohomology

📌 **Companion-tensor based method**

📌 D-modules and Pfaffians

📌 Applications

📌 Hypergeometric functions

📌 Feynman Integrals

📌 Matrix elements in Quantum Mechanics

📌 Green's functions and Wick's theorem

📌 Kontsevich-Witten tau-function

📌 Fourier integrals

📌 **Cosmological wave function integrals**

📌 Conclusions

Based on:

- **PM**, Mizera
Feynman Integral and Intersection Theory
JHEP 1902 (2019) 139 [arXiv: 1810.03818]
- Frellesvig, Gasparotto, Laporta, Mandal, **PM**, Mattiazzi, Mizera
Decomposition of Feynman Integrals in the Maximal Cut by Intersection Numbers
JHEP 1095 (2019) 153 [arXiv: 1901.11510]
- Frellesvig, Gasparotto, Mandal, **PM**, Mattiazzi, Mizera
Vector Space of Feynman Integrals and Multivariate Intersection Numbers
Phys. Rev. Lett. 123 (2019) 20, 201602 [arXiv 1907.02000]
- Frellesvig, Gasparotto, Laporta, Mandal, **PM**, Mattiazzi, Mizera
Decomposition of Feynman Integrals by Multivariate Intersection Numbers.
JHEP 03 (2021) 027 [arXiv 2008.04823]
- Chestnov, Gasparotto, Mandal, **PM**, Matsubara-Heo, Munch, Takayama
Macaulay Matrix for Feynman Integrals: linear relations and intersection numbers.
JHEP09 (2022) 187 [arXiv: 2204.12983]
- Cacciatori & **PM**,
Intersection Numbers in Quantum Mechanics and Field Theory.
2211.03729 [hep-th].
- **Brunello, Chestnov, Crisanti**, Frellesvig, Mandal & **PM**
Intersection Numbers, Polynomial Division & Relative Cohomology
JHEP09(2024)015 [arXiv: 2401.01897]
- **Brunello, Crisanti, Giroux, Smith & PM**,
Fourier Calculus from Intersection Theory
Phys.Rev.D 109 (2024) 9, 094047 [arXiv: 2311.14432]
- **Brunello, Chestnov, & PM**,
Intersection Numbers from Companion Tensor Algebra
2408.16668 [hep-th].
- **Benincasa, Brunello, Mandal, Vazão, & PM**,
On one-loop corrections to the Bunch-Davies wavefunction of the universe
2408.16386 [hep-th].

What we have found

Vector Space Structure of *Feynman* [- *Euler-Mellin* - *GKZ* - *A-hypergeometric*] Integrals

- **Vector decomposition**

$$I = \sum_{i=1}^{\nu} c_i J_i$$

 Master Integral = basis

ν = dimension of the vector space

- **Projections**

$$c_i = I \cdot J_i, \quad J_i \cdot J_j = \delta_{ij}$$

- **Completeness**

$$\sum_i J_i J_i = \mathbb{I}_{\nu \times \nu}$$

Vector Space Structure of *Feynman* [- *Euler-Mellin* - *GKZ* - *A-hypergeometric*] Integrals

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The two questions:

- 1) what is the vector space dimension ν ?
- 2) what is the *scalar product* “.” between integrals ?

Integrals as a Pairing

- **Twisted Period Integrals**

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \dots, z_m)$

$$I = \int_{\text{domain}} \text{integrand } d^m \mathbf{z}$$

Integrals as a Pairing

● Twisted Period Integrals

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \dots, z_m)$

$$I = \int_{\text{domain}} \text{integrand } d^m \mathbf{z} \qquad \text{integrand } d^m \mathbf{z} \equiv \left(\text{multivalued } f'n \right) \times \left(\text{differential form} \right)$$
$$= \int_{\text{domain}} \left(\text{multivalued } f'n \right) \left(\text{differential form} \right)$$

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$$= \int_{\text{domain}} \text{multivalued f'n} \quad \equiv \quad \text{differential form}$$

● Pairing

The **domain** and the **diff. form** are elements of *certain vector spaces*

Integrals as a Pairing

• Twisted Period Integrals

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \dots, z_m)$

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$$= \int_{\text{domain}} (\text{multivalued f'n}) (\text{differential form})$$

$$= \int_{\text{domain}} \text{multivalued f'n} \quad \equiv \quad \text{differential form}$$

↑
• Pairing

The **domain** and the **diff. form** are elements of *certain vector spaces*

• Important property:

$$\left. (\text{multivalued f'n}) \right|_{\partial(\text{domain})} = 0 \quad \Longrightarrow \quad \int_{\text{domain}} d(\text{integrand}) d^m \mathbf{z} = 0 = \int_{\partial(\text{domain})} (\text{integrand}) d^m \mathbf{z}$$

Basics of Intersection Theory

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Ohara, Yoshida,...

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \dots, z_m)$

$$I = \int_{\mathcal{C}} u(\mathbf{z}) \varphi_m(\mathbf{z})$$

$u(\mathbf{z})$ is a multivalued function
 $u(\partial\mathcal{C}) = 0$
 $\varphi_m(\mathbf{z})$ is a differential m -form

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Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \dots, z_m)$

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Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \dots, z_m)$

$$I = \int_{\mathcal{C}} u(\mathbf{z}) \cdot \varphi_m(\mathbf{z})$$

$u(\mathbf{z})$ is a multivalued function
 $u(\partial\mathcal{C}) = 0$
 $\varphi_m(\mathbf{z})$ is a differential m -form

• **Gamma function**

$$u = z^{s-1} e^{-z}, \quad \mathcal{C} = (0, \infty)$$

$$\varphi_1 = \frac{dz}{z^{n_1}}$$

• **Euler Beta function**

$$u = z^a (z-1)^b, \quad \mathcal{C} = (0, 1)$$

$$\varphi_1 = \frac{dz}{z^{n_1} (z-1)^{n_2}}$$

• **2F1 Hypergeometric**

$$u = z^a (z-1)^b (z-1/x)^c, \quad \mathcal{C} = (0, 1)$$

$$\varphi_1 = \frac{dz}{z^{n_1} (z-1)^{n_2} (z-1/x)^{n_3}}$$

• **... and many more**

.....

.....

.....

$n_i \in \mathbb{Z}$

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$u(\mathbf{z})$ is a multivalued function
 $u(\partial\mathcal{C}) = 0$
 $\varphi_m(\mathbf{z})$ is a differential m -form

- **The dawn of Integration by parts identities:**

- **Equivalence Classes of DIFFERENTIAL FORMS**

- There could exist many forms φ_m that upon integration give the same result I

- **Equivalence Classes of INTEGRATION CONTOURS**

- There could exist many contours \mathcal{C} that do not alter the the result of I

Vector Space Structure of Twisted Period Integrals

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Consider the $(m - 1)$ -differential form φ_{m-1} ,

$$0 = \int_{\mathcal{C}} d(u \varphi_{m-1}) = \int_{\mathcal{C}} u (\nabla_{\omega} \varphi_{m-1})$$

• **Covariant Derivative** $\omega \equiv d \log u$ $\nabla_{\omega} \equiv d + \omega \wedge \equiv u^{-1} \cdot d \cdot u$

$$u \rightarrow u^{-1}$$

$$0 = \int_{\mathcal{C}} d(u^{-1} \varphi_{m-1}) = \int_{\mathcal{C}} u^{-1} (\nabla_{-\omega} \varphi_{m-1})$$

• **Dual Covariant Derivative** $\nabla_{-\omega} \equiv d - \omega \wedge \equiv u \cdot d \cdot u^{-1}$

De Rham Twisted Co-Homology Groups

(dual) Homology groups $H_m^{\pm\omega}$ and (dual) Co-homology groups $H_{\pm\omega}^m$ are **isomorphic** [same dimension]
[same # of generators]

- **Cohomology group**

$$H_{\omega}^m(X) = \frac{\text{Ker}(\nabla_{\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$$

- **Dual Cohomology group**

$$H_{-\omega}^m(X) = \frac{\text{Ker}(\nabla_{-\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{-\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$$

- **Homology group**

$$H_m^{\omega}(X) = \frac{\text{Ker}(\partial \otimes u : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}{\text{Im}(\partial \otimes u : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}$$

- **Dual Homology group**

$$H_m^{-\omega}(X) = \frac{\text{Ker}(\partial \otimes u^{-1} : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}{\text{Im}(\partial \otimes u^{-1} : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}$$

De Rham Twisted Co-Homology Groups / Elements

(dual) Homology groups $H_m^{\pm\omega}$ and (dual) Co-homology groups $H_{\pm\omega}^m$ are **isomorphic** [same dimension]
[same # of generators]

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H_{\omega}^m$$

$$|\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H_{-\omega}^m$$

$$|\mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H_m^{\omega}$$

$$[\mathcal{C}_L| \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H_m^{-\omega}$$

De Rham Twisted Co-Homology Groups / Pairing / Integrals

(dual) Homology groups $H_m^{\pm\omega}$ and (dual) Co-homology groups $H_{\pm\omega}^m$ are **isomorphic** [same dimension]
[same # of generators]

• **Integrals** :: pairings of cycles and co-cycles

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H_{\omega}^m$$

$$|\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H_{-\omega}^m$$

$$\langle \varphi_L | \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \varphi_L(\mathbf{z}) = I$$

$$|\mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H_m^{\omega}$$

$$[\mathcal{C}_L| \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H_m^{-\omega}$$

De Rham Twisted Co-Homology Groups / Pairing / Dual Integrals

(dual) Homology groups $H_m^{\pm\omega}$ and (dual) Co-homology groups $H_{\pm\omega}^m$ are **isomorphic** [same dimension]
[same # of generators]

- **Dual Integrals** :: pairings of cycles and co-cycles

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H_{\omega}^m$$

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$$[\mathcal{C}_L | \varphi_R \rangle \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \varphi_R(\mathbf{z}) = \tilde{I}$$

$$|\mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H_m^{\omega}$$

$$[\mathcal{C}_L| \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H_m^{-\omega}$$

De Rham Twisted Co-Homology Groups / Pairing / Homology Intersection Number

(dual) Homology groups $H_m^{\pm\omega}$ and (dual) Co-homology groups $H_{\pm\omega}^m$ are **isomorphic** [same dimension]
[same # of generators]

- **Intersection numbers for cycles** :: pairings of cycles

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H_{\omega}^m$$

$$| \varphi_R \rangle \equiv \varphi_R(\mathbf{z}) \in H_{-\omega}^m$$

$$[\mathcal{C}_L | \mathcal{C}_R] \equiv \text{intersection number}$$

$$| \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H_m^{\omega}$$

$$[\mathcal{C}_L | \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H_m^{-\omega}$$

De Rham Twisted Co-Homology Groups / Pairing / Cohomology Intersection Number

(dual) Homology groups $H_m^{\pm\omega}$ and (dual) Co-homology groups $H_{\pm\omega}^m$ are **isomorphic** [same dimension]
[same # of generators]

- **Intersection numbers for co-cycles** :: pairings of co-cycles

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H_{\omega}^m$$

$$|\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H_{-\omega}^m$$

$$\langle \varphi_L | \varphi_R \rangle \equiv \frac{1}{2\pi i} \int_{\mathcal{X}} \iota(\varphi_L) \wedge \varphi_R$$

$$[\mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H_m^{\omega}$$

$$[\mathcal{C}_L] \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H_m^{-\omega}$$

Identity Resolution

$$\dim H_{\pm\omega}^n = \dim H_n^{\pm\omega} \equiv \nu$$

● Cohomology Space

[vector space of differential forms]

Cohomology basis

$$\langle e_i | \in H_{\omega}^n$$

Dual Cohomology basis

$$|h_i\rangle \in H_{-\omega}^n$$

$$i = 1, \dots, \nu$$

Identity resolution

$$\mathbb{I}_c = \sum_{i,j=1}^{\nu} |h_i\rangle (\mathbf{C}^{-1})_{ij} \langle e_j|$$

Metric matrix for Forms

$$\mathbf{C}_{ij} \equiv \langle e_i | h_j \rangle$$

● Homology Space

[vector space of integration contours]

Homology basis

$$|\gamma_i\rangle \in H_n^{\omega}$$

Dual Homology basis

$$[\eta_i] \in H_n^{-\omega}$$

$$i = 1, \dots, \nu$$

Identity resolution

$$\mathbb{I}_h = \sum_{i,j=1}^{\nu} |\gamma_i\rangle (\mathbf{H}^{-1})_{ij} [\eta_j]$$

Metric Matrix for Contours

$$\mathbf{H}_{ij} \equiv [\eta_i | \gamma_j]$$

Linear Relations

Linear Relations / IBPs identity / Gauss contiguity relations

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

• Master Integrals from Master Forms

Consider a set of ν MIs,

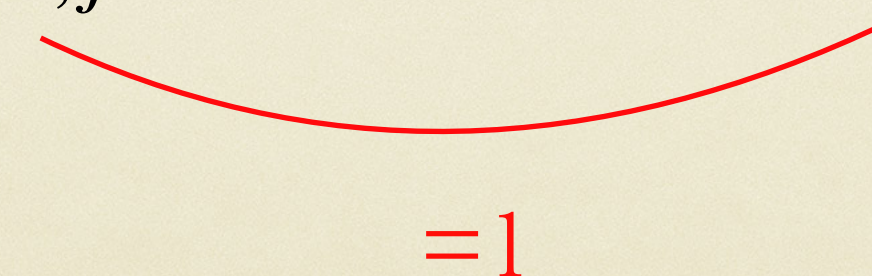
$$J_i = \int_{\mathcal{C}_R} u(\mathbf{z}) e_i(\mathbf{z}) = \langle e_i | \mathcal{C}_R \rangle, \quad i = 1, \dots, \nu,$$

• Integral Decomposition

$$I = \int_{\mathcal{C}_R} u(\mathbf{z}) \varphi_L(\mathbf{z}) = \langle \varphi_L | \mathcal{C}_R \rangle = \sum_{i=1}^{\nu} c_i J_i.$$

• Decomposition of Differential Forms

$$\langle \varphi_L | = \langle \varphi_L | \mathbb{I}_c = \langle \varphi_L | \sum_{i,j=1}^{\nu} |h_i\rangle \left(\mathbf{C}^{-1} \right)_{ij} \langle e_j |$$



Linear Relations / IBPs identity / Gauss contiguity relations

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

• Master Integrals from Master Forms

Consider a set of ν MIs,

$$J_i = \int_{\mathcal{C}_R} u(\mathbf{z}) e_i(\mathbf{z}) = \langle e_i | \mathcal{C}_R \rangle, \quad i = 1, \dots, \nu,$$

• Integral Decomposition

$$I = \int_{\mathcal{C}_R} u(\mathbf{z}) \varphi_L(\mathbf{z}) = \langle \varphi_L | \mathcal{C}_R \rangle = \sum_{i=1}^{\nu} c_i J_i.$$

• Decomposition of Differential Forms

• Master Decomposition Formula

$$\langle \varphi_L | = \langle \varphi_L | \mathbb{I}_c = \sum_{i=1}^{\nu} c_i \langle e_i |, \quad \text{with} \quad c_i = \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji}$$

coefficients depend on the basis choice
but **do not depend** on the dual basis choice

Quadratic Relations

Riemann Bilinear Relations

Riemann bilinear relations for periods of closed holomorphic (non-twisted) differential forms

$$\langle \phi_L | \phi_R \rangle = \int_{\Sigma} \phi_L \wedge \phi_R = \sum_{i=1}^g \left(\int_{a_i} \phi_L \int_{b_i} \phi_R - \int_{b_i} \phi_L \int_{a_i} \phi_R \right)$$

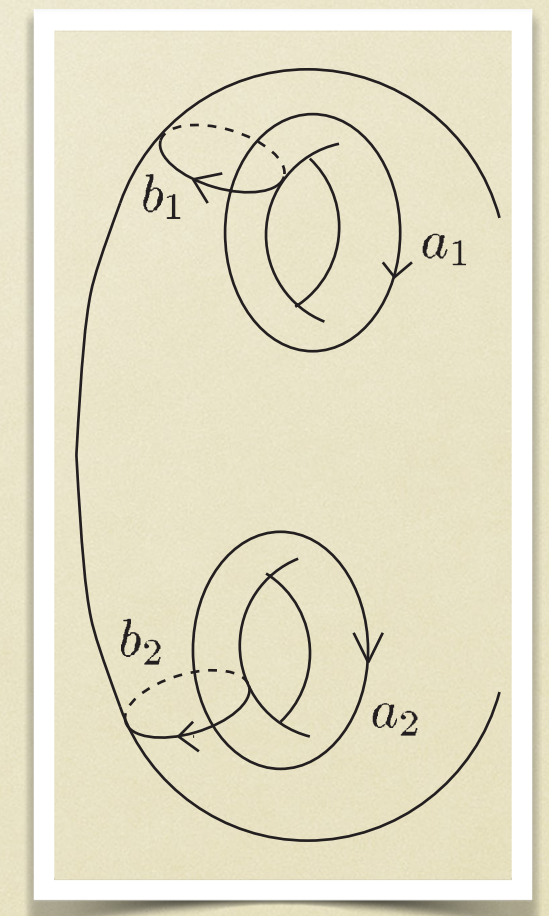
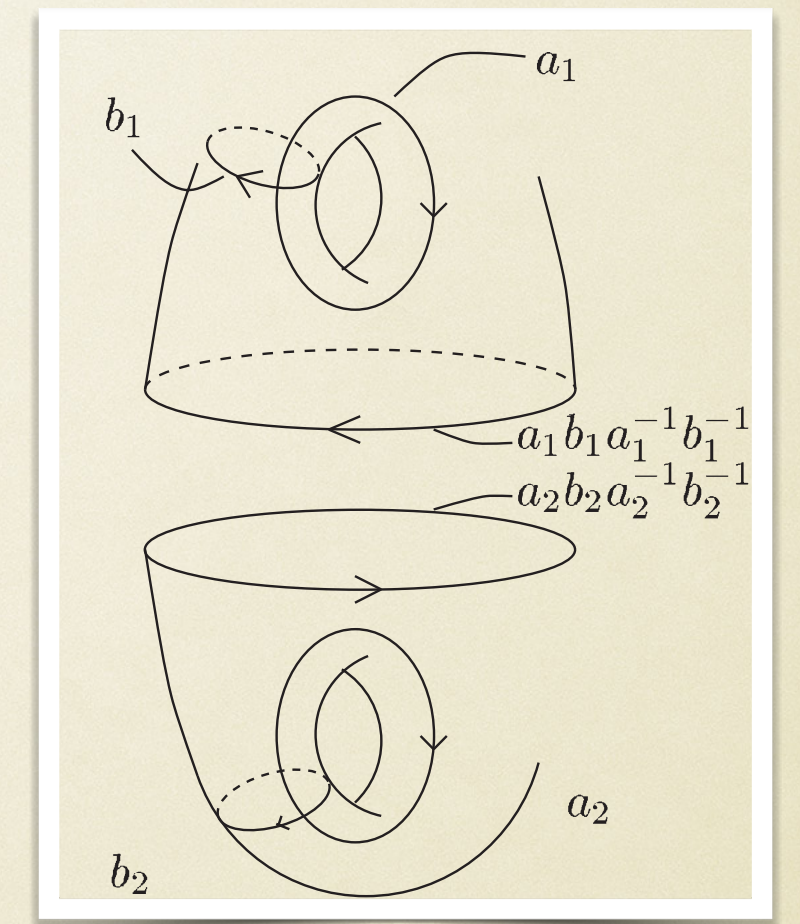
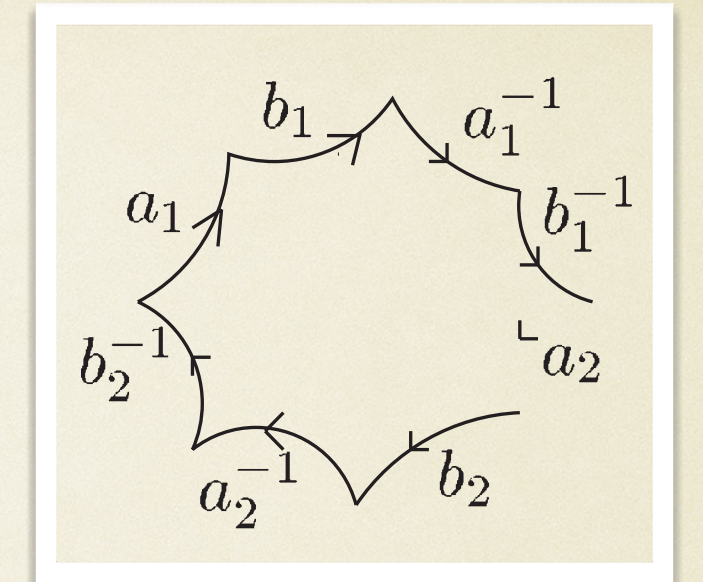
where Σ is an oriented Riemann surface of genus $g > 0$, built out of a $4g$ -gon with edges $\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1}$ (where the exponent ± 1 stands for clock/anticlockwise orientation) and gluing each edge with its inverse. The integration contours a_i and b_i , for $i = 1, \dots, g$, are a canonical bases of cycles, hence intersect *transversally*, *i.e.* their pairwise intersection numbers are: $a_i \cdot a_j = b_i \cdot b_j = 0$, and $a_i \cdot b_j = -b_j \cdot a_i = \delta_{ij}$. Riemann bilinear relation can be cast as,

$$\langle \phi_L | \phi_R \rangle = \sum_{i,j} \int_{\gamma_i} \phi_L (\mathbf{H}^{-1})_{ij} \int_{\gamma_j} \phi_R,$$

where $\{\gamma_i\}_{i=1, \dots, g} = a_i$ and $\{\gamma_i\}_{i=g+1, \dots, 2g} = b_i$, and $\mathbf{H}_{ij} = [\gamma_i | \gamma_j]$, namely

$$\mathbf{H} = \begin{pmatrix} 0 & \mathbb{I}_{g \times g} \\ -\mathbb{I}_{g \times g} & 0 \end{pmatrix}, \quad \text{yielding} \quad \mathbf{H}^{-1} = \begin{pmatrix} 0 & -\mathbb{I}_{g \times g} \\ \mathbb{I}_{g \times g} & 0 \end{pmatrix},$$

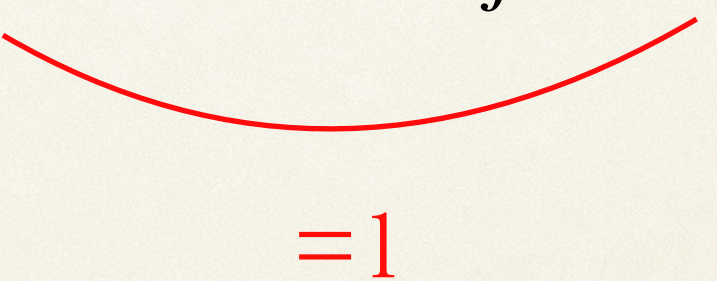
and $\mathbb{I}_{g \times g}$ is the identity matrix in the $(g \times g)$ -space.



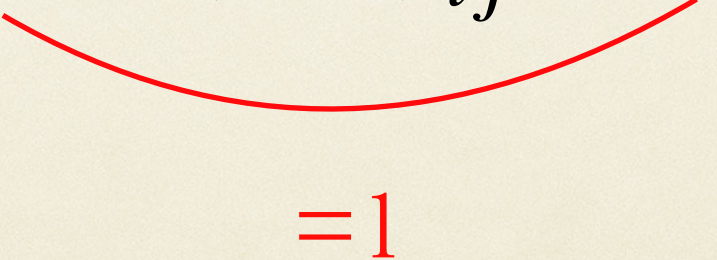
Twisted Riemann Periods Relations (TRPR)

Cho, Matsumoto (1995)

$$\langle \varphi_L | \varphi_R \rangle = \langle \varphi_L | \mathbb{I}_h | \varphi_R \rangle = \sum_{i,j=1}^{\nu} \langle \varphi_L | \gamma_i \rangle \left(\mathbf{H}^{-1} \right)_{ij} [\eta_j | \phi_R] = \sum_{i,j} \int_{\gamma_i} u \varphi_L \left(\mathbf{H}^{-1} \right)_{ij} \int_{\eta_j} u^{-1} \varphi_R$$



$$[C_L | C_R] = [C_L | \mathbb{I}_c | C_R] = \sum_{i,j=1}^{\nu} [C_L | h_i] \left(\mathbf{C}^{-1} \right)_{ij} \langle e_j | C_R \rangle = \sum_{i,j} \int_{C_L} u^{-1} h_i \left(\mathbf{C}^{-1} \right)_{ij} \int_{C_R} u e_j$$



● Generalising Riemann Bilinear Relations

Vector Space Structure of Feynman Integrals

Vector Space Dimensions / counting “holes”

Chetyrkin, Tkachov (1981); Remiddi, Laporta (1996); Laporta (2000)

Number of **Master Integrals**

Betti numbers

Number of **Critical Points** Lee, Pomeranski (2013)

Relation with χ_E

Aluffi, Marcolli (2008)

Bitoun, Bogner, Klausen, Panzer (2018)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

**De Rahm
Co-Homology**
 $\nu = \dim H$

Maximum likelihood degree

Agostini, Brysiewicz, Fevola, Sturmfels, Tellen (2021)

Bosma, Sogaard, Zhang (2017) Primo, Tancredi (2017)

Number of **Independent Contours**

Holonomic rank of GKZ systems

Gelfand Kapranov Zelevinski

Number of **Independent Forms**

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

Mixed volume of Newton Polyhedra

Bernstein-Khobaskii-Kushnirenko

Saito Sturmfels Takayama

Ideal saturation / dimension of quotient space

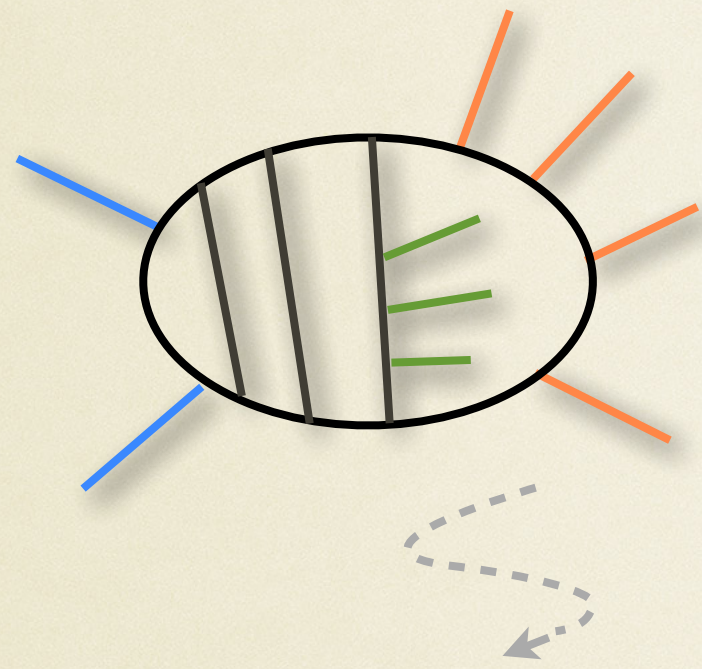
Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2020)

Parametric Representation(s)

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019, 2020)

- Upon a change of integration variables



N-denominator
generic Integral

$$I_{a_1, \dots, a_N}^{[d]} = \int_{\mathcal{C}} u(\mathbf{z}) \varphi_N(\mathbf{z})$$

$$\varphi_N(\mathbf{z}) = \hat{\varphi}(\mathbf{z}) d^N \mathbf{z} \quad \text{differential } N\text{-form}$$

$$d^N \mathbf{z} = dz_1 \wedge \dots \wedge dz_N$$

$$\hat{\varphi}_N(\mathbf{z}) = f(\mathbf{z}) \prod_i z_i^{-a_i}$$

$$u(\mathbf{z}) = \mathcal{P}(\mathbf{z})^\gamma$$

$$\mathcal{P}(\mathbf{z}) = \text{graph-Polynomial}$$

$$\gamma(d) = \text{generic exponent}$$

Feynman Integrals :: Baikov Representation

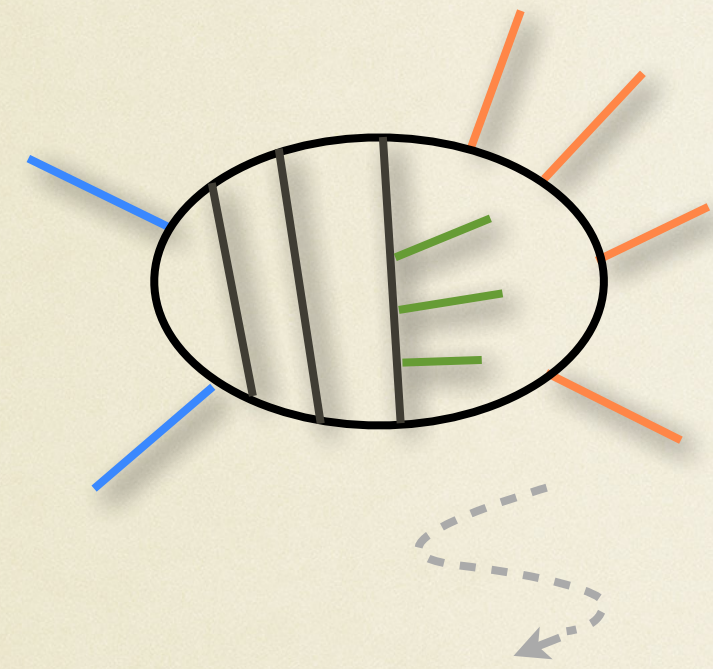
Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019, 2020)

• Denominators as integration variables

Baikov (1996)

Frellesvig and Papadopoulos (2017)



N-denominator
generic Integral

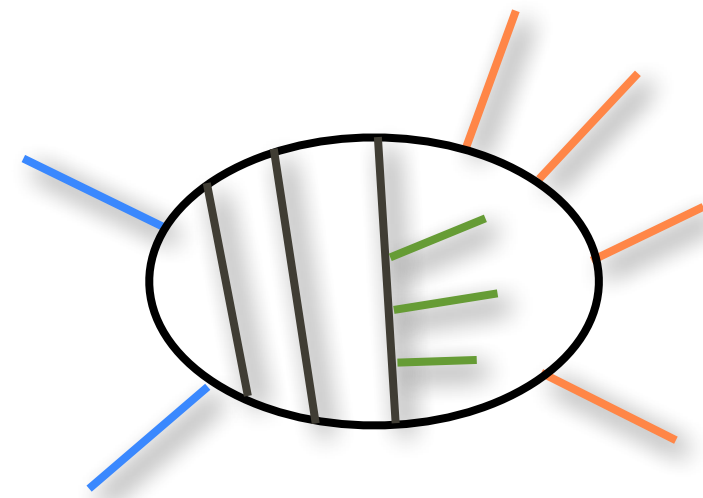
$$\{D_1, \dots, D_N\} \rightarrow \{z_1, \dots, z_N\} \equiv \mathbf{z}$$

$$I_{a_1, \dots, a_N}^{[d]} = \int_{\mathcal{C}} B(\mathbf{z})^\gamma \frac{d^n \mathbf{z}}{z_1^{a_1} z_2^{a_2} \dots z_N^{a_N}}$$

$$B(p_i, k_j) = \begin{vmatrix} k_1^2 & \dots & (k_1 \cdot p_{E-1}) \\ \vdots & \ddots & \vdots \\ (p_{E-1} \cdot k_1) & \dots & p_{E-1}^2 \end{vmatrix} = B(\mathbf{z})$$

Gram determinant

$$\gamma \equiv (d - E - L - 1)/2$$



$$I_{a_1, \dots, a_N}^{[d]} = \int_{\mathcal{C}} B(\mathbf{z})^\gamma \frac{d^n \mathbf{z}}{z_1^{a_1} z_2^{a_2} \dots z_N^{a_N}} \quad u = B^\gamma, \quad \gamma \equiv (d - E - L - 1)/2$$

• Vector space dimension

$$\omega \equiv \sum_{i=1}^n \hat{\omega}_i dz_i = d \log(u)$$

$$\mathcal{Z}_\omega = \{\text{zeroes of } \omega\}$$

$$\mathcal{P}_\omega = \{\text{poles of } \omega\} \cup \{\infty\}$$

$$\nu \equiv \dim(H_{\pm\omega}^n) = \dim(\mathcal{Z}_\omega) = (-1)^n (n + 1 - \chi(\mathcal{P}_\omega)) = \text{number of solutions of the system} \begin{cases} \omega_1 = 0 \\ \vdots \\ \omega_n = 0 \end{cases} \quad (\text{Zero-dimensional})$$

• (dual) bases choices: Master Forms for Master Integrals

$$\langle e_i | \in H_\omega^n$$

$$|h_i\rangle \in H_{-\omega}^n$$

$$i = 1, \dots, \nu$$

• Decomposing Forms for Decomposing Integrals

$$\langle \varphi | = c_1 \langle e_1 | + c_2 \langle e_2 | + c_3 \langle e_3 | + \dots + c_\nu \langle e_\nu |$$

$$c_i = \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle (\mathbf{C}^{-1})_{ji}$$

Four special applications:

i) Differential Equations / Pfaffian system

Mizera & P.M. (2018)
Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

- External Derivative

$$\partial_x I = \partial_x \langle \varphi | \mathcal{C} \rangle = \partial_x \int_{\mathcal{C}} u \varphi = \int_{\mathcal{C}} u \left(\frac{\partial_x u}{u} \wedge + \partial_x \right) \varphi = \langle (\partial_x + \sigma) \varphi | \mathcal{C} \rangle$$

- External (connection) dLog-form

$$\nabla_{x,\sigma} \equiv \partial_x + \sigma \quad \sigma = \partial_x \log u$$

- Derivative of Master Forms

$$\partial_x \langle e_i | = \langle \nabla_{x,\sigma} e_i | = \langle \nabla_{x,\sigma} e_i | h_k \rangle \underbrace{(C^{-1})_{kj}}_{=1} \langle e_j | = \Omega_{ij} \langle e_j |$$

- System of DEQ for Master Forms

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |, \quad \Omega = \Omega(d, x)$$

An analogous System of DEQ can be derived for dual forms: $u \rightarrow u^{-1} \implies \nabla_{x,\sigma} \rightarrow \nabla_{x,-\sigma}$

ii) Differential Equations / Higher-Order DEQ

- Generic Bases

$$\begin{pmatrix} \langle e_1 | \\ \langle e_2 | \\ \langle e_3 | \\ \vdots \\ \langle e_\nu | \end{pmatrix}$$

- Special Bases 1

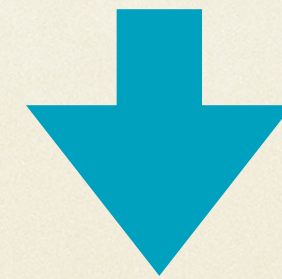
$$\begin{pmatrix} \langle e_i | \\ \partial_x \langle e_i | \\ \partial_x^2 \langle e_i | \\ \vdots \\ \partial_x^{\nu-1} \langle e_i | \end{pmatrix}$$

- Decomposition

$$\langle \varphi | = c_1 \langle e_1 | + c_2 \langle e_2 | + c_3 \langle e_3 | + \dots + c_\nu \langle e_\nu |$$

- Decomposition

$$\partial_x^\nu \langle e_i | = a_{i,0} \langle e_i | + a_{i,1} \partial_x \langle e_i | + a_{i,2} \partial_x^2 \langle e_i | + \dots + a_{i,\nu-1} \partial_x^{\nu-1} \langle e_i |$$



- Higher-order Diff.Eq. for the i-th Master Form (Master Integral)

$$\sum_{j=0}^{\nu} a_{i,j} \partial_x^j \langle e_i | = 0, \quad (a_{i,\nu} \equiv -1)$$

iii) Finite Difference Equation / Dimension-shift equation

- Generic Bases

$$\begin{pmatrix} \langle e_1 | \\ \langle e_2 | \\ \langle e_3 | \\ \vdots \\ \langle e_\nu | \end{pmatrix}$$

- Decomposition

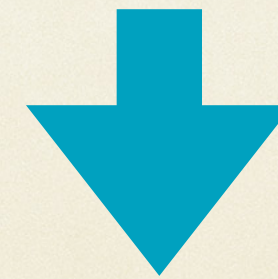
$$\langle \varphi | = c_1 \langle e_1 | + c_2 \langle e_2 | + c_3 \langle e_3 | + \dots + c_\nu \langle e_\nu |$$

- Special Bases 2

$$\begin{pmatrix} \langle e_i | \\ \langle B e_i | \\ \langle B^2 e_i | \\ \vdots \\ \langle B^{\nu-1} e_i | \end{pmatrix}$$

- Decomposition

$$\langle B^\nu e_i | = b_{i,0} \langle e_i | + b_{i,1} \langle B e_i | + b_{i,2} \langle B^2 e_i | + \dots + b_{i,\nu-1} \langle B^{\nu-1} e_i |$$



$$u = B^\gamma, \quad \gamma \equiv (d-E-L-1)/2$$

$$J_i^{[d]} = \int_C u e_i = \langle e_i | C \rangle$$

$$J_i^{[d+2j]} = \int_C u B^j e_i = \langle B^j e_i | C \rangle$$

- Finite Difference Equation for the i-th Master Form (Master Integral)

$$\sum_{j=0}^{\nu} b_{i,j} \langle B^j e_i | = 0, \quad (b_{i,\nu} \equiv -1)$$

iv) Secondary Equation

Matsubara-Heo, Takayama (2019)

Weinzierl (2020)

Chestnov, Gasparotto, Munch, Matsubara-Heo, Takayama & P.M. (2022)

- DEQ for forms

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

$$\Omega_{ij} = \langle (\partial_x + \sigma_x) e_i | h_k \rangle (\mathbf{C}^{-1})_{kj}$$

- DEQ dual-forms

$$\partial_x |h_i\rangle = \tilde{\Omega}_{ji} |h_j\rangle$$

$$\tilde{\Omega}_{ji} = (\mathbf{C}^{-1})_{jk} \langle e_k | (\partial_x - \sigma_x) h_i \rangle$$

- Secondary Equation for the Intersection Matrix

$$\mathbf{C}_{ij} \equiv \langle e_i | h_j \rangle$$

$$\partial_x \mathbf{C} = \mathbf{\Omega} \cdot \mathbf{C} + \mathbf{C} \cdot \tilde{\mathbf{\Omega}}, \quad \partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}} \cdot \mathbf{C}^{-1} - \mathbf{C}^{-1} \cdot \mathbf{\Omega}$$

Intersection Numbers for 1-forms

Intersection Numbers for **1-forms** Cho and Matsumoto (1998)

● **1-form** $\langle \varphi | \equiv \hat{\varphi}(z) dz$ $\hat{\varphi}(z)$ rational function

● **Zeroes and Poles of ω**

$$\omega \equiv d \log u$$

$\nu = \{ \text{the number of solutions of } \omega = 0 \}$

$\mathcal{P} \equiv \{ z \mid z \text{ is a pole of } \omega \}$

\mathcal{P} can also include the pole at infinity if $\text{Res}_{z=\infty}(\omega) \neq 0$.

● **Intersection Numbers**

1-forms φ_L and φ_R

$$\langle \varphi_L | \varphi_R \rangle := \frac{1}{2\pi i} \int_{\mathcal{X}} \iota(\varphi_L) \wedge \varphi_R = \sum_{p \in \mathcal{P}} \text{Res}_{z=p}(\psi_p \varphi_R)$$

ψ_p is a function (0-form), solution to the differential equation $\nabla_{\omega} \psi = \varphi_L$, around p

Intersection Numbers for n-forms :: Iterative Method

Nested Integrations / Fibration-based approach

- **Multivariate integral decomposition**

$$I = \sum_{i=1}^{\nu} c_i J_i$$

$$I = \int dz_n \dots \int dz_3 \int dz_2 \int dz_1 f(z_n, \dots, z_3, z_2, z_1)$$

$$J_i \equiv \int dz_n \dots \int dz_3 \int dz_2 \int dz_1 f_i(z_n, \dots, z_1) \quad \text{[master integrals]}$$

- **Fibrations: decompositions' tower**

$$I = \int dz_n \dots \int dz_3 \int dz_2 \underbrace{\int dz_1 f(z_n, \dots, z_3, z_2, z_1)}_{\exists \nu^{(1)} \text{ master integrals in } z_1}$$

$$I = \int dz_n \dots \int dz_3 \underbrace{\int dz_2 \sum_{i_1=1}^{\nu^{(1)}} c_{i_1}(z_n, \dots, z_3, z_2) J_{i_1}(z_n, \dots, z_3, z_2)}_{\exists \nu^{(2)} \text{ master integrals in } z_2}$$

$$I = \int dz_n \dots \underbrace{\int dz_3 \sum_{i_2=1}^{\nu^{(2)}} c_{i_2}(z_n, \dots, z_3) J_{i_2}(z_n, \dots, z_3)}_{\exists \nu^{(3)} \text{ master integrals in } z_3}$$

⋮

$$I = \underbrace{\int dz_n \sum_{i_n=1}^{\nu^{(n-1)}} c_{i_n}(z_n) J_{i_n}(z_n)}_{\exists \nu \text{ master integrals in } z_n}$$

$$I = \sum_{i=1}^{\nu} c_i J_i$$

Intersection Numbers for **n-forms** (I)

Ohara (1998) Mizera (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

• by *Induction*:

• **(n-1)-form Vector Space: known!**

$$\nu_{\mathbf{n}-1} \quad \langle e_i^{(\mathbf{n}-1)} | \quad | h_i^{(\mathbf{n}-1)} \rangle \quad (\mathbf{C}_{(\mathbf{n}-1)})_{ij} \equiv \langle e_i^{(\mathbf{n}-1)} | h_j^{(\mathbf{n}-1)} \rangle$$

• **n-form decomposition: $\mathbf{n} = (\mathbf{n}-1) + (\mathbf{n})$**

$$\langle \varphi_L^{(\mathbf{n})} | = \sum_{i=1}^{\nu_{\mathbf{n}-1}} \langle e_i^{(\mathbf{n}-1)} | \wedge \langle \varphi_{L,i}^{(\mathbf{n})} | ,$$

$$\langle \varphi_{L,i}^{(\mathbf{n})} | = \langle \varphi_L^{(\mathbf{n})} | h_j^{(\mathbf{n}-1)} \rangle (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ji} ,$$

$$\langle \varphi_{L,i}^{(\mathbf{n})} | (\mathbf{C}_{(\mathbf{n}-1)})_{ij} = \langle \varphi_L^{(\mathbf{n})} | h_j^{(\mathbf{n}-1)} \rangle$$

$$| \varphi_R^{(\mathbf{n})} \rangle = \sum_{i=1}^{\nu_{\mathbf{n}-1}} | h_i^{(\mathbf{n}-1)} \rangle \wedge | \varphi_{R,i}^{(\mathbf{n})} \rangle ,$$

$$| \varphi_{R,i}^{(\mathbf{n})} \rangle = (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ij} \langle e_j^{(\mathbf{n}-1)} | \varphi_R^{(\mathbf{n})} \rangle ,$$

$$(\mathbf{C}_{(\mathbf{n}-1)})_{ij} | \varphi_{R,j}^{(\mathbf{n})} \rangle = \langle e_i^{(\mathbf{n}-1)} | \varphi_R^{(\mathbf{n})} \rangle$$

🔔 Intersection Numbers for **n-forms** :: Recursive Formula

$$\begin{aligned} \langle \varphi_L^{(\mathbf{n})} | \varphi_R^{(\mathbf{n})} \rangle &= \sum_{i,j} \langle \varphi_L^{(\mathbf{n})} | h_j^{(\mathbf{n}-1)} \rangle (\mathbf{C}_{(\mathbf{n}-1)})_{ji}^{-1} \langle e_i^{(\mathbf{n}-1)} | \varphi_R^{(\mathbf{n})} \rangle \\ &= \sum_{i,j} \langle \varphi_{L,i}^{(\mathbf{n})} | (\mathbf{C}_{(\mathbf{n}-1)})_{ij} \varphi_{R,j}^{(\mathbf{n})} \rangle \end{aligned}$$

= 1

$$\partial_{z_n} \psi_i^{(\mathbf{n})} + \psi_j^{(\mathbf{n})} \hat{\Omega}_{ji}^{(\mathbf{n})} = \hat{\varphi}_{L,i}^{(\mathbf{n})} ,$$

$\hat{\Omega}^{(\mathbf{n})}$ is a $\nu_{\mathbf{n}-1} \times \nu_{\mathbf{n}-1}$ matrix, whose entries are given by

$$\hat{\Omega}_{ji}^{(\mathbf{n})} = \langle (\partial_{z_n} + \hat{\omega}_n) e_j^{(\mathbf{n}-1)} | h_k^{(\mathbf{n}-1)} \rangle (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ki}$$

Intersection Numbers for **n-forms** (I)

Ohara (1998) Mizera (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

● Property of Intersection Number

invariance under differential forms redefinition within the same equivalence classes,

$$\langle \varphi_L | \varphi_R \rangle = \langle \varphi'_L | \varphi'_R \rangle, \quad \varphi'_L = \varphi_L + \nabla_{\omega} \xi_L, \quad \varphi'_R = \varphi_R + \nabla_{-\omega} \xi_R$$

● Global Residue Thm Weinzierl (2020)

choose ξ_L and ξ_R , to build φ'_L and φ'_R that contain only simple poles, and if $\hat{\Omega}^{(n)}$ is reduced to Fuchsian form



the computation of multivariate intersection number can benefit of the evaluation of intersection numbers for dlog forms at each step of the iteration.

● Special dual basis choice CaronHuot Pokraka (2019-2021)

Relative Dirac-delta basis elements trivialise the evaluation of the intersection numbers

● Multi-pole ansatz Fontana Peraro (2023)

Solving $\nabla_{\omega} \psi = \varphi_L$, bypassing the pole factorisation, and using FF reconstruction methods.
(avoiding irrational functions which would disappear in the intersection numbers)

Contiguity relations & Differential Equations of Special Functions

- ✓ Gamma Functions
- ✓ Beta Functions
- ✓ Hypergeometric ${}_2F_1$
- ✓ Appel F_D
- ✓ Lauricella functions
- ✓ Hypergeometric ${}_3F_2$

Lauricella F_D Functions

$$\beta(a, c - a) F_D(a, b_1, b_2, \dots, b_m, c; x_1, \dots, x_m) = \int_{\mathcal{C}} u \varphi = \omega \langle \varphi | \mathcal{C} \rangle$$

$$u = z^{a-1} (1 - z)^{-a+c-1} \prod_{i=1}^m (1 - x_i z)^{-b_i},$$

$$\mathcal{C} = [0, 1], \quad \varphi = dz, \quad \omega = d \log(u),$$

$$\nu = m+1, \quad \mathcal{P} = \left\{ 0, \frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_m}, 1, \infty \right\}$$

$$\nu = \dim H_{\pm\omega}^1 = [\text{number of P-poles} - 2] = [\text{number of P-poles} - (1+1)]$$

Feynman Integrals Decomposition

Intersection Numbers for **1-forms** (II)

Brunello, Chestnov, Crisanti, Frellesvig, Mandal & P.M. (2023)

● Polynomial Division Fontana Peraro (2023)

$$\langle \varphi_L | \varphi_R \rangle = -\text{Res}_{\langle B \rangle}(g) - \text{Res}_{z=\infty}(g) \quad g = \psi_R \varphi_L$$

$$\text{Res}_{\langle B \rangle}(g) = \frac{g_{-1, \kappa-1}}{\ell_c}$$

$$\left[\partial_z \psi_R(z, \beta) + \partial_\beta \psi_R(z, \beta) \partial_z B(z) - \omega \psi_R(z, \beta) - \varphi_R \right]_{\mathcal{B}} = 0,$$

$$\psi_R = \sum_{i=\min}^{\max} \sum_{j=0}^{\kappa-1} \psi_{R,ij} z^j \beta^i \quad \beta = B(z)$$

where κ and ℓ_c are the degree and the leading coefficient of B

☑ Series expansion by polynomial division modulo $\langle \mathcal{B} \rangle \equiv \langle \mathcal{B}(z) - \beta \rangle$

☑ Bypassing the knowledge of the poles' position, hence avoiding algebraic extension and explicit polynomial factorisation

Intersection Numbers for **1-forms** (II)

Brunello, Chestnov, Crisanti, Frellesvig, Mandal & P.M. (2023)

● **Polynomial Division** Fontana Peraro (2023)

$$\langle \varphi_L | \varphi_R \rangle = -\text{Res}_{\langle B \rangle}(g) - \text{Res}_{z=\infty}(g) \quad g = \psi_R \varphi_L$$

$$\text{Res}_{\langle B \rangle}(g) = \frac{g_{-1, \kappa-1}}{\ell_c}$$

$$\left[\partial_z \psi_R(z, \beta) + \partial_\beta \psi_R(z, \beta) \partial_z B(z) - \omega \psi_R(z, \beta) - \varphi_R \right]_{\mathcal{B}} = 0,$$

$$\psi_R = \sum_{i=\min}^{\max} \sum_{j=0}^{\kappa-1} \psi_{R,ij} z^j \beta^i \quad \beta = B(z)$$

where κ and ℓ_c are the degree and the leading coefficient of B

☑ Series expansion by polynomial division modulo $\langle \mathcal{B} \rangle \equiv \langle \mathcal{B}(z) - \beta \rangle$

☑ Bypassing the knowledge of the poles' position, hence avoiding algebraic extension and explicit polynomial factorisation

● **Delta-bases** Caron-Huot and Pokraka (2021)

$$\delta_z := \frac{u(z)}{u(0)} d\theta_{z,0}, \quad \langle \varphi_L | \delta_z \rangle := \frac{-1}{2\pi i} \int_{\mathcal{X}} \varphi_L \wedge \delta_z = \text{Res}_{z=0} \left(\frac{u(z)}{u(0)} \varphi_L \right)$$

Intersection Numbers for **1-forms** (II)

Brunello, Chestnov, Crisanti, Frellesvig, Mandal & P.M. (2023)

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● Ordinary Cohomology vs Relative Cohomology

✓ Evanescent regulator limit

$$c_i = \lim_{\rho \rightarrow 0} \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle \mathbf{C}_{ji}^{-1} = \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle_{\text{LT}} (\mathbf{C}_{\text{LT}}^{-1})_{ji}$$

$h_j \sim z^\tau$ with $\tau < 0$, around $z = 0$

$$\langle \eta | h_j \rangle_{\text{LT}} = \langle \eta | \delta_z^{(-\tau)} \rangle \quad \delta_z^{(k)} \sim \frac{\partial_k^{(k-1)} u(z)}{u(0)} d\theta$$

Intersection Numbers for **1-forms** (II)

Brunello, Chestnov, Crisanti, Frellesvig, Mandal & P.M. (2023)

● Polynomial Division Fontana Peraro (2023)

$$\langle \varphi_L | \varphi_R \rangle = -\text{Res}_{\langle B \rangle}(g) - \text{Res}_{z=\infty}(g) \quad g = \psi_R \varphi_L$$

$$\text{Res}_{\langle B \rangle}(g) = \frac{g_{-1, \kappa-1}}{\ell_c}$$

$$\left[\partial_z \psi_R(z, \beta) + \partial_\beta \psi_R(z, \beta) \partial_z B(z) - \omega \psi_R(z, \beta) - \varphi_R \right]_{\mathcal{B}} = 0,$$

$$\psi_R = \sum_{i=\min}^{\max} \sum_{j=0}^{\kappa-1} \psi_{R,ij} z^j \beta^i \quad \beta = B(z)$$

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Simplifying Intersection Numbers for **n-forms**

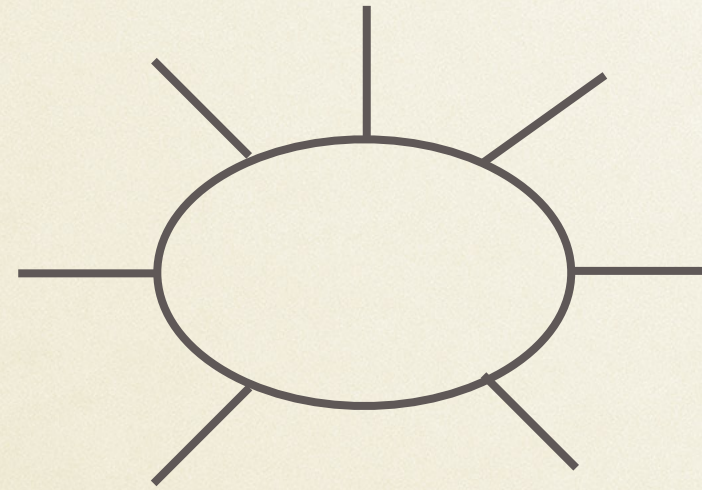
Complete decomposition @ 1- & 2-Loop

Brunello, Chestnov, Crisanti, Frellesvig, Gasparotto, Mandal & P.M. (2023)

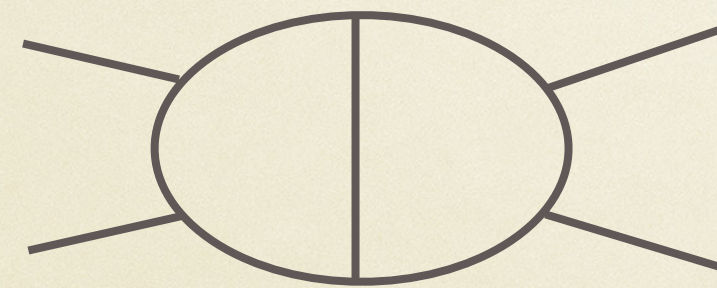
☑ 1-Loop 6-point



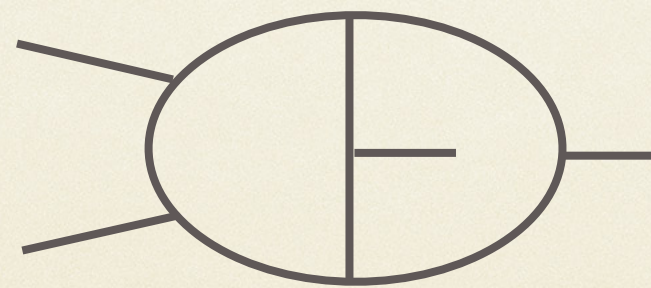
☑ 1-Loop 7-point



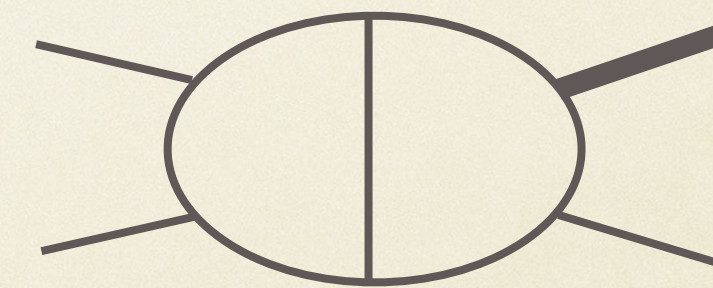
☑ 2-loop 4-point



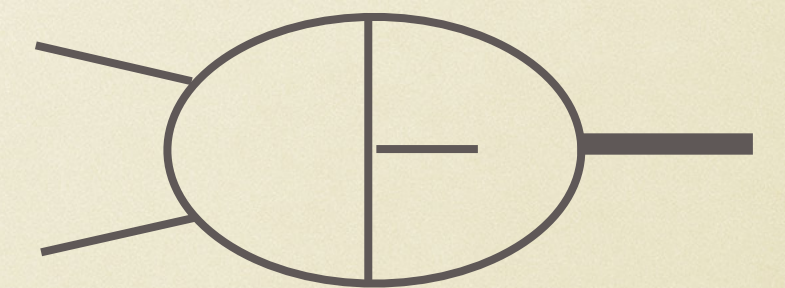
planar diagram



non-planar diagram



planar diagram



non-planar diagram

Orthogonal Bases for **quadratic** twists

Crisanti, Smith (2024)

- Quadratic polynomial in the twist

$$u(\mathbf{z}) = b(\mathbf{z})^\gamma \quad \text{for } b(\mathbf{z}) \text{ quadratic}$$

- Master Decomposition Formula

$$\langle \varphi_L | = \langle \varphi_L | \mathbb{I}_C = \sum_{i=1}^{\nu} c_i \langle e_i |, \quad \text{with} \quad \boxed{c_i} = \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle (\mathbf{C}^{-1})_{ji} \quad \mathbf{C}_{ij} \equiv \langle e_i | h_j \rangle$$

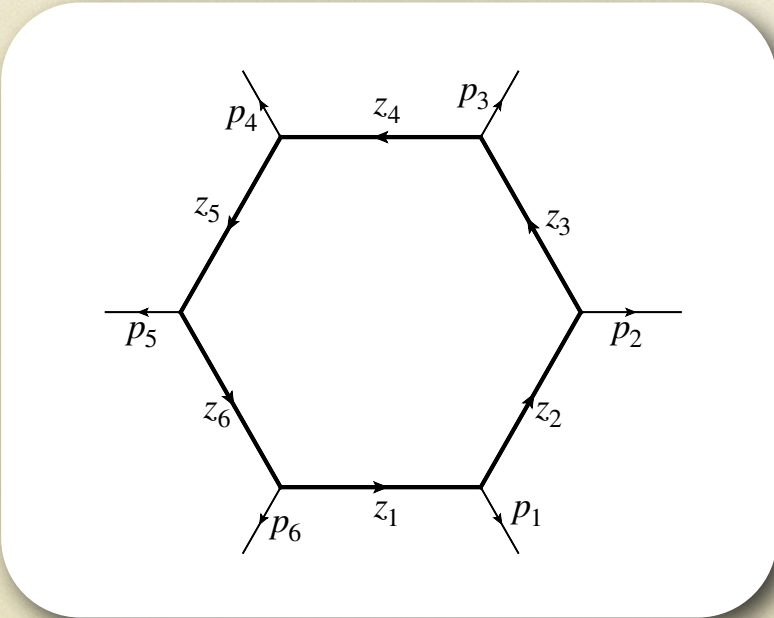
coefficients depend on the basis choice but **do not depend** on the **dual basis** choice

● **Special Dual Bases** Combination of δ -forms and **inverse powers of $b(\mathbf{z})$** \rightarrow **Diagonal C matrix**

- Intersection Numbers and Resultants

- 1-loop Feynman integrals

Quadratic Baikov polynomial $b(\mathbf{z})$



- Bubbles
- Triangles
- Boxes
- Pentagons
- Hexagons

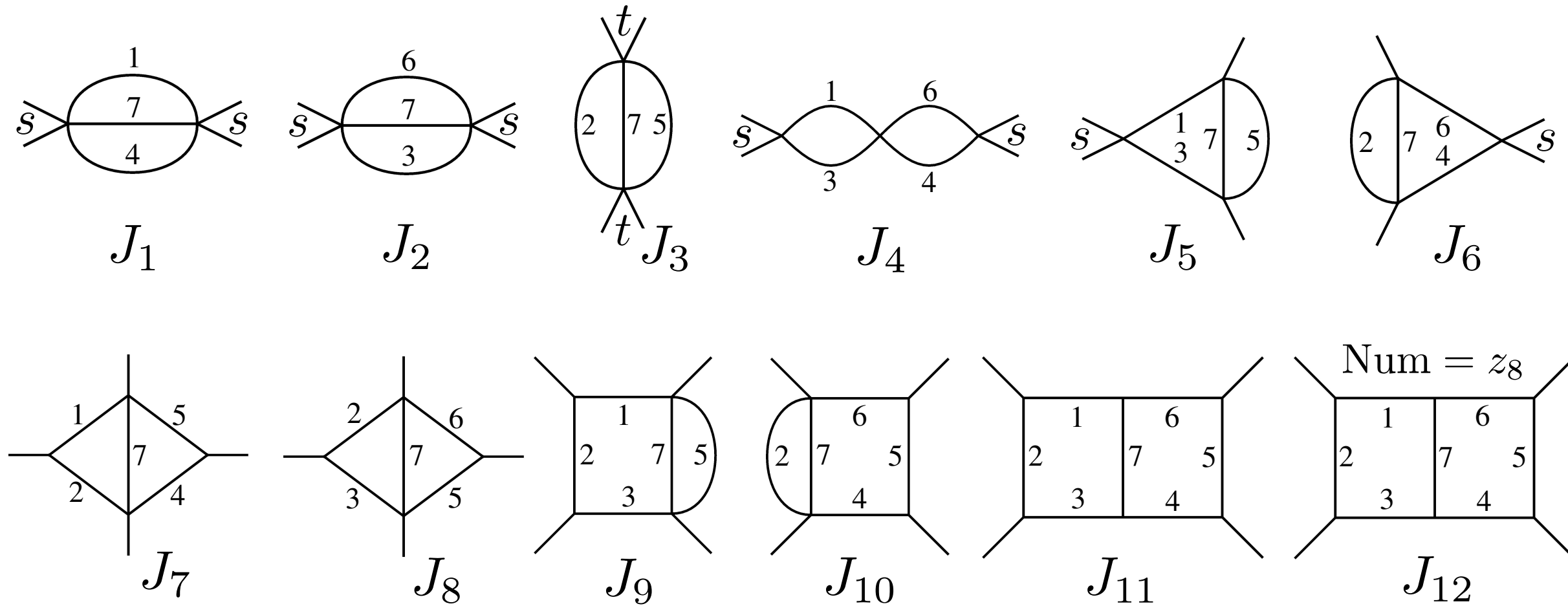
Cut $\{z_6\}$ $\nu = 32$ Master Integrals

$$e = \left\{ 1, \frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}, \frac{1}{z_5}, \frac{1}{z_1 z_2}, \frac{1}{z_1 z_3}, \frac{1}{z_1 z_4}, \frac{1}{z_1 z_5}, \frac{1}{z_2 z_3}, \frac{1}{z_2 z_4}, \frac{1}{z_2 z_5}, \frac{1}{z_3 z_4}, \frac{1}{z_3 z_5}, \frac{1}{z_4 z_5}, \frac{1}{z_1 z_2 z_3}, \frac{1}{z_1 z_2 z_4}, \frac{1}{z_1 z_2 z_5}, \frac{1}{z_1 z_3 z_4}, \frac{1}{z_1 z_3 z_5}, \frac{1}{z_1 z_4 z_5}, \frac{1}{z_2 z_3 z_4}, \frac{1}{z_2 z_3 z_5}, \frac{1}{z_2 z_4 z_5}, \frac{1}{z_3 z_4 z_5}, \frac{1}{z_1 z_2 z_3 z_4}, \frac{1}{z_1 z_2 z_3 z_5}, \frac{1}{z_1 z_2 z_4 z_5}, \frac{1}{z_1 z_3 z_4 z_5}, \frac{1}{z_2 z_3 z_4 z_5}, \frac{1}{z_1 z_2 z_3 z_4 z_5} \right\}$$

$$h = \left\{ \frac{1}{b^5}, \frac{\delta_1}{b^4}, \frac{\delta_2}{b^4}, \frac{\delta_3}{b^4}, \frac{\delta_4}{b^4}, \frac{\delta_5}{b^4}, \frac{\delta_{12}}{b^3}, \frac{\delta_{13}}{b^3}, \frac{\delta_{14}}{b^3}, \frac{\delta_{15}}{b^3}, \frac{\delta_{23}}{b^3}, \frac{\delta_{24}}{b^3}, \frac{\delta_{25}}{b^3}, \frac{\delta_{34}}{b^3}, \frac{\delta_{35}}{b^3}, \frac{\delta_{45}}{b^3}, \frac{\delta_{123}}{b^2}, \frac{\delta_{124}}{b^2}, \frac{\delta_{125}}{b^2}, \frac{\delta_{134}}{b^2}, \frac{\delta_{135}}{b^2}, \frac{\delta_{145}}{b^2}, \frac{\delta_{234}}{b^2}, \frac{\delta_{235}}{b^2}, \frac{\delta_{245}}{b^2}, \frac{\delta_{345}}{b^2}, \frac{\delta_{1234}}{b}, \frac{\delta_{1235}}{b}, \frac{\delta_{1245}}{b}, \frac{\delta_{1345}}{b}, \frac{\delta_{2345}}{b}, \delta_{12345} \right\}$$

Complete decomposition @ Planar double-box integral

Brunello, Chestnov, Crisanti, Frellesvig, Gasparotto, Mandal & P.M.



$$z_1 = k_1^2, \quad z_2 = (k_1 - p_1)^2, \quad z_3 = (k_1 - p_1 - p_2)^2, \quad z_4 = (k_2 - p_1 - p_2)^2, \quad z_5 = (k_2 + p_4)^2,$$

$$z_6 = k_2^2, \quad z_7 = (k_1 - k_2)^2, \quad z_8 = (k_1 + p_4)^2, \quad z_9 = (k_2 - p_1)^2$$

$$p_i^2 = 0, \quad s = (p_1 + p_2)^2, \quad t = (p_1 + p_4)^2, \quad s + t + u = 0$$

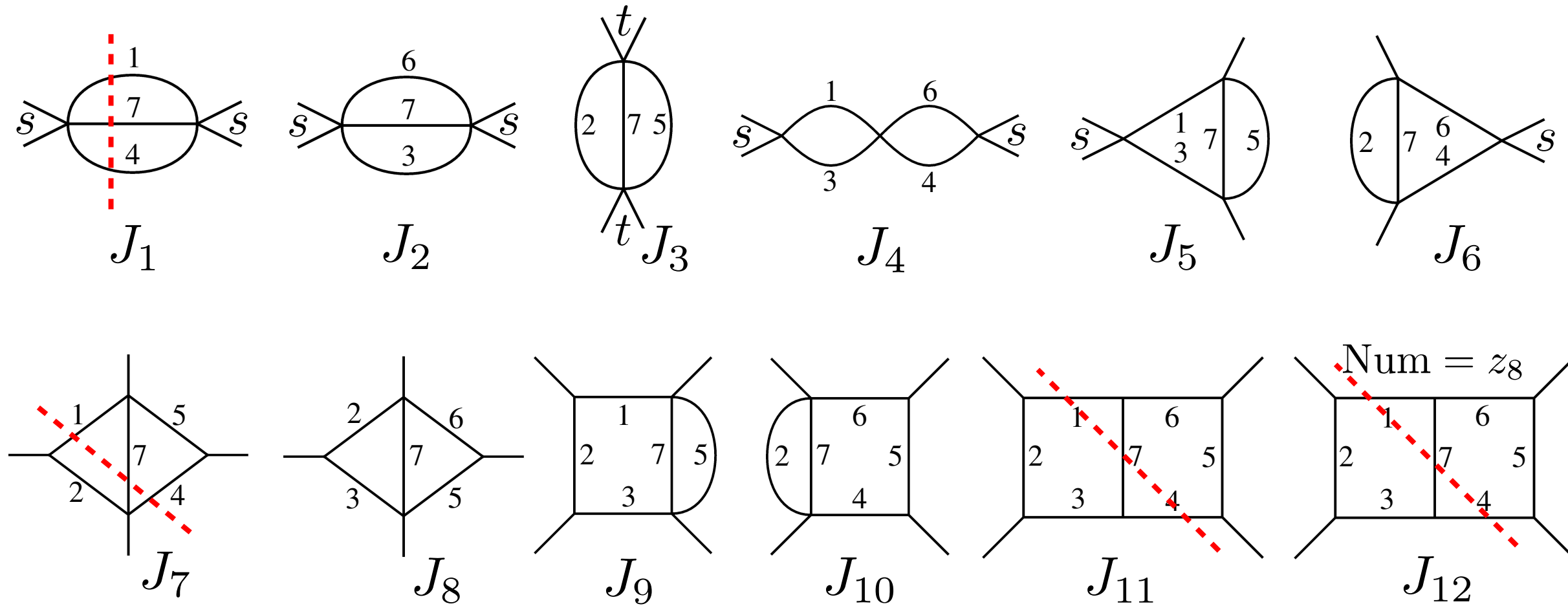
intersection numbers of (up to) 6-forms (instead of 9-forms)

spanning cuts = maximal cuts of $\{J_1, \dots, J_6\}$

$$I = \sum_{i=1}^{12} c_i J_i$$

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Brunello, Chestnov, Crisanti, Frellesvig, Gasparotto, Mandal & P.M.



$$z_1 = k_1^2, \quad z_2 = (k_1 - p_1)^2, \quad z_3 = (k_1 - p_1 - p_2)^2, \quad z_4 = (k_2 - p_1 - p_2)^2, \quad z_5 = (k_2 + p_4)^2,$$

$$z_6 = k_2^2, \quad z_7 = (k_1 - k_2)^2, \quad z_8 = (k_1 + p_4)^2, \quad z_9 = (k_2 - p_1)^2$$

$$p_i^2 = 0, \quad s = (p_1 + p_2)^2, \quad t = (p_1 + p_4)^2, \quad s + t + u = 0$$

intersection numbers of (up to) 6-forms (instead of 9-forms)

spanning cuts = maximal cuts of $\{J_1, \dots, J_6\}$

Cut 147, maximal cut of J_1

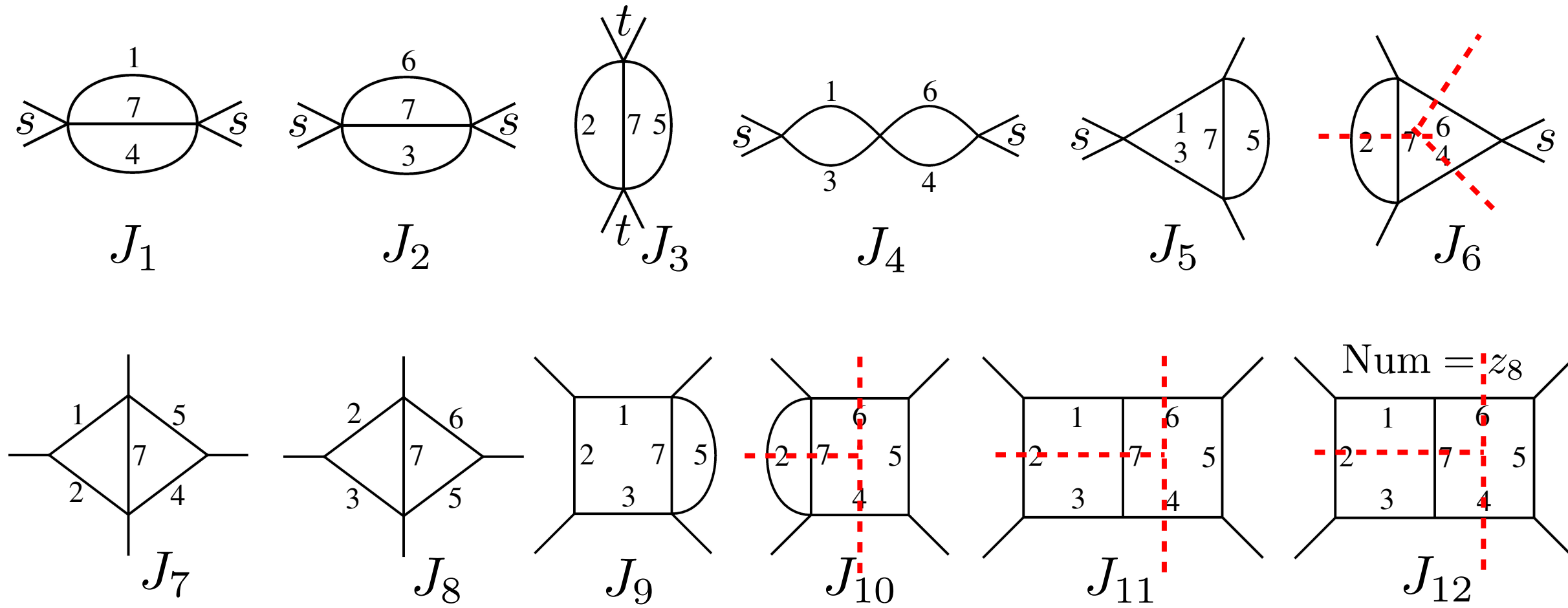
$$\nu^{(9)} = 1, \quad \nu^{(59)} = 2, \quad \nu^{(659)} = 2, \quad \nu^{(2659)} = 4, \quad \nu^{(82659)} = 5, \quad \nu^{(382659)} = 4$$

$$e^{(9)} = \{1\}, \quad e^{(59)} = \left\{1, \frac{1}{z_5}\right\}, \quad e^{(659)} = \left\{1, \frac{1}{z_5 z_6}\right\}, \quad e^{(2659)} = \left\{1, \frac{1}{z_2}, \frac{1}{z_5 z_6}, \frac{1}{z_2 z_5 z_6}\right\}, \quad e^{(82659)} = \left\{1, \frac{1}{z_5}, \frac{1}{z_2 z_5}, \frac{1}{z_2 z_5 z_6}, \frac{z_8}{z_2 z_5 z_6}\right\}, \quad e^{(382659)} = \left\{1, \frac{1}{z_2 z_5}, \frac{1}{z_2 z_3 z_5 z_6}, \frac{z_8}{z_2 z_3 z_5 z_6}\right\}$$

$$h^{(9)} = \{1\}, \quad h^{(59)} = \{1, \delta_5\}, \quad h^{(659)} = \{1, \delta_{56}\}, \quad h^{(2659)} = \{1, \delta_2, \delta_{56}, \delta_{256}\}, \quad h^{(82659)} = \{1, \delta_5, \delta_{25}, \delta_{256}, z_8 \delta_{256}\}, \quad h^{(382659)} = \{1, \delta_{25}, \delta_{2356}, z_8 \delta_{2356}\}$$

Complete decomposition @ Planar double-box integral

Brunello, Chestnov, Crisanti, Frellesvig, Gasparotto, Mandal & P.M.



$$z_1 = k_1^2, \quad z_2 = (k_1 - p_1)^2, \quad z_3 = (k_1 - p_1 - p_2)^2, \quad z_4 = (k_2 - p_1 - p_2)^2, \quad z_5 = (k_2 + p_4)^2,$$

$$z_6 = k_2^2, \quad z_7 = (k_1 - k_2)^2, \quad z_8 = (k_1 + p_4)^2, \quad z_9 = (k_2 - p_1)^2$$

$$p_i^2 = 0, \quad s = (p_1 + p_2)^2, \quad t = (p_1 + p_4)^2, \quad s + t + u = 0$$

intersection numbers of (up to) 6-forms (instead of 9-forms)

spanning cuts = maximal cuts of $\{J_1, \dots, J_6\}$

Cut 147, maximal cut of J_1

Cut 367, maximal cut of J_2

...

Cut 2467, maximal cut of J_6

$$\nu^{(8)} = 1, \quad \nu^{(58)} = 2, \quad \nu^{(358)} = 4, \quad \nu^{(1358)} = 4, \quad \nu^{(91358)} = 4$$

$$e^{(8)} = \{1\}, \quad e^{(58)} = \left\{1, \frac{1}{z_5}\right\}, \quad e^{(358)} = \left\{1, \frac{1}{z_3}, \frac{1}{z_5}, \frac{1}{z_3 z_5}\right\}, \quad e^{(1358)} = \left\{1, \frac{1}{z_5}, \frac{1}{z_1 z_3}, \frac{1}{z_1 z_3 z_5}\right\}, \quad e^{(91358)} = \left\{1, \frac{1}{z_5}, \frac{1}{z_1 z_3 z_5}, \frac{z_8}{z_1 z_3 z_5}\right\}$$

$$h^{(8)} = \{1\}, \quad h^{(58)} = \{1, \delta_5\}, \quad h^{(358)} = \{1, \delta_3, \delta_5, \delta_{35}\}, \quad h^{(1358)} = \{1, \delta_5, \delta_{13}, \delta_{135}\}, \quad h^{(91358)} = \{1, \delta_5, \delta_{135}, z_8 \delta_{135}\}$$

- **Polynomial ideal**

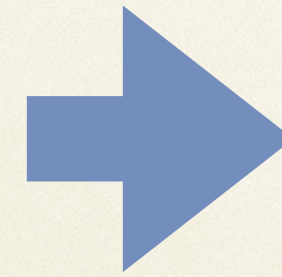
$$\langle \mathcal{B} \rangle \equiv \langle \mathcal{B}(z) - \beta \rangle = \langle b_0 - \beta + z b_1 + \dots + z^{\kappa-1} b_{\kappa-1} + z^\kappa \rangle$$

$$\langle \varphi | \varphi^\vee \rangle + \text{Res}_{\langle \mathcal{B} \rangle} (\varphi \psi) = 0,$$

$$\left[\widehat{\nabla}_{-\omega} \psi - \widehat{\varphi}^\vee \right]_{\langle \mathcal{B} \rangle} = 0,$$

$$\widehat{\nabla}_{-\omega} \equiv (\partial_z \mathcal{B}) \partial_\beta - \widehat{\omega} + \partial_z$$

$$\psi(\beta, z) = \sum_{a=0}^{\kappa-1} \sum_{n \in \mathbb{Z}} z^a \beta^n \psi_{an}$$



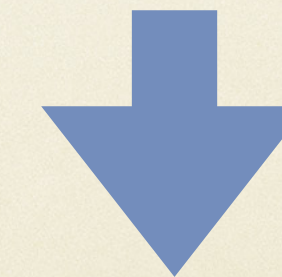
- **Companion Tensor Algebra**

$$\langle \varphi | \varphi^\vee \rangle + R \cdot \mathcal{T}_\varphi \cdot \psi = 0,$$

$$\mathcal{T}_{\widehat{\nabla}_{-\omega}} \cdot \psi - \widehat{\varphi}^\vee = 0,$$

$$\mathcal{T}_{\widehat{\nabla}_{-\omega}} \equiv \mathcal{T}_{\partial_z \mathcal{B}} \cdot \mathcal{T}_{\partial_\beta} - \mathcal{T}_{\widehat{\omega}} + \mathcal{T}_{\partial_z}$$

$$\psi_i^{(m)} = \sum_{a,n} z^a \beta^n \psi_{ian}$$



- **Three vector spaces**

$$\psi^{(m)} \in \mathbb{K}^\nu \otimes \mathcal{Q} \otimes \mathcal{L}$$

\mathbb{K}^ν Vector space of ν -dimensional vectors labeled by the first index $i = 1, \dots, \nu$

$\mathcal{Q} = \text{Span}_{\mathbb{K}}(1, \dots, z^{\kappa-1})$, $\kappa := \deg(\mathcal{B}(z))$

$\mathcal{L} = \text{Span}_{\mathbb{K}}(\dots, \beta^{-1}, \beta^0, \beta^1, \dots)$

Intersection Numbers for **1-forms** (III)

Brunello, Chestnov, & P.M. (2024)

Companion Tensor Algebra

$$\langle \varphi | \varphi^\vee \rangle + R \cdot \mathcal{T}_\varphi \cdot \psi = 0,$$

$$\mathcal{T}_{\widehat{\nabla}_{-\omega}} \cdot \psi - \widehat{\varphi}^\vee = 0,$$

$$\mathcal{T}_{\widehat{\nabla}_{-\omega}} \equiv \mathcal{T}_{\partial_z \mathcal{B}} \cdot \mathcal{T}_{\partial_\beta} - \mathcal{T}_{\widehat{\omega}} + \mathcal{T}_{\partial_z}$$

$$\psi^{(m)} \in \mathbb{K}^\nu \otimes \mathcal{Q} \otimes \mathcal{L}$$

Companion Tensor Representation

$$z \rightsquigarrow \mathcal{T}_z = \mathbb{1} \otimes Q_{z,0} + L_\beta \otimes Q_{z,1},$$

$$\partial_z \rightsquigarrow \mathcal{T}_{\partial_z} = \mathbb{1} \otimes Q_{\partial_z},$$

$$\beta \rightsquigarrow \mathcal{T}_\beta = L_\beta \otimes \mathbb{1},$$

$$\partial_\beta \rightsquigarrow \mathcal{T}_{\partial_\beta} = L_{\partial_\beta} \otimes \mathbb{1},$$

$$\text{Res}\langle \mathcal{B} \rangle \rightsquigarrow R = E_{\kappa-1} \otimes E_{-1}, = \left[0 \cdots 0 \underset{\substack{\uparrow \\ |\mu| \kappa}}{1} 0 \cdots 0 \right],$$

$$f(z, \beta) \Big|_{\beta \rightarrow 0} = \sum_{a,n} z^a \beta^n f_{an} \rightsquigarrow \mathcal{T}_f = \sum_{a,n} (\mathcal{T}_z)^a \cdot (\mathcal{T}_\beta)^n f_{an} = \sum_{a,n} \mathbb{1} \otimes (Q_{z,0} + L_\beta \otimes Q_{z,1})^a \cdot (L_\beta \otimes \mathbb{1})^n f_{an}.$$

Q-space operators

$$Q_z := \begin{bmatrix} 0 & & & & -b_0 + \beta \\ 1 & 0 & & & -b_1 \\ & 1 & 0 & & -b_2 \\ & & \ddots & \ddots & \\ & & & 1 & 0 & -b_{\kappa-2} \\ & & & & 1 & -b_{\kappa-1} \end{bmatrix}$$

$\xleftarrow{\kappa}$ $\xrightarrow{\kappa}$

$$Q_{\partial_z} := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & 3 & \\ & & & \ddots & \ddots \\ & & & & \kappa - 1 \\ & & & & & 0 \end{bmatrix}$$

$\xleftarrow{\kappa}$ $\xrightarrow{\kappa}$

L-space operators

$$L_\beta := \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & \ddots & \\ -1 & & 1 & 0 & \\ 0 & & & 1 & 0 \\ 1 & & & & 1 & 0 \\ & & & & & \ddots \\ -\mu^\vee & & & & & 1 & 0 \end{bmatrix}$$

$\xleftarrow{-\mu^\vee - \mu + 1}$ $\xrightarrow{-\mu^\vee - \mu + 1}$

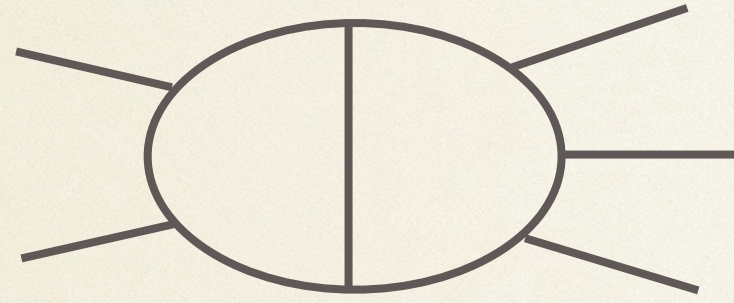
$$L_{\partial_\beta} := \begin{bmatrix} 0 & \mu & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ -1 & & & 0 & 0 \\ 0 & & & 0 & 1 \\ 1 & & & & 0 & 2 \\ & & & & & \ddots \\ -\mu^\vee & & & & & 0 & -\mu^\vee \\ & & & & & & & 0 \end{bmatrix}$$

$\xleftarrow{-\mu^\vee - \mu + 1}$ $\xrightarrow{-\mu^\vee - \mu + 1}$

Simplifying Intersection Numbers for **n-forms**

Complete decomposition @ 1- & 2-Loop

✓ 2-loop 5-point

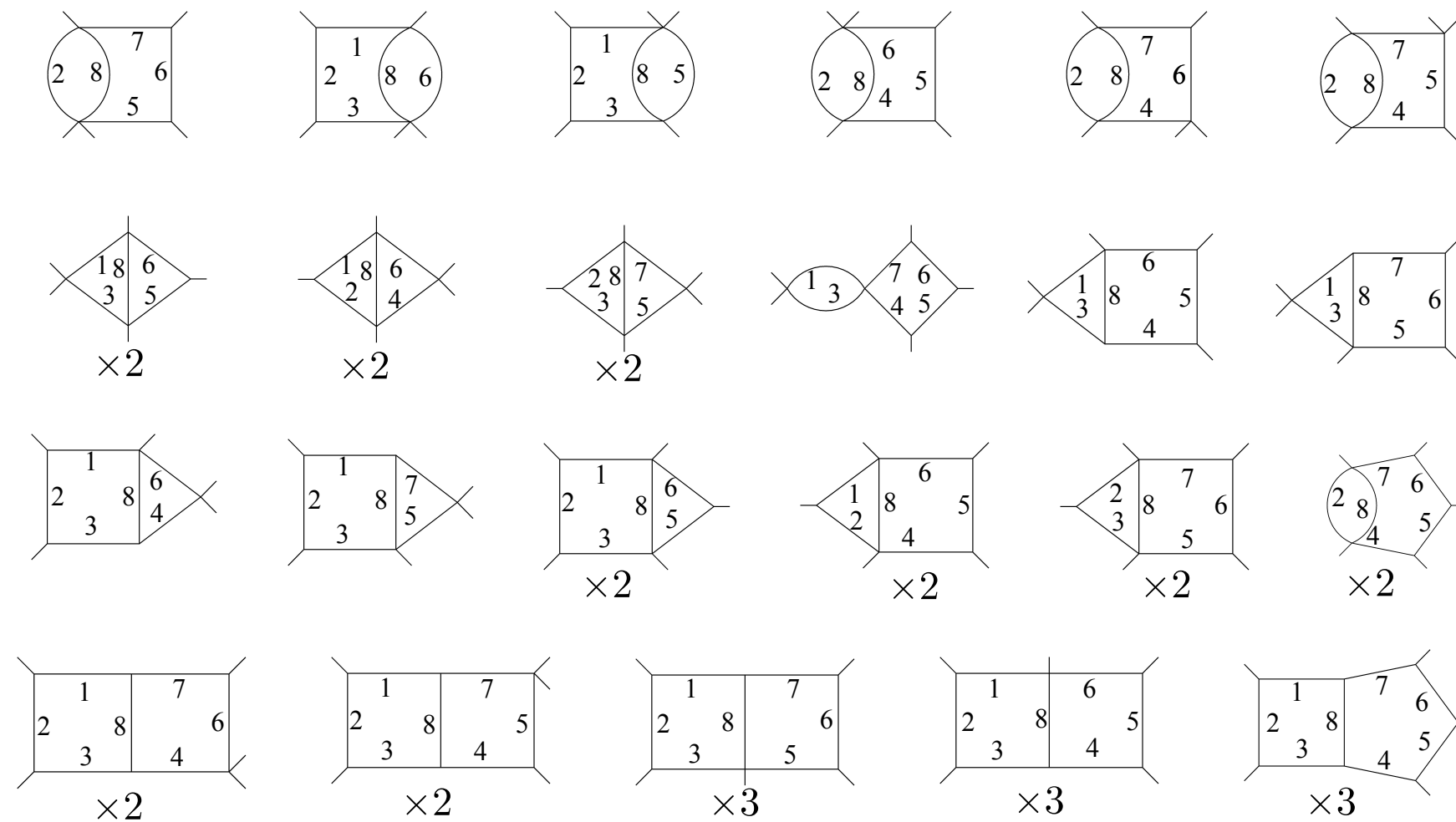
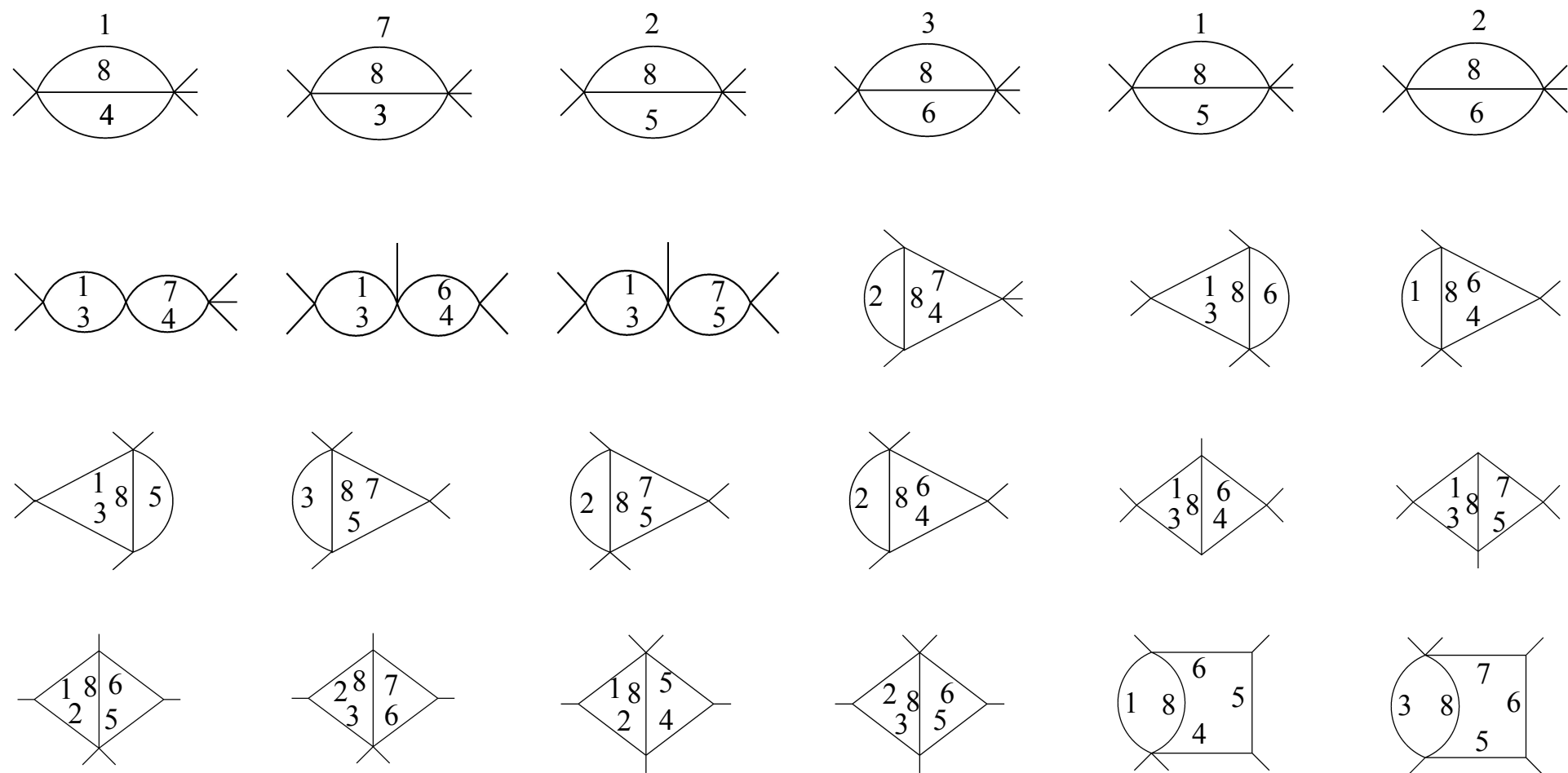


Brunello, Chestnov, Crisanti, Frellesvig, Gasparotto, Mandal & P.M. (2023)

Brunello, Chestnov, & P.M. (2024)

$$I_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11}} = \int d^{11} z u(\mathbf{z}) \frac{z_9^{-a_9} z_{10}^{-a_{10}} z_{11}^{-a_{11}}}{z_1^{a_1} z_2^{a_2} z_3^{a_3} z_4^{a_4} z_5^{a_5} z_6^{a_6} z_7^{a_7} z_8^{a_8}}$$

● 62 MIs and 47 sectors



Intersection Numbers for n-forms :: nPDE

Chestnov, Frellesvig, Gasparotto, Mandal & P.M. (2022)

Intersection Numbers for **n-forms** (IV)

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, Mandal & P.M. (2022)

$$\langle \varphi_L^{(\mathbf{n})} | \varphi_R^{(\mathbf{n})} \rangle = (2\pi i)^{-n} \int_X (u \varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \text{Res}_{z=p}(\psi \varphi_R^{(\mathbf{n})})$$

● nPDE

$$\nabla_{\omega_1} \nabla_{\omega_2} \dots \nabla_{\omega_n} \psi = \varphi_L^{(\mathbf{n})}$$

Proof.

$$\eta := \bar{h}_1 \dots \bar{h}_n (u \psi) (u^{-1} \varphi_R^{(\mathbf{n})}) \quad d_{z_1} \dots d_{z_n} \eta = (u \varphi_{L,c}) \wedge (u^{-1} \varphi_R),$$

$$\bar{h}_i := 1 - h_i$$

$$h_i \equiv h(z_i) := \begin{cases} 1 & \text{for } |z_i| < \epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_{L,c} := \bar{h}_1 \dots \bar{h}_n \varphi_L + \dots + (-1)^n \psi dh_1 \wedge \dots \wedge dh_n \equiv \nabla_{\omega_1} \dots \nabla_{\omega_n} (\bar{h}_1 \dots \bar{h}_n \psi)$$

$$\begin{aligned} \int_X (u \varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) &= \sum_{p \in \mathbb{P}_\omega} \int_{D_p} d_{z_1} \dots d_{z_n} \eta = (-1)^n \sum_{p \in \mathbb{P}_\omega} \int_{D_p} (u \psi) dh_1 \wedge \dots \wedge dh_n \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) \\ &= \sum_{p \in \mathbb{P}_\omega} \int_{\mathcal{O}_1 \wedge \dots \wedge \mathcal{O}_n} \psi \varphi_R^{(\mathbf{n})} = (2\pi i)^n \sum_{p \in \mathbb{P}_\omega} \text{Res}_{z=p}(\psi \varphi_R^{(\mathbf{n})}) \end{aligned}$$

It avoids fibrations

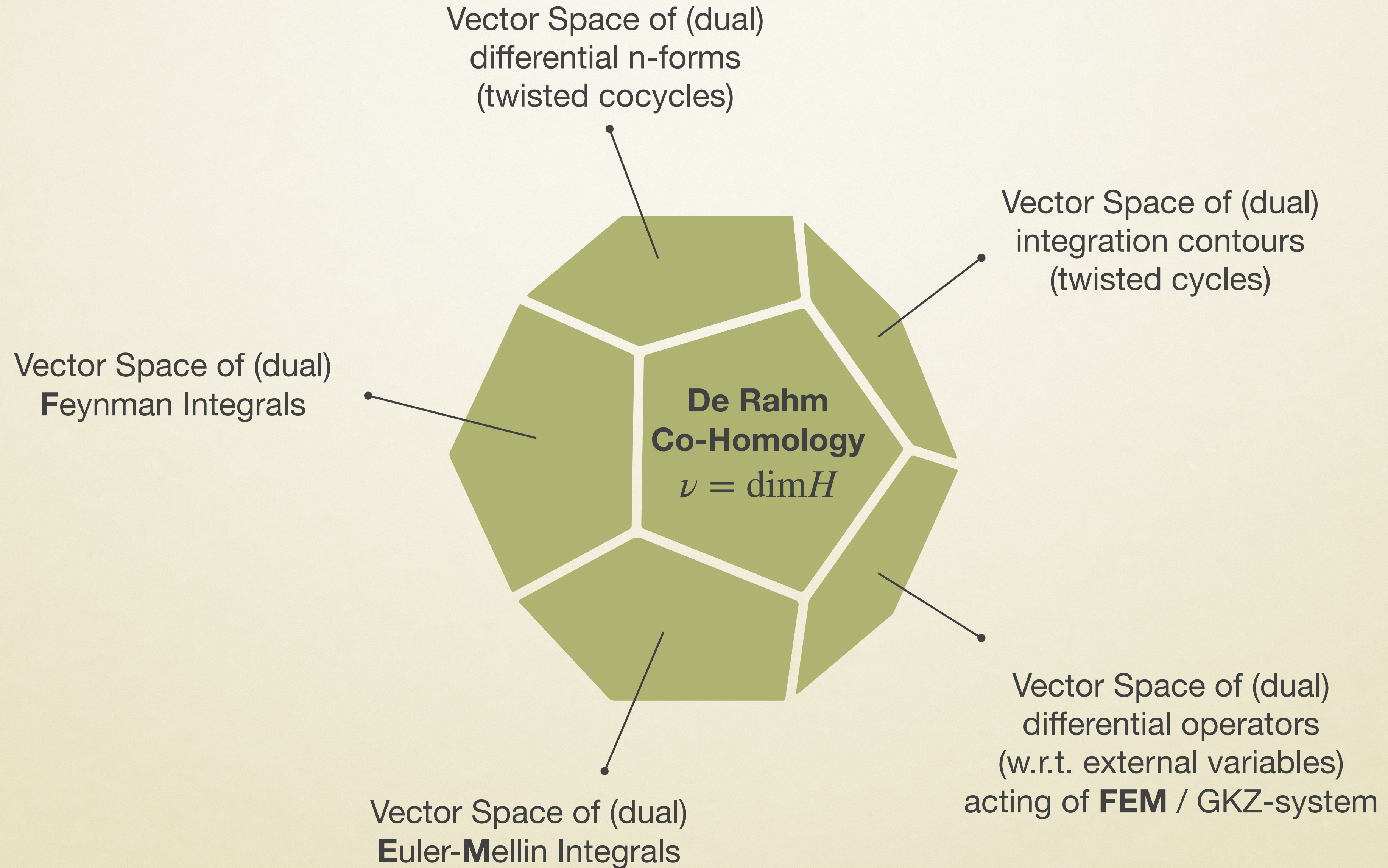
It requires the knowledge of the poles' position: ok for hyperplane arrangement

It requires blow-ups

Intersection Numbers for n-forms: Pfaffian systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & **P.M.** (2022)

De Rham Thm & Vector Spaces *Isomorphism*



GKZ Hypergeometric Systems

- Euler-Mellin Integral / A-Hypergeometric function

$$f_{\Gamma}(z) = \int_{\Gamma} g(z; x)^{\beta_0} x_1^{-\beta_1} \cdots x_n^{-\beta_n} \frac{dx}{x}$$

Bernstein, Saito, Sturmfels, Takayama, Matsubara-Heo,
Agostini, Fevola, Sattelberger, Tellen,
De La Cruz,...

$$\frac{dx}{x} := \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

$$u(\mathbf{x}) = g(z, x)^{\beta_0} x_1^{-\beta_1} \cdots x_n^{-\beta_n}$$

$$g(z; x) = \sum_{i=1}^N z_i x^{\alpha_i}$$

$$x^{\alpha_i} := x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}}$$

$$A = (a_1 \ \dots \ a_N) \quad (n+1) \times N \text{ matrix}$$

$$a_i := (1, \alpha_i)$$

$$\text{Ker}(A) = \left\{ u = (u_1, \dots, u_N) \in \mathbb{Z}^N \mid \sum_{j=1}^N u_j a_j = \mathbf{0} \right\}$$

- Gelfand-Kapranov-Zelevinsky (GKZ) system of PDEs

$$E_j f_{\Gamma}(z) = 0 ,$$

$$\square_u f_{\Gamma}(z) = 0 ,$$

- Generators

$$E_j = \sum_{i=1}^N a_{j,i} z_i \frac{\partial}{\partial z_i} - \beta_j , \quad j = 1, \dots, n+1$$

$$\square_u = \prod_{u_i > 0} \left(\frac{\partial}{\partial z_i} \right)^{u_i} - \prod_{u_i < 0} \left(\frac{\partial}{\partial z_i} \right)^{-u_i} , \quad \forall u \in \text{Ker}(A) .$$

GKZ D-Module and De Rham Cohomology group

• **Weyl Algebra:** E_j \square_u can be regarded as elements of a Weyl algebra

$$\mathcal{D}_N = \mathbb{C}[z_1, \dots, z_N] \langle \partial_1, \dots, \partial_N \rangle \quad , \quad [\partial_i, \partial_j] = 0 \quad , \quad [\partial_i, z_j] = \delta_{ij}$$

GKZ system as the left \mathcal{D}_N -module $\mathcal{D}_N/H_A(\beta)$

$$H_A(\beta) = \sum_{j=1}^{n+1} \mathcal{D}_N \cdot E_j + \sum_{u \in \text{Ker}(A)} \mathcal{D}_N \cdot \square_u$$

• **Standard Monomials** $\text{Std} := \{\partial^k\}$ found by Groebner basis Hibi, Nishiyama, Takayama (2017)

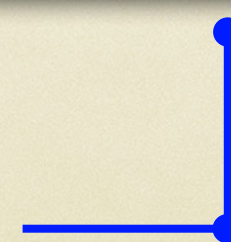
The holonomic rank equals the number of independent solutions to the system of PDEs

$$r = n! \cdot \text{vol}(\Delta_A)$$

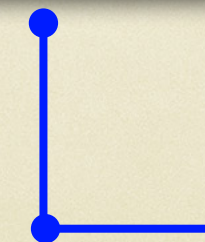
$$\mathcal{D}_N/H_A(\beta) \simeq \mathbb{H}^n$$

• **Isomorphism**

GKZ D-module



n th-Cohomology group



Intersection Numbers for **n-forms** (V) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & P.M. (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for \mathbb{H}^{n^\vee}

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

● **Thm : Isomorphism**



Gelfand Kapranov Zelevinsky (1990)

Intersection Numbers for **n-forms** (V) from Pfaffian D-module systems

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$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

● **Thm : Isomorphism**

nth-Cohomology group \cong

GKZ D-module

Gelf'and Kapranov Zelevinsky (1990)

Euler-Mellin Integrals

Differential Operators
(w.r.t. external variables)

Pfaffian Systems: for **Master Integrals** (alias **Master forms**)

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

Basis of the Cohomology group

$$\Omega = \Omega(d, x) \quad \bullet \text{ Pfaffian Matrix}$$

Integral decomposition (IBP/InterX)

Intersection Numbers for **n-forms** (V) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & P.M. (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

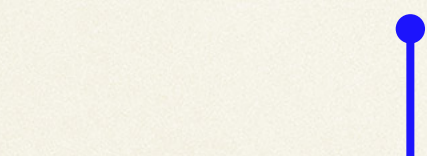
$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

● **Thm : Isomorphism**

nth-Cohomology group \cong

GKZ D-module

Gelf'and Kapranov Zelevinsky)1990)



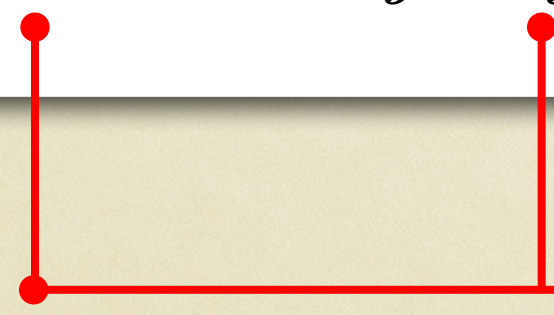
Euler-Mellin Integrals



**Differential Operators
(w.r.t. external variables)**

Pfaffian Systems: for **Master Integrals** (alias **Master forms**) & for **D-operators** (alias **Std mon's**)

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$



Basis of the D-Operators

$$\Omega = \Omega(d, x) \quad \bullet \text{ Pfaffian Matrix}$$



Macaulay Matrix method

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & P.M. (2022)

Intersection Numbers for **n-forms** (V) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & P.M. (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for \mathbb{H}^{n^\vee}

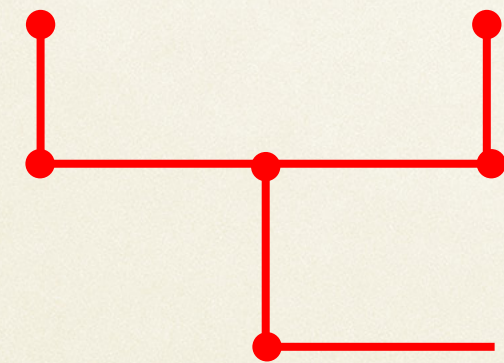
$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

● **Thm : Isomorphism**

nth-Cohomology group \cong **GKZ D-module**

● **Secondary Equations**

$$\partial_x \mathbf{C} = \mathbf{\Omega} \cdot \mathbf{C} + \mathbf{C} \cdot \tilde{\mathbf{\Omega}}, \quad \partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}} \cdot \mathbf{C}^{-1} - \mathbf{C}^{-1} \cdot \mathbf{\Omega}$$



1) **Build them from Macaulay Matrix for D-module**

Intersection Numbers for **n-forms** (V) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & P.M. (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

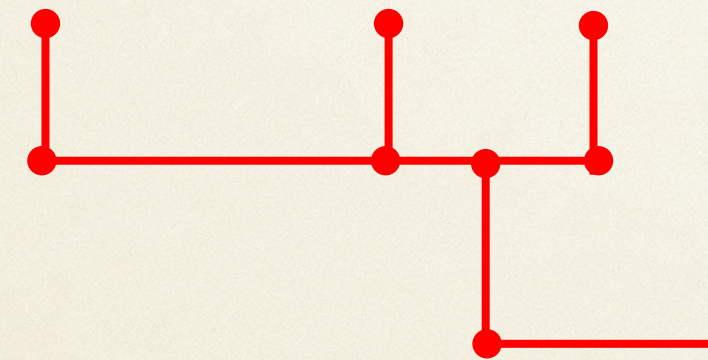
$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

● **Thm : Isomorphism**

nth-Cohomology group \cong **GKZ D-module**

● **Secondary Equations**

$$\partial_x \mathbf{C} = \mathbf{\Omega} \cdot \mathbf{C} + \mathbf{C} \cdot \tilde{\mathbf{\Omega}}, \quad \partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}} \cdot \mathbf{C}^{-1} - \mathbf{C}^{-1} \cdot \mathbf{\Omega}$$



2) Rational Solutions of Secondary Equations
[integrable connections]

Barkatou et al. @ MAPLE

Direct determination of Intersection Matrices

Intersections Numbers beyond Feynman Integrals

Intersection Numbers beyond Feynman Integrals

Extending the range of applicability of techniques developed in the context of Feynman integrals:

- *searching for* problems admitting **twisted period integrals** representations
- if needed, *modify* integrals to become **twisted period integrals**: *analytic continuation/regularisation*

$$\int_0^\infty f(z) \rightarrow \lim_{\rho \rightarrow 0} \int_0^\infty z^\rho f(z)$$
$$\int_a^b f(z) \rightarrow \lim_{\rho \rightarrow 0} \int_a^b ((z-a)(z-b))^\rho f(z)$$

Regulators

Intersections Numbers @ QM and QFT

Cacciatori & P.M. (2022)

(Special) Applications of Intersection Numbers for 1-forms

- Looking at a known landscapes with new eyes

1. Identify a univariate twisted period integral $\int_{\Gamma} \mu \varphi$

If μ is not multivalued, replace it with the regulated twist $u = u(\rho)$ by introducing a regulator ρ , so that, $\lim_{\rho \rightarrow \rho_0} u(\rho) = \mu$

- Dimension of cohomology group $\nu = \#$ of solutions of $\omega = d \log(u) = 0$ (critical points)

2. After choosing the bases of forms $e_i \equiv \hat{e}_i dz$ and dual forms $h_i \equiv \hat{h}_i dz$, with $\hat{h}_i = \hat{e}_i$, such that $\hat{e}_1 = \hat{h}_1 = 1$, decompose φ

- Master Decomposition formula $\varphi = c_1 e_1 + c_2 e_2 + \dots + c_\nu e_\nu$

3. Translate the decomposition of φ to the one of the corresponding integral, (eventually, taking the $\rho \rightarrow \rho_0$ limit)

$$\int_{\Gamma} \mu \varphi = c_1 E_1 + c_2 E_2 + \dots + c_\nu E_\nu, \quad \text{with} \quad E_1 \equiv \int_{\Gamma} \mu dz, \quad \text{and} \quad E_j = \int_{\Gamma} \mu e_j, \quad (j \neq 1),$$

and compare the result with the literature.

Orthogonal Polynomials and Matrix Elements in QM

Case i)
$$I_{nm} \equiv \int_{\Gamma} P_n(z) P_m(z) f(z) dz,$$

Case ii)
$$I_{nm} \equiv \langle n | \mathcal{O} | m \rangle = \int_{\Gamma} \psi_n^*(z) \mathcal{O}(z) \psi_m(z) f(z) dz$$

- **Master Decomposition formula**

For the considered cases, we obtain:

corresponding to:

$$\varphi = c_1 e_1, \quad \text{in terms of just one basic form, } e_1 = dz$$

$$I_{nm} = c_1 E_1 \quad (\text{one master integral})$$

i) Orthogonal Polynomials

Laguerre $L_n^{(\rho)}$, Legendre P_n , Tchebyshev T_n , Gegenbauer $C_n^{(\rho)}$, and Hermite H_n polynomials:

$$I_{nm} \equiv \int_{\Gamma} \mu P_n P_m dz = f_n \delta_{nm} = \int_{\Gamma} \mu \varphi = c_1 E_1$$

$$\varphi \equiv P_n P_m dz$$

Type	u	ν	\hat{e}_i	C-matrix	ρ_0	E_1	c_1
$L_n^{(\rho)}$	$z^\rho \exp(-z)$	1	1	ρ	-	$\Gamma(1 + \rho)$	$(\rho + 1)(\rho + 2) \cdots (\rho + n)/n!$
P_n	$(z^2 - 1)^\rho$	1	1	$2\rho/(4\rho^2 - 1)$	0	2	$1/(2n + 1)$
T_n	$(1 - z^2)^\rho$	1	1	$2\rho/(4\rho^2 - 1)$	-1/2	π	1/2
$C_n^{(\rho)}$	$(1 - z^2)^{\rho-1/2}$	1	1	$(1 - 2\rho)/(4\rho(\rho - 1))$	-	$\sqrt{\pi}\Gamma(1/2 + \rho)/\Gamma(1 + \rho)$	$\rho(2\rho(2\rho + 1) \cdots (2\rho + n - 1))/((n + \rho)n!)$
H_n	$z^\rho \exp(-z^2)$	2	1, 1/z	diagonal(1/2, 1/ ρ)	0	$\sqrt{\pi}$	$2^n n!$

Let us observe that, in the case of Hermite polynomials, $\nu = 2$, yielding $\varphi = c_1 e_1 + c_2 e_2$, but $c_2 = 0$, due to the adopted basis

ii) Matrix Elements in QM

Harmonic Oscillator. (for unitary mass and pulsation, $m = 1 = \omega$)

$$\langle z|n\rangle = \psi_n(z) = e^{-\frac{z^2}{2}} W_n(z), \quad \text{with} \quad W_n(z) \equiv N_n H_n(z), \quad N_n \equiv 1/\sqrt{(2^n n! \sqrt{\pi})}$$

● **Position operator**

$$\langle m|z^k|n\rangle = \int_{-\infty}^{\infty} dz \psi_m(z) z^k \psi_n(z) = \int_{\Gamma} \mu \varphi = c_1 E_1, \quad \text{with} \quad \mu \equiv e^{-z^2}, \quad \text{and} \quad \varphi \equiv W_m(z) z^k W_n(z) dz.$$

Type	u	ν	\hat{e}_i	C-matrix	ρ_0	E_1
W_n	$z^\rho \exp(-z^2)$	2	1, 1/z	diagonal(1/2, 1/ ρ)	0	$\sqrt{\pi}$

$$\begin{aligned} \langle n|m\rangle &= \delta_{nm}, \\ \langle n|z^{2k+1}|n\rangle &= 0, \\ \langle n|z^4|n\rangle &= \frac{3}{4}(2n^2 + 2n + 1), \\ \langle n|z^3|n-3\rangle &= \sqrt{n(n-1)(n-2)/8}, \\ \langle n|z^3|n-1\rangle &= \sqrt{9n^3/8}. \end{aligned}$$

● **Hamiltonian operator**

$$\langle n|H|n\rangle = (n + 1/2) \quad H \equiv (1/2)(-\nabla^2 + z^2) \quad \varphi = \sum_{k=0}^n b_k z^{2k}$$

ii) Matrix Elements in QM

Hydrogen Atom. (for unitary Bohr radius $a_0 = 1$)

$$\langle z|n, \ell\rangle = R_{n, \ell}(z) = e^{-\frac{z}{n}} W_{n, \ell}(z), \quad \text{with} \quad W_{n, \ell}(z) \equiv N_{n\ell} \left(\frac{2z}{n}\right)^\ell L_{(n-\ell-1)}^{2\ell+1} \left(\frac{2z}{n}\right) \quad N_{n\ell} = (2/n)^{3/2} \sqrt{(n-\ell-1)!/(2n(n+\ell)!)}$$

● Position operator

$$\langle n_1, \ell|z^k|n_2, \ell\rangle = \int_0^\infty dz z^2 R_{n_1, \ell}(z) z^k R_{n_2, \ell}(z) = \int_\Gamma \mu \varphi = c_1 E_1, \quad \text{with} \quad \mu \equiv z^2 e^{-z\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}, \quad \text{and} \quad \varphi \equiv W_{n_1, \ell}(z) z^k W_{n_2, \ell}(z)$$

Type	u	v	\hat{e}_i	C-matrix	ρ_0	E_1
$W_{n, \ell}$	$z^{\rho+2} \exp(-z(n_1 + n_2)/(n_1 n_2))$	1	1	$(n_1 n_2 / (n_1 + n_2))^2 (2 + \rho)$	0	$2(n_1 n_2 / (n_1 + n_2))^3$

$$\langle n_1, \ell|n_2, \ell\rangle = \delta_{n_1 n_2},$$

$$\langle n, \ell|z|n, \ell\rangle = \frac{1}{2} [3n^2 - \ell(\ell + 1)],$$

$$\langle n, \ell|z^{-1}|n, \ell\rangle = \frac{1}{n^2},$$

$$\langle n, \ell|z^{-2}|n, \ell\rangle = \frac{2}{n^3(2\ell + 1)},$$

$$\langle n, \ell|z^{-3}|n, \ell\rangle = \frac{2}{n^3 \ell(\ell + 1)(2\ell + 1)}$$

Green's Function and Kontsevich-Witten tau-function

Case iii)
$$G_n \equiv \frac{\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) \exp[-S_E]}{\int \mathcal{D}\phi \exp[-S_E]}$$

Weinzierl (2020)

Case iv)
$$Z_{KW} \equiv \frac{\int d\Phi \exp \left[-\text{Tr} \left(-\frac{i}{3!} \Phi^3 + \frac{\Lambda}{2} \Phi^2 \right) \right]}{\int d\Phi \exp \left[-\text{Tr} \left(\frac{\Lambda}{2} \Phi^2 \right) \right]}$$

$$c_1 = \frac{\int_{\Gamma} \mu \varphi}{\int_{\Gamma} \mu e_1}, \quad \text{equivalently rewritten as} \quad \int_{\Gamma} \mu \varphi = c_1 E_1 \quad \bullet \text{ Master Decomposition formula}$$

• Toy models univariate integrals

i) Green's Function

Single field, ϕ^4 -theory

real scalar field $\phi(x)$ $S_E \equiv S_0 + \epsilon S_1$, with $S_0 = (\gamma/2) \phi^2(x)$, and $S_1 = \phi^4(x)$

$$\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) e^{-S_E} = G_n \int \mathcal{D}\phi e^{-S_E}$$

$$\int_{\Gamma} \mu \varphi = G_n E_1, \quad \text{with} \quad \mu \equiv e^{-S_E}, \quad \varphi \equiv \phi(x_1) \cdots \phi(x_n) \mathcal{D}\phi, \quad E_1 \equiv \int_{\Gamma} \mu e_1, \quad \text{and} \quad e_1 \equiv \mathcal{D}\phi$$

Free theory. The n -point Green's function $G_n^{(0)}$ $\phi(x) \equiv z$ $\mu \equiv e^{-S_0}$ $\varphi = z^n dz$

Type	u	v	\hat{e}_i	C-matrix	ρ_0	E_1	c_1
$G_n^{(0)}$	$z^\rho \exp(-\gamma z^2/2)$	2	1, 1/z	diagonal(1/ γ , 1/ ρ)	0	not needed	$(n-1)!!/\gamma^{n/2}$

for even n

• **2-point function: the propagator** $G_2^{(0)} = 1/\gamma$

Perturbation Theory. The n -point correlation function G_n in the full theory can be computed perturbatively, in the small coupling limit, $\epsilon \rightarrow 0$, and expressed in terms of $G_n^{(0)}$. For example, the determination of the next-to-leading order (NLO) corrections to the 2-point function, proceeds as follows,

$$\begin{aligned} G_2 &= \frac{\int dz z^2 e^{-S_0 - \epsilon S_1}}{\int dz e^{-S_0 - \epsilon S_1}} = \frac{\int dz z^2 e^{-S_0} (1 - \epsilon S_1 + \dots)}{\int dz e^{-S_0} (1 - \epsilon S_1 + \dots)} = \left(G_2^{(0)} - \epsilon G_6^{(0)} + \dots \right) \left(1 + \epsilon G_4^{(0)} + \dots \right) = G_2^{(0)} + \epsilon \left(G_2^{(0)} G_4^{(0)} - G_6^{(0)} \right) + \mathcal{O}(\epsilon^2) \\ &= \frac{1}{\gamma} \left(1 - 12\epsilon \frac{1}{\gamma^2} \right) + \mathcal{O}(\epsilon^2) \end{aligned}$$

i) Green's Function

Single field, ϕ^4 -theory

real scalar field $\phi(x)$ $S_E \equiv S_0 + \epsilon S_1$, with $S_0 = (\gamma/2) \phi^2(x)$, and $S_1 = \phi^4(x)$

Exact theory. $\phi(x) \equiv z$ $\mu \equiv e^{-S_E}$ $\varphi = z^n dz$

$$u \equiv z^\rho \mu \quad \nu = 4,$$

$$\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\} = \{1, 1/z, z, z^2\},$$

$$\{\hat{h}_i\}_{i=1}^4 = \{\hat{e}_i\}_{i=1}^4,$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4\gamma} \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & \frac{1}{4\gamma} & 0 \\ \frac{1}{4\gamma} & 0 & 0 & -\frac{\gamma}{16\epsilon^2} \end{pmatrix}$$

For instance, let us consider the decomposition:

$$\varphi = z^4 dz = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 \quad c_1 = \frac{1}{4\epsilon}, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = -\frac{\gamma}{4\epsilon}$$

$$\int_{\Gamma} dz z^4 e^{-S_E} = c_1 \int_{\Gamma} dz e^{-S_E} + c_4 \int_{\Gamma} dz z^2 e^{-S_E}$$

$$G_4 = c_1 + c_4 G_2$$

$$G_2 = \frac{1}{\gamma} (1 - 4\epsilon G_4)$$

ii) Kontsevich-Witten tau-function

$$Z_{KW} \equiv \frac{\int d\Phi \exp \left[-\text{Tr} \left(-\frac{i}{3!} \Phi^3 + \frac{\Lambda}{2} \Phi^2 \right) \right]}{\int d\Phi \exp \left[-\text{Tr} \left(\frac{\Lambda}{2} \Phi^2 \right) \right]}$$

• Univariate Model

Itzykson-Zuber (1992)

$$Z_{KW} = \sum_{n=0}^{\infty} Z_{KW}^{(n)} \quad \int_{\Gamma} \mu \varphi = c_1 E_1 \quad c_1 = Z_{KW}^{(n)}$$

$$\varphi \equiv N_n z^{6n}, \quad N_n \equiv \varepsilon^{2n} \quad \varepsilon \equiv i/(3!)(\Lambda/2)^{-3/2}$$

Type	u	v	\hat{e}_i	C-matrix	ρ_0	E_1	c_1
$Z_{KW}^{(n)}$	$z^\rho \exp(-z^2)$	2	1, 1/z	diagonal(1/2, 1/ρ)	0	not needed	$(-2/9)^n (\Lambda^{-3n}/(2n)!) \prod_{j=0}^{3n-1} (j+1/2)$

Intersection Numbers @ Fourier Integrals

Brunello, Crisanti, Giroux, Smith & P.M. (2023)

Fourier integrals from Intersection Theory

Brunello, Crisanti, Giroux, Smith & P.M. (2023)

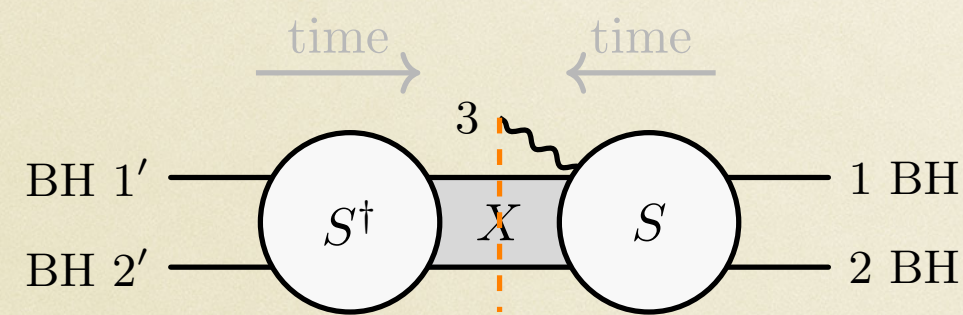
- Fourier integrals in Baikov representation as twisted periods

$$\tilde{f}(\{x_i\}) = \int f(\{q_i\}) \prod_{j=1}^L e^{iq_j \cdot x_j} \frac{d^D q_j}{(2\pi)^{D/2}} = \int_{C_R} u(\mathbf{z}) \varphi_L(\mathbf{z}) \quad u(\mathbf{z}) = \kappa e^{ig(\mathbf{z})} B(\mathbf{z})^{\frac{D-L-E-1}{2}}$$

- Application-1: Feynman propagator in position-space

$$I_n = \int_{\mathcal{M}} d^D q \frac{e^{iq \cdot x}}{(q^2 + m^2 - i\varepsilon)^n}$$

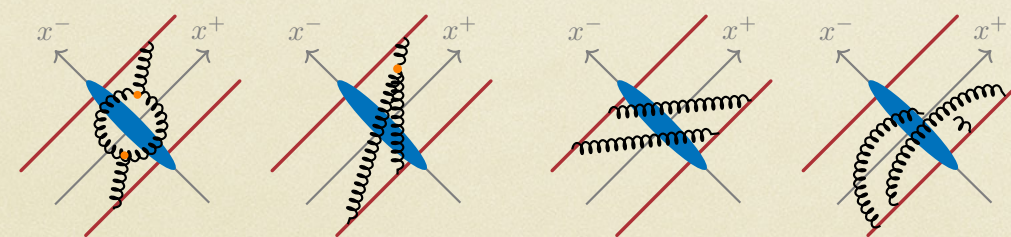
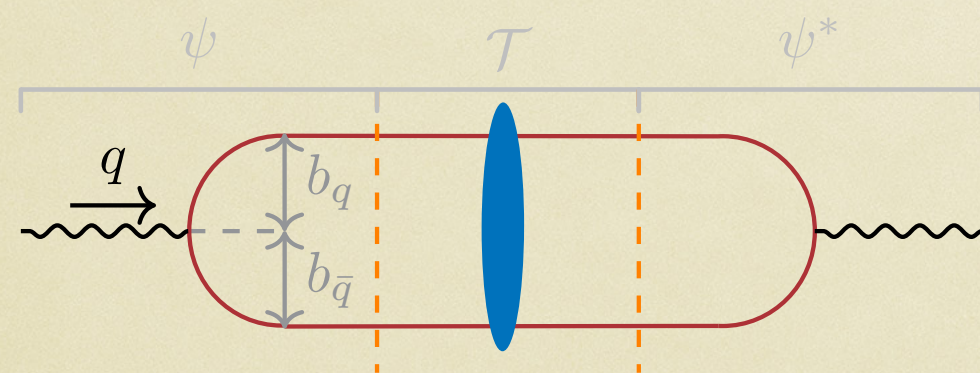
- Application-2: Spectral gravitation wave form in KMOC formalism



$$\mathcal{I}_{\beta_1 \beta_2}^{\nu_{2m}} = \int_{\mathcal{M}} d^D q \frac{\delta(u_1 \cdot q) \delta(u_2 \cdot (q-k)) q^{\nu_1} \dots q^{\nu_{2m}} e^{-iq \cdot b}}{[q^2 - i\varepsilon]^{\beta_1} [(q-k)^2 - i\varepsilon]^{\beta_2}}$$

$$\text{Exp}_3 = \text{in} \langle 2'1' | S^\dagger a_3 S | 12 \rangle_{\text{in}}$$

- Application-3: QCD Color Dipole Scattering and Balitski-Kovchegov Equations



$$I^{ij} = \int_{\mathbb{R}^{2D}} d^D q_1 d^D q_2 \frac{N_I^{ij}(q_1, q_2) e^{i(q_1 \cdot x_1 + q_2 \cdot x_2)}}{q_1^2 (q_1^2 \tau + q_2^2)}$$

$$G^{ij} = \int_{\mathbb{R}^{2D}} d^D q_1 d^D q_2 \frac{N_G^{ij}(q_1, q_2) e^{i(q_1 \cdot x_1 + q_2 \cdot x_2)}}{(q_1 + q_2)^2 (q_1^2 \tau + q_2^2)}$$

$$N_I^{ij} = q_1^i q_2^j,$$

$$N_G^{ij} = \delta^{ij} (q_1^2 - q_2^2) - \frac{2q_1^i (q_1 + q_2)^j}{u} + \frac{2(q_1 + q_2)^i q_2^j}{u\tau}$$

Intersection Numbers @ Cosmological Integrals

Cosmological wavefunctions

Arkani-Hamed, Benincasa, Postnikov

Arkani-Hamed, Baumann, Hillmann, Joyce, Lee, Pimentel

Benincasa, Vazao

- **Toy-model:** conformally coupled scalar field (with polynomial self-interactions),

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{12}R\phi^2 - \sum_{p>2} \frac{\lambda_p}{p!} \phi^p \right]$$

- **Goal:** correlation functions in an FRW cosmology. $a(\eta) = (\eta/\eta_0)^{-(1+\varepsilon)}$

$$\Psi_{\text{FRW}}(E_v, E_I) = \int_0^\infty \prod_v d\omega_v \left(\prod_v \omega_v \right)^\varepsilon \Psi_{\text{flat}}(E_v + \omega_v, E_I)$$

rational function of E_v and E_I

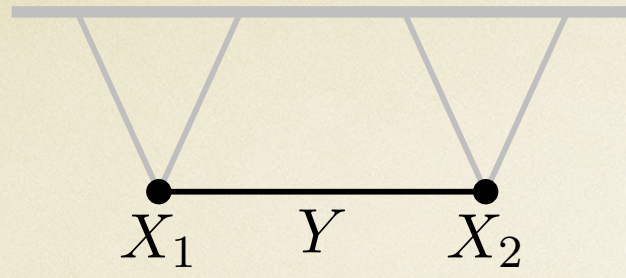
(“energies” associated with the vertices and the internal edges)

- **Twisted period integrals**

$$I(C, D; n; \varepsilon) = \int_0^\infty dx_1 \cdots dx_m P(x) \prod_I (C_{Ij}x_j + D_I)^{-n_I + \varepsilon_I}$$

The cosmological wavefunction satisfies a differential equation, which governs how it changes as the external kinematics are varied.

Arkani-Hamed, Baumann, Hillmann, Joyce, Lee, Pimentel



$$I = \int dz_1 \wedge dz_2 \frac{(z_1 z_2)^\epsilon}{(z_1 + y_1 + 1)(z_2 + y_2 + 1)(z_1 + z_2 + y_1 + y_2)}$$

● Twisted Period Integrals

$$I = \int_{\mathcal{C}} u(z_1, z_2) \varphi(z_1, z_2)$$

$$u = (z_1 z_2)^\epsilon (D_1 D_2 D_3)^\gamma \quad D_1 = (z_1 + y_1 + 1), \quad D_2 = (z_2 + y_2 + 1), \quad D_3 = (z_1 + z_2 + y_1 + y_2)$$

γ is a regulator

$$\omega = d \log(u) = \omega_1 dz_1 + \omega_2 dz_2$$

$$\omega_1 = \frac{\gamma(2y_1 + y_2 + 2z_1 + z_2 + 1)}{(y_1 + z_1 + 1)(y_1 + y_2 + z_1 + z_2)} + \frac{\epsilon}{z_1} \quad \omega_2 = \frac{\gamma(y_1 + 2y_2 + z_1 + 2z_2 + 1)}{(y_2 + z_2 + 1)(y_1 + y_2 + z_1 + z_2)} + \frac{\epsilon}{z_2}$$

● Number of MIs = dimH and bases choice

$$\omega_2 = 0$$

$$\nu_2 = 2$$

$$e^{(2)} = h^{(2)} = \left\{ \frac{1}{D_1}, \frac{1}{D_2} \right\}$$

● 2 MIs in the internal layer

$$\begin{cases} \omega_1 = 0 \\ \omega_2 = 0 \end{cases}$$

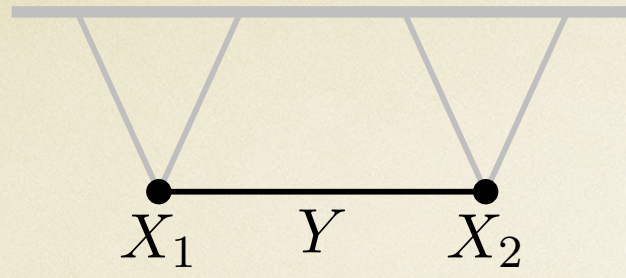
$$\nu_{21} = 4$$

$$e^{(21)} = h^{(21)} = \left\{ \frac{1}{\epsilon D_3^2}, \frac{1}{D_1 D_3}, \frac{1}{D_2 D_3}, \frac{1}{D_1 D_2 D_3} \right\}$$

● 4 MIs in the external layer

● Intersection Matrix

$$C = \begin{pmatrix} \frac{(\gamma+\epsilon)^2}{\gamma(\gamma^2-1)\epsilon^2(3\gamma+2\epsilon)} & -\frac{\gamma+\epsilon}{(\gamma-1)\gamma\epsilon(3\gamma+2\epsilon)} & -\frac{\gamma+\epsilon}{(\gamma-1)\gamma\epsilon(3\gamma+2\epsilon)} & \frac{1}{\gamma\epsilon-\gamma^2\epsilon} \\ -\frac{\gamma+\epsilon}{\gamma(\gamma+1)\epsilon(3\gamma+2\epsilon)} & \frac{2(\gamma+\epsilon)^2}{\gamma^2(2\gamma+\epsilon)(3\gamma+2\epsilon)} & \frac{1}{3\gamma^2+2\gamma\epsilon} & \frac{1}{\gamma^2} \\ -\frac{\gamma+\epsilon}{\gamma(\gamma+1)\epsilon(3\gamma+2\epsilon)} & \frac{1}{3\gamma^2+2\gamma\epsilon} & \frac{2(\gamma+\epsilon)^2}{\gamma^2(2\gamma+\epsilon)(3\gamma+2\epsilon)} & \frac{1}{\gamma^2} \\ -\frac{1}{\gamma^2\epsilon+\gamma\epsilon} & \frac{1}{\gamma^2} & \frac{1}{\gamma^2} & \frac{3}{\gamma^2} \end{pmatrix}$$



$$I = \int dz_1 \wedge dz_2 \frac{(z_1 z_2)^\epsilon}{(z_1 + y_1 + 1)(z_2 + y_2 + 1)(z_1 + z_2 + y_1 + y_2)}$$

• 4 MIs

$$e^{(21)} = \left\{ \frac{1}{\epsilon D_3^2}, \frac{1}{D_1 D_3}, \frac{1}{D_2 D_3}, \frac{1}{D_1 D_2 D_3} \right\}$$

• System of Differential Equations

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

• Master Decomposition Formula

$$\Omega_{ij} = \langle (\partial_x + \sigma_x) e_i | h_k \rangle (\mathbf{C}^{-1})_{kj}$$

after taking the limit $\gamma \rightarrow 0$:

• Canonical system

$$\Omega_{y_1} = \begin{pmatrix} \frac{2\epsilon}{y_1 + y_2 + 1} & 0 & 0 & 0 \\ -\frac{\epsilon}{y_1 + 1} & \frac{\epsilon}{y_1 + 1} & 0 & 0 \\ \frac{\epsilon}{y_1} & 0 & \frac{\epsilon}{y_1} & 0 \\ \frac{\epsilon}{y_1(y_1 + 1)} & 0 & \frac{\epsilon}{y_1(y_1 + 1)} & \frac{\epsilon}{y_1 + 1} \end{pmatrix}$$

$$\Omega_{y_2} = \begin{pmatrix} \frac{2\epsilon}{y_1 + y_2 + 1} & 0 & 0 & 0 \\ \frac{\epsilon}{y_2} & \frac{\epsilon}{y_2} & 0 & 0 \\ -\frac{\epsilon}{y_2 + 1} & 0 & \frac{\epsilon}{y_2 + 1} & 0 \\ \frac{\epsilon}{y_2(y_2 + 1)} & \frac{\epsilon}{y_2(y_2 + 1)} & 0 & \frac{\epsilon}{y_2 + 1} \end{pmatrix}$$

✓ Cohomology-based methods for cosmological correlations @ tree level

Pokraka et al. (2023)

✓ Differential Equations for cosmological correlations @ tree level

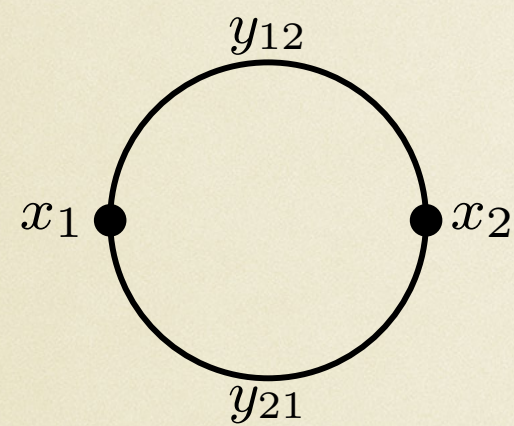
Arkani-Hamed, Baumann, Hillmann, Joyce, Lee, Pimentel (2023)

Cosmological Integrals @ 1-loop

Benincasa, Brunello, Mandal, Vazão, & PM (2024)

- 📍 Mapping cosmological integrals to **QFT-like integrals in momentum space**, with **semi-integer denominator powers**
- 📍 From momentum-space to **Baikov representation** to cast them as **twisted period integrals**

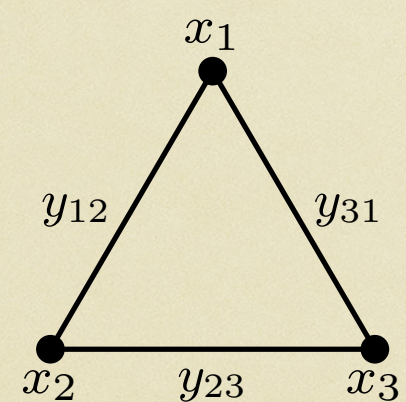
● Two-site graph



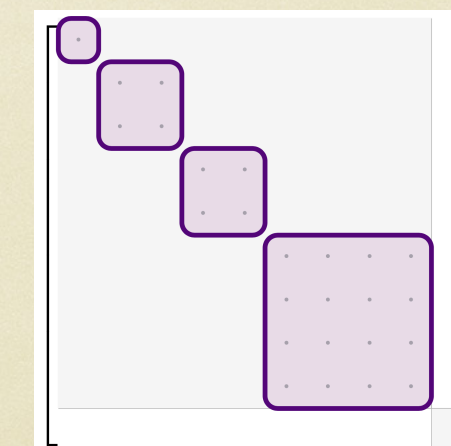
- ✓ Linear algebra from **Algebraic Geometry and Syzygy equations**
- ✓ Linear algebra from **Intersection Theory**
- ✓ (y-integration) **Canonical Differential Equations for $\nu = 6$ MIs: polylog structure**
- ✓ (y-integration) **Analytic solution**
- ✓ **Site-weight** x-integration: Mellin Transform and **Method of Brackets**
- ✓ **Analytic solution**: back of a envelope result

$$\begin{aligned} \mathcal{I}_{(2,1)} = & \frac{2^{-3-2\alpha} \pi^{3/2} (X_1 + X_2)^{1+2\alpha} \csc(\pi\alpha)^2 \Gamma(-\frac{1}{2} - \alpha)}{\Gamma[-\alpha]} \left(2 - \frac{1}{\epsilon} - \log(4\pi e^{\gamma_E} P^2) \right) \\ & + \frac{\pi^{3/2} \csc^2(\pi\alpha)}{8(\alpha + 1)^2 P} \left[-4\sqrt{\pi} \left((P + X_1)^{\alpha+1} - 2(X_1 - P)^{\alpha+1} \right) (P + X_2)^{\alpha+1} \right. \\ & \left. - \frac{4^{-\alpha} \Gamma(-\alpha - \frac{1}{2}) (X_1 + X_2)^{2\alpha+2}}{\Gamma(-\alpha)} {}_2F_1 \left(1, -2(\alpha + 1); -\alpha; \frac{P + X_1}{X_1 + X_2} \right) \right] \\ & + \frac{\pi^2 \csc(\pi\alpha) \csc(2\pi\alpha) (P + X_1)^\alpha}{4\alpha + 2} \left[-2(P + X_1) \left((P - X_2)^\alpha + (-1)^\alpha (P + X_2)^\alpha \right) \right. \\ & \left. + (-1)^\alpha (X_1 - X_2) (P + X_1)^\alpha {}_2F_1 \left(1 - \alpha, -2\alpha; 1 - 2\alpha; \frac{X_1 - X_2}{P + X_1} \right) \right. \\ & \left. + (X_1 + X_2) (P + X_1)^\alpha {}_2F_1 \left(1 - \alpha, -2\alpha; 1 - 2\alpha; \frac{X_1 + X_2}{P + X_1} \right) \right] \\ & - \frac{\pi^{5/2} 4^{-\alpha-1} \csc(\pi\alpha) \csc(2\pi\alpha)}{\Gamma(-\alpha) \Gamma(\alpha + \frac{3}{2}) (P + X_1)} \left[(-1)^\alpha (X_1 - X_2)^{2\alpha+2} {}_3F_2 \left(1, 1, \alpha + 2; 2, 2\alpha + 3; \frac{X_1 - X_2}{P + X_1} \right) \right. \\ & \left. + (X_1 + X_2)^{2\alpha+2} {}_3F_2 \left(1, 1, \alpha + 2; 2, 2\alpha + 3; \frac{X_1 + X_2}{P + X_1} \right) \right] \\ & + \frac{\pi^{5/2} 2^{-2\alpha-1} \csc(\pi\alpha) \csc(2\pi\alpha) \left((-1)^\alpha (X_1 - X_2)^{2\alpha+1} + (X_1 + X_2)^{2\alpha+1} \right)}{\Gamma(-\alpha) \Gamma(\alpha + \frac{3}{2})} \log \left(\frac{P + X_1}{P} \right) \\ & + (X_1 \leftrightarrow X_2). \end{aligned}$$

● Three-site graph



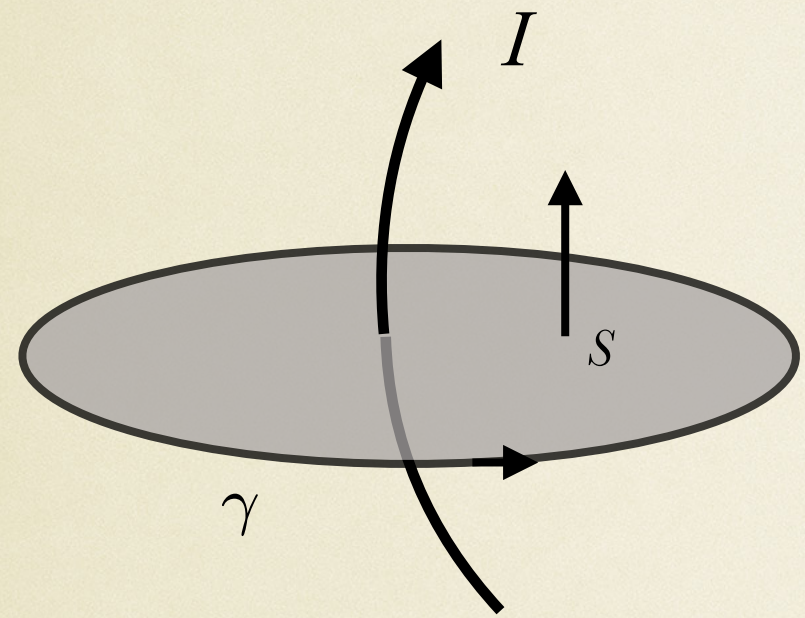
- ✓ Linear algebra from **Algebraic Geometry and Syzygy equations**
- ✓ Linear algebra from **Intersection Theory**
- ✓ (y-integration) **Differential Equations for $\nu = 41$ MIs: polylog and elliptic structure**



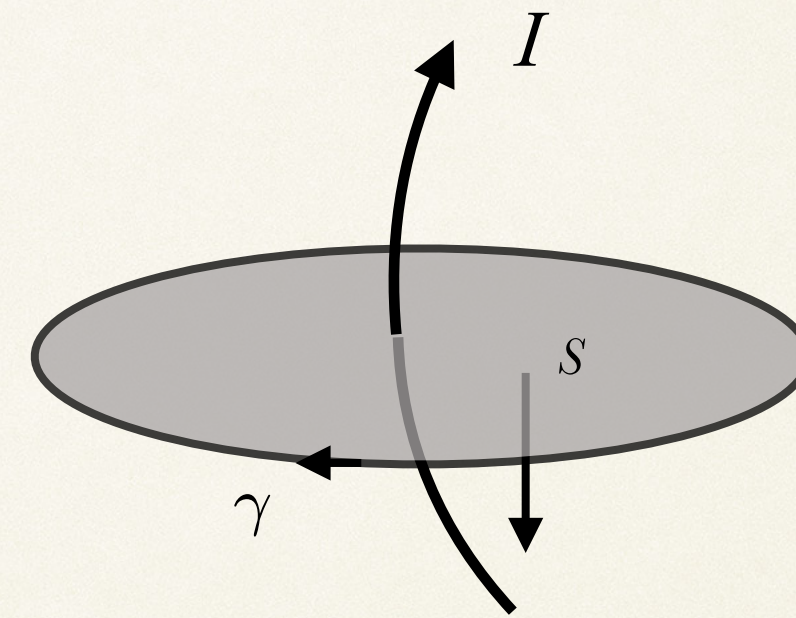
DEQ:
structure of the
elliptic sector
(4x4)-block

To Conclude:

Ampere's Law

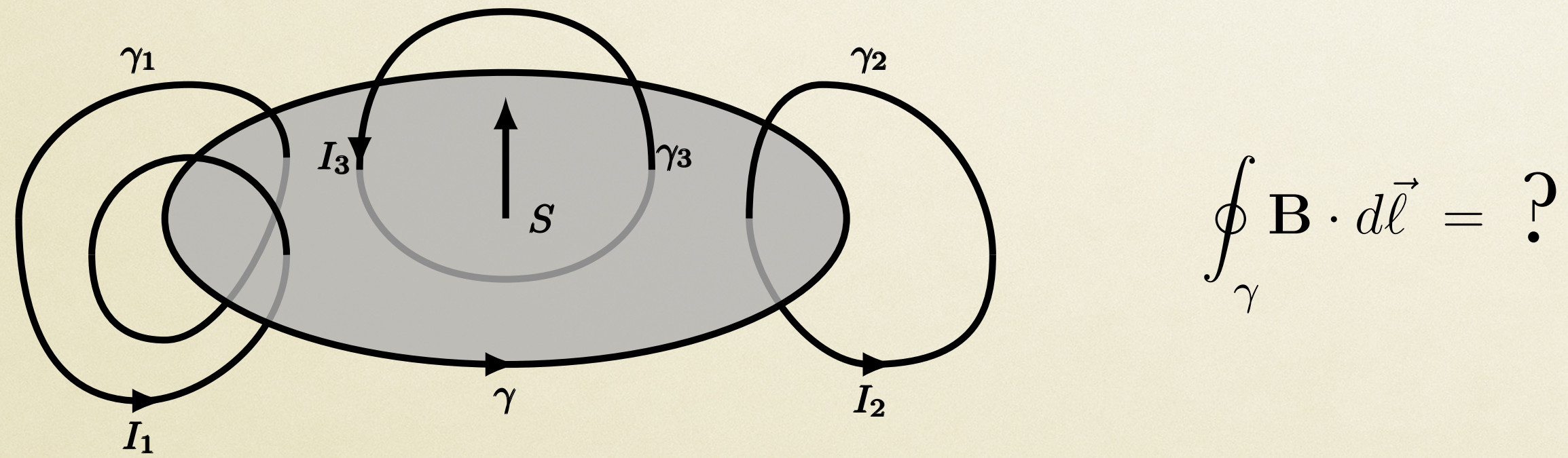
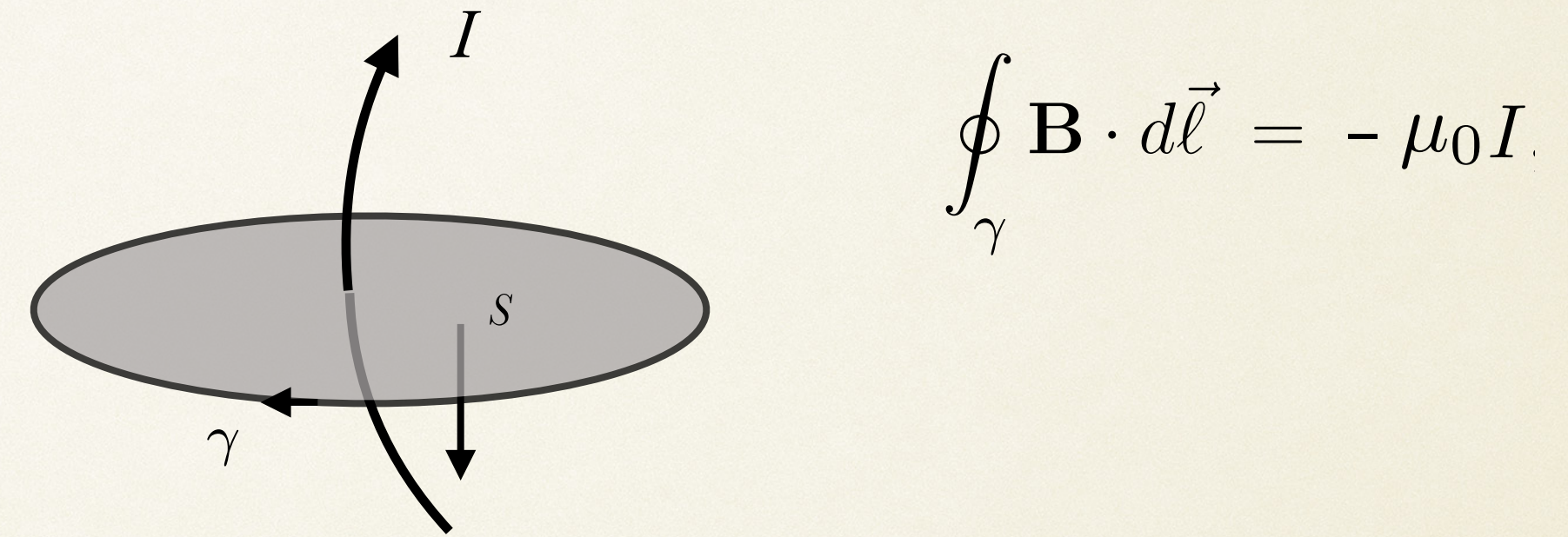
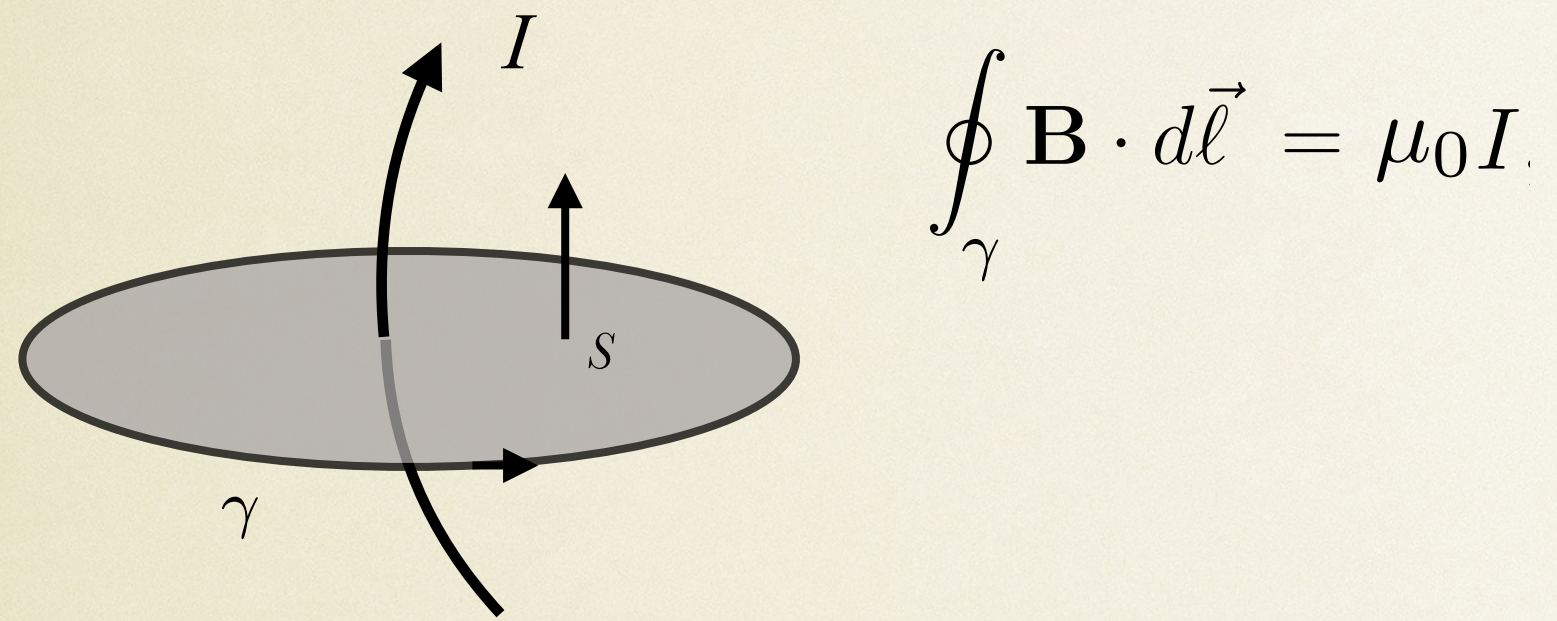


$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = \mu_0 I.$$

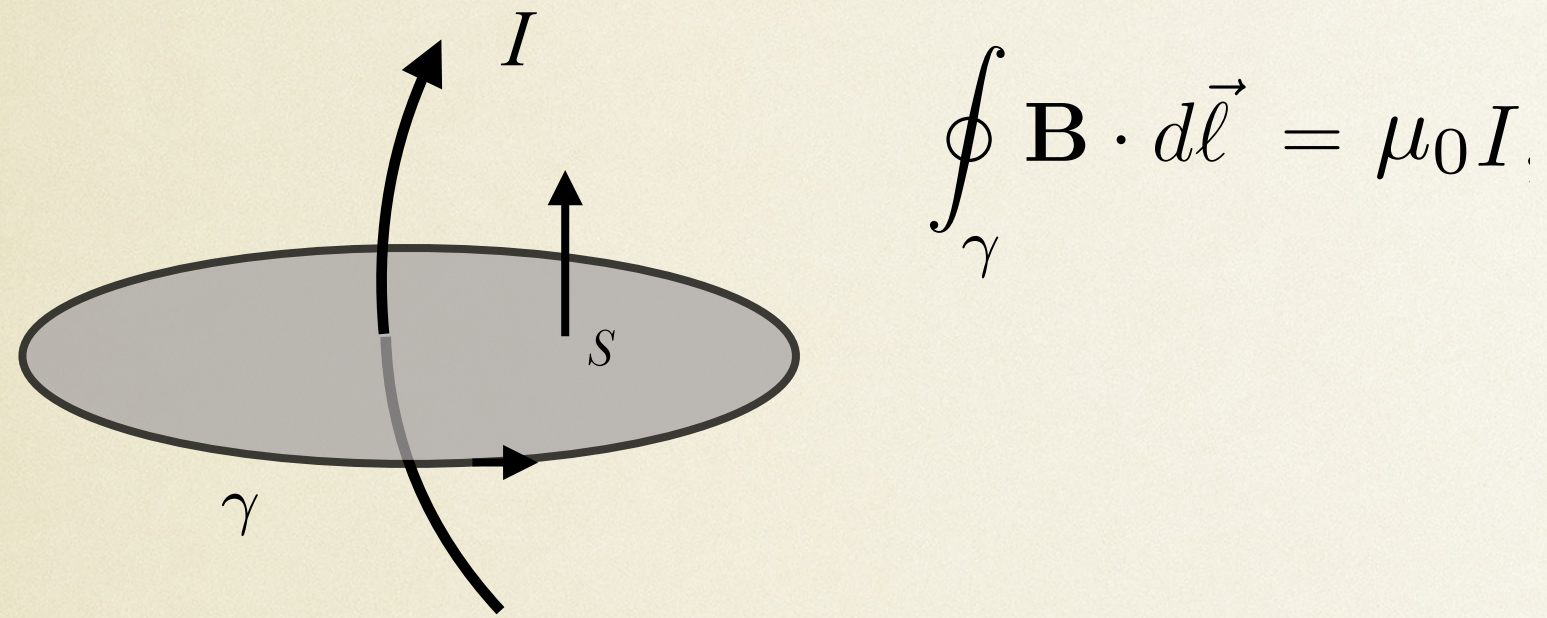


$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = -\mu_0 I.$$

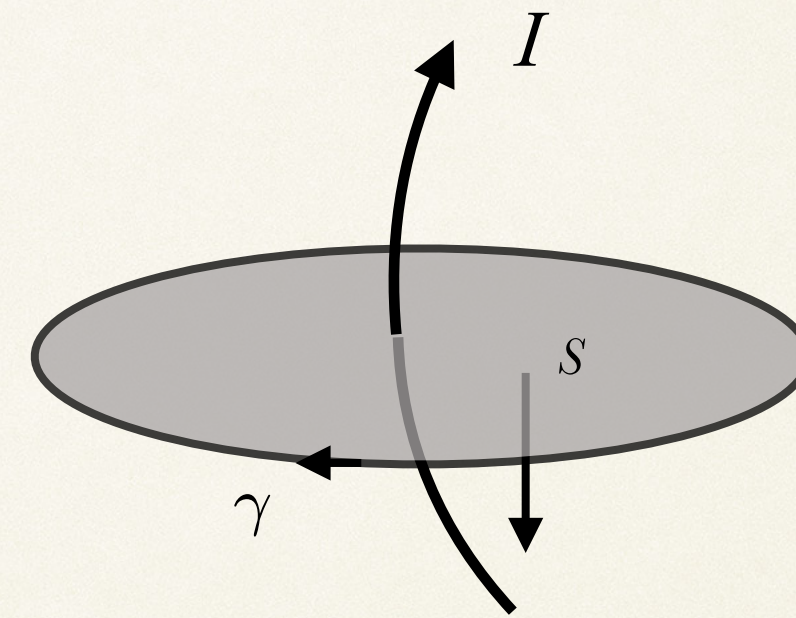
Ampere's Law



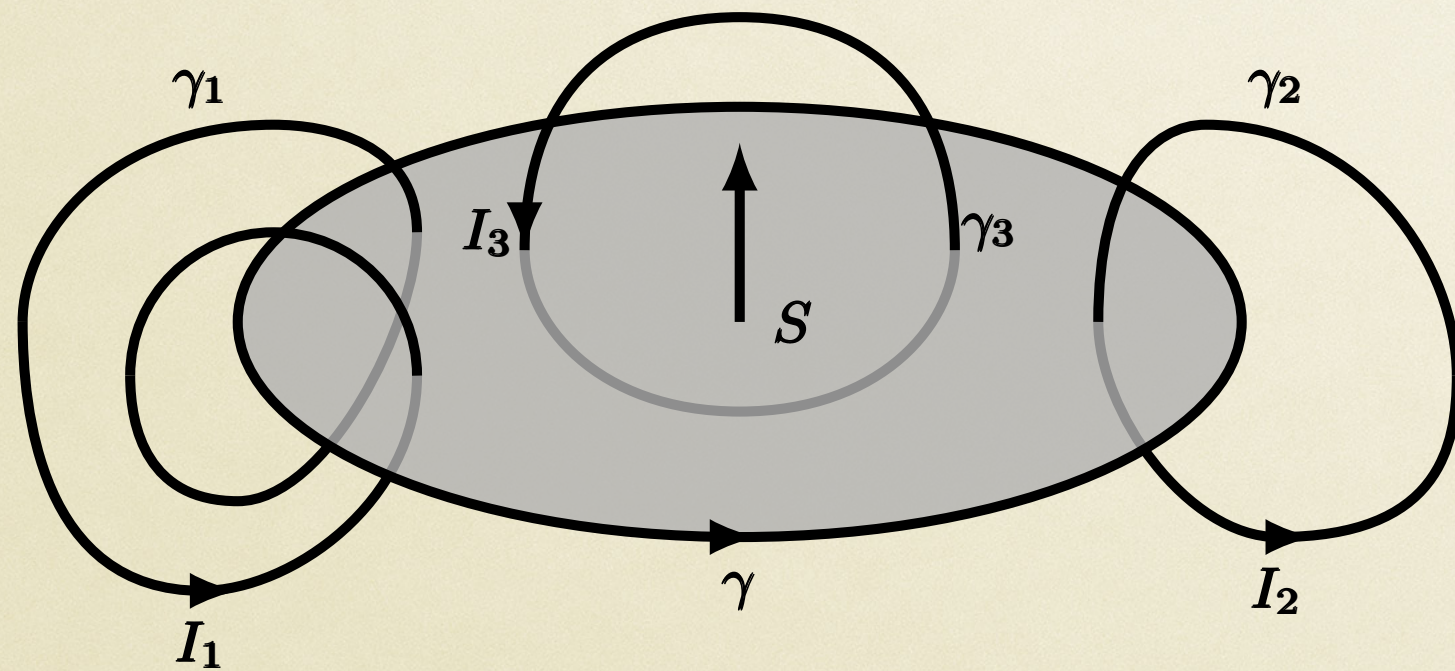
Ampere's Law



$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = \mu_0 I.$$



$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = -\mu_0 I.$$



$$\text{Link}(\gamma_1, \gamma) = +2, \text{Link}(\gamma_2, \gamma) = -1, \text{and } \text{Link}(\gamma_3, \gamma) = 0$$

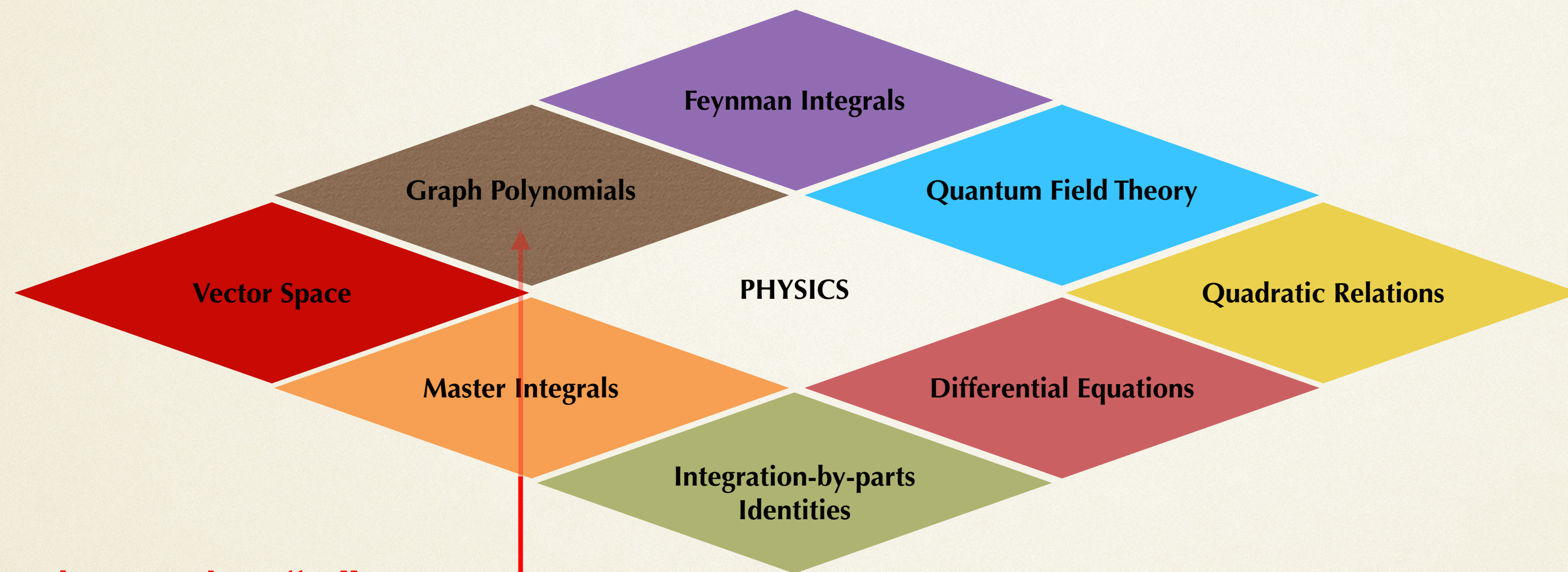
• Integral decomposition by geometry

$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = \sum_k (\pm n_k) \oint_{\gamma_k} \mathbf{B} \cdot d\vec{\ell} = \mu_0 \sum_k (\pm n_k) I_k$$

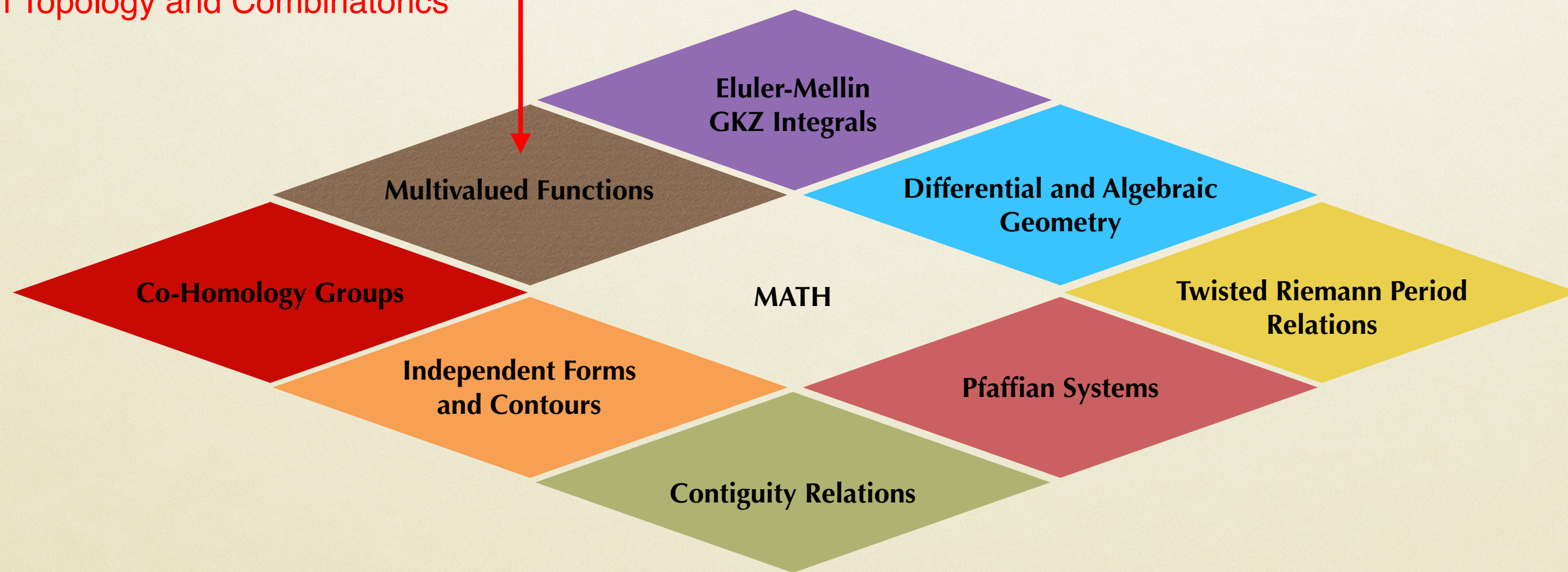
Gauss' Linking Number

$$n_k = \text{Link}(\gamma_k, \gamma)$$

Master Contributions



the twist "u"
 Analyticity and Algebra
 from Topology and Combinatorics



Summary

- **The ubiquitous De Rham Theory**

- 📍 Intersection Theory for Twisted de Rham co-homology

- 📍 Analyticity & Unitarity vs Differential and Algebraic Geometry, Topology, Number Theory, Combinatorics, Statistics

- **Novel Concepts: Vector Space Structures**

- 📍 Vector-space dimensions = dimension of co-homology group = *counting holes* = number of independent Integrals

- 📍 Intersection Numbers ~ **Scalar Product** for Feynman (Twisted Period) Integrals

- **New Methods for Multivariate Intersection number**

- 📍 Relation between Ordinary and Relative Cohomogy

- 📍 Fibration-based method and Companion Tensor Algebra

- **General algorithm for Physics and Math applications**

- 📍 key: Co-Homology Group Isomorphisms

- 📍 Feynman Integrals, Euler-Mellin Integrals, D-Module and GKZ hypergeometric theory, Orthogonal Polynomials, QM matrix elements, Correlator functions in QFT.

- **Modern Multi-Loop diagrammatic techniques and Amplitudes calculus useful beyond Particle Physics**

- 📍 Triggering interdisciplinarity

- 📍 Interwinement between Fundamental Physics, Geometry and Statistics: fluxes ~ period integrals ~ statistical moments

- 📍 Interesting implications in QM, QFT and Cosmology: invariance and independent moments of distributions, perturbative and non-perturbative approaches

- **Bilinear relations (not covered)**

- 📍 Recent interesting applications

Duhr, Porkert, Semper, Stawiński (2024)

Loebbert, Stawiński (2024)

- **Work in progress (not covered)**

- 📍 Intersection theory and Gluing method in N=4 SYM

Eden, Crisanti Gottwald, Scherdin, & P.M.

- 📍 Novel structures in D-module theory and De Rham Co-homology groups

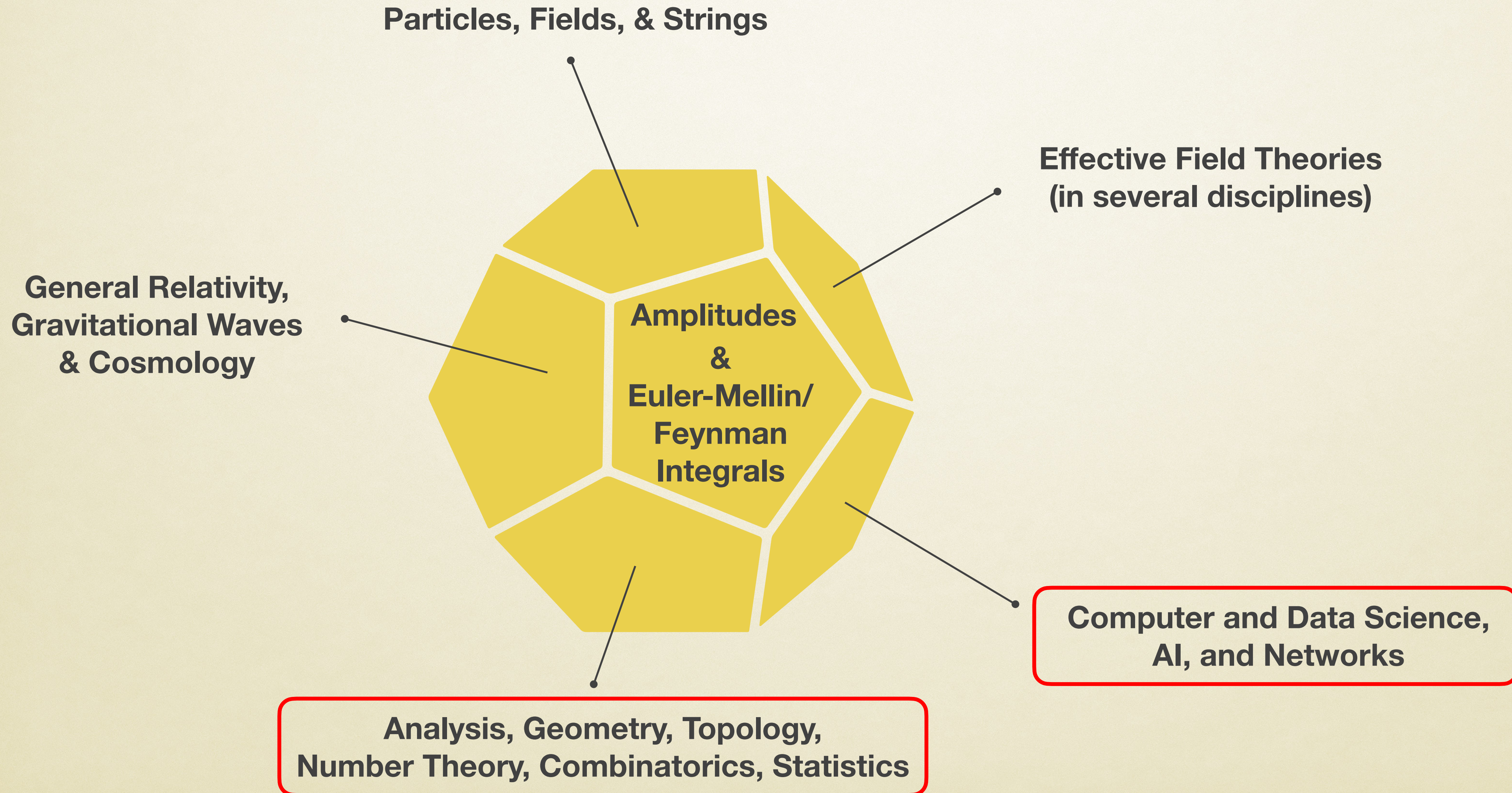
Chestnov, Flieger, & P.M.

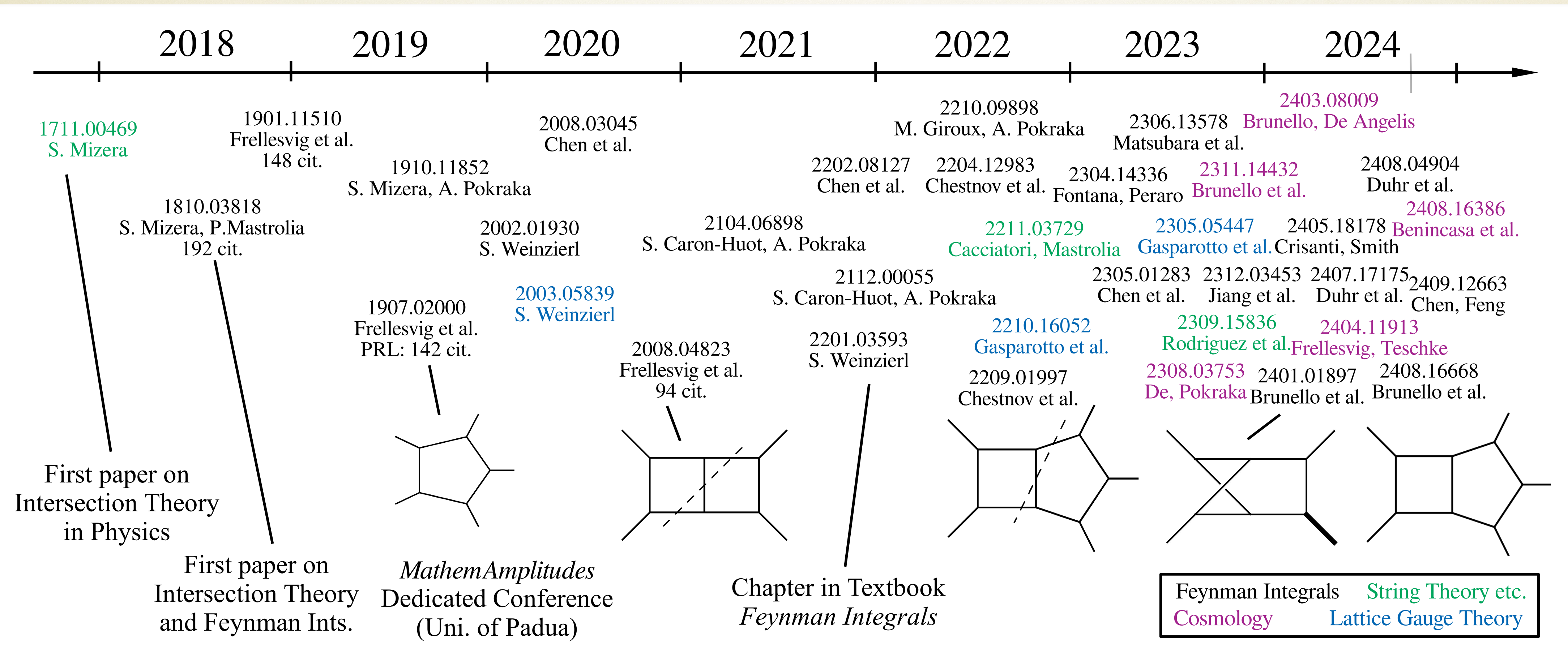
- 📍 Numerical methods for Feynman integrals and Neural Networks

Calisto, Moodie, Zoia (2023)

Boni, Mandal, & P.M.

Scattering Amplitudes & Multiloop Calculus: interdisciplinary toolbox





courtesy: Hjalte Frellesvig

MathemAmplitudes 2025

Co-homology and Combinatorics of GKZ Systems, Euler-Mellin-Feynman Integrals, and Scattering Amplitudes.

September 22-26, 2025

MITP Mainz

Bringing together mathematicians and theoretical physicists with interdisciplinary expertise, the workshop will cover a broad range of topics, including Differential and Algebraic Geometry, Number Theory, Combinatorics, Statistics, Feynman Integrals, and Scattering Amplitudes.

Definition. *Physics is a part of mathematics devoted to the calculation of integrals of the form $\int g(x)e^{f(x)}dx$. Different branches of physics are distinguished by the range of the variable x and by the names used for $f(x)$, $g(x)$ and for the integral. [...]*

Of course this is a joke, physics is not a part of mathematics. However, it is true that the main mathematical problem of physics is the calculation of integrals of the form

$$I(g) = \int g(x)e^{-f(x)}dx$$

[...] If f can be represented as $f_0 + \lambda V$ where f_0 is a negative quadratic form, then the integral $\int g(x)e^{f(x)}dx$ can be calculated in the framework of perturbation theory with respect to the formal parameter λ . We will fix f and consider the integral as a functional $I(g)$ taking values in $\mathbb{R}[[\lambda]]$. It is easy to derive from the relation

$$\int \partial_a(h(x)e^{f(x)})dx = 0$$

that the functional $I(g)$ vanishes in the case when g has the form

$$g = \partial_a h + (\partial_a f)h.$$

The unreasonable effectiveness of mathematics

E. Wigner

Wigner was referring to the mysterious phenomenon in which areas of pure mathematics, originally constructed without regard to application, are suddenly discovered to be exactly what is required to describe the structure of the physical world.

M. Berry

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For collaboration and/or discussions:

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V. Chestnov, H. Munch, T. Peraro, G. Fontana,

G. Brunello, G. Crisanti, S. Smith, M. Giroux, W. Flieger,

W. J. Torres, J. Ronca, R. Patil, M. Bigazzi, G. Boni,

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C. Fevola, M. Bertola

Extra Slides

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Consider the $(m - 1)$ -differential form φ_{m-1} ,

$$0 = \int_{\mathcal{C}} d(u \varphi_{m-1}) = \int_{\mathcal{C}} u (\nabla_{\omega} \varphi_{m-1})$$

• **Covariant Derivative** $\omega \equiv d \log u$ $\nabla_{\omega} \equiv d + \omega \wedge \equiv u^{-1} \cdot d \cdot u$

• **Integrals** $I = \int_{\mathcal{C}} u \varphi_m = \int_{\mathcal{C}} u (\varphi_m + \nabla_{\omega} \varphi_{m-1}) = \int_{\mathcal{C} + \partial \Gamma} u \varphi_m$

• **Twisted Cohomology Group** $H_{\omega}^m(X) = \frac{\text{Ker}(\nabla_{\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

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• **Twisted Homology Group** $H_m^{\omega}(X) = \frac{\text{Ker}(\partial \otimes u : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}{\text{Im}(\partial \otimes u : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}$

Basics of Intersection Theory / De Rham Twisted **Dual** Co-Homology Groups: $u \rightarrow u^{-1}$

Consider the $(m - 1)$ -differential form φ_{m-1} ,

$$0 = \int_{\mathcal{C}} d(u^{-1} \varphi_{m-1}) = \int_{\mathcal{C}} u^{-1} (\nabla_{-\omega} \varphi_{m-1})$$

● **Dual Covariant Derivative**

$$\nabla_{-\omega} \equiv d - \omega \wedge \equiv u \cdot d \cdot u^{-1}$$

● **Dual Integrals**

$$\tilde{I} = \int_{\mathcal{C}} u^{-1} \phi_m = \int_{\mathcal{C}} u^{-1} (\phi_m + \nabla_{-\omega} \phi_{m-1}) = \int_{\mathcal{C} + \partial\Gamma} u^{-1} \phi_m$$

● **Dual Twisted Cohomology Group**

$$H_{-\omega}^m(X) = \frac{\text{Ker}(\nabla_{-\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{-\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$$

● **Dual Twisted Homology Group**

$$H_m^{-\omega}(X) = \frac{\text{Ker}(\partial \otimes u^{-1} : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}{\text{Im}(\partial \otimes u^{-1} : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}$$

Intersection Numbers for *Logarithmic n-forms*

Matsumoto (1998), Mizera (2017)

If $\langle \varphi_L |$ and $\langle \varphi_R |$ are dLog n -forms (hence contain only simple poles)

$$\langle \varphi_L | \varphi_R \rangle = \int dz_1 \cdots dz_n \delta(\omega_1) \cdots \delta(\omega_n) \hat{\varphi}_L \hat{\varphi}_R =$$

$$= \sum_{(z_1^*, \dots, z_n^*)} \det^{-1} \begin{bmatrix} \frac{\partial \omega_1}{\partial z_1} & \cdots & \frac{\partial \omega_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \omega_n}{\partial z_1} & \cdots & \frac{\partial \omega_n}{\partial z_n} \end{bmatrix} \hat{\varphi}_L \hat{\varphi}_R \Big|_{(z_1, \dots, z_n) = (z_1^*, \dots, z_n^*)}$$

[Global Residue Theorem]

(z_1^*, \dots, z_n^*) *critical points*, namely the solutions of the system $\omega_i = 0, \quad i = 1, \dots, n.$

In the 1-variate case: $\langle \varphi_L | \varphi_R \rangle = \text{Res}_{z \in \mathcal{P}_{\omega_1}} \left(\frac{\hat{\varphi}_L \hat{\varphi}_R}{\omega} \right) = \int dz_1 \delta(\omega_1) \hat{\varphi}_L \hat{\varphi}_R = \sum_{(z_1^*)} \frac{\hat{\varphi}_L \hat{\varphi}_R}{\partial \omega_1 / \partial z_1}$ [Residue Theorem]

● Efficiently implemented also *via Companion Matrix* credit Salvatori

Intersection Numbers for **n-forms** (V) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama & P.M. (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for \mathbb{H}^n and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

• **Secondary Equations** $\partial_x \mathbf{C} = \mathbf{\Omega} \cdot \mathbf{C} + \mathbf{C} \cdot \tilde{\mathbf{\Omega}}$, $\partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}} \cdot \mathbf{C}^{-1} - \mathbf{C}^{-1} \cdot \mathbf{\Omega}$

• **Master Decomposition** $\langle \varphi | = \sum_{\lambda=1}^r c_\lambda \langle e_\lambda |$,

$$\begin{bmatrix} e_1 \\ \vdots \\ e_{r-1} \\ \varphi \end{bmatrix} = C^{\text{aux}} \cdot C^{-1} \begin{bmatrix} e_1 \\ \vdots \\ e_{r-1} \\ e_r \end{bmatrix} \implies C^{\text{aux}} \cdot C^{-1} = \left[\begin{array}{c|c} \text{id}_{r-1} & \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \\ \hline c_1 & \cdots & c_{r-1} & c_r \end{array} \right]$$

Coefficients from matrix multiplication

Quadratic Relations

Twisted Riemann Periods Relations (TRPR)

- Completeness for forms

$$\sum_{i,j=1}^{\nu} |e_j\rangle (\mathbf{C}^{-1})_{ji} \langle e_i| = \mathbb{I}_c \quad \mathbf{C}_{ij} \equiv \langle e_i|e_j\rangle$$

- Completeness for contours

$$\sum_{i,j=1}^{\nu} |\mathcal{C}_j] (\mathbf{H}^{-1})_{ji} [\mathcal{C}_i| = \mathbb{I}_h \quad \mathbf{H}_{ij} \equiv [\mathcal{C}_i|\mathcal{C}_j]$$

- Riemann Twisted Period Relations Cho, Matsumoto (1995)

$$\langle \varphi_L | \varphi_R \rangle = \sum_{i,j} \langle \varphi_L | \mathcal{C}_{R,j}] [\mathcal{C}_{L,j} | \mathcal{C}_{R,i}]^{-1} [\mathcal{C}_{L,i} | \varphi_R \rangle$$

$$[\mathcal{C}_L | \mathcal{C}_R] = \sum_{i,j} [\mathcal{C}_L | \varphi_{R,j} \rangle \langle \varphi_{L,j} | \varphi_{R,i} \rangle^{-1} \langle \varphi_L | \mathcal{C}_R]$$

TRPR for Gauss Hypergeometric Function

Cho, Matsumoto (1995)

$$u = t^\alpha (1-t)^{\gamma-\alpha} (1-xt)^{-\beta}, \quad \varphi_1 = \left(\frac{dt}{t-x_1} - \frac{dt}{t-x_2} \right) = \frac{dt}{t(1-t)}, \quad \varphi_3 = \left(\frac{dt}{t-x_3} - \frac{dt}{t-x_4} \right) = \frac{-xdt}{1-xt},$$

$$P^+ = \begin{pmatrix} \int_0^1 u \varphi_1 & \int_{1/x}^\infty u \varphi_1 \\ \int_0^1 u \varphi_3 & \int_{1/x}^\infty u \varphi_3 \end{pmatrix}, \quad P^- = \begin{pmatrix} \int_0^1 u^{-1} \varphi_1 & \int_{1/x}^\infty u^{-1} \varphi_1 \\ \int_0^1 u^{-1} \varphi_3 & \int_{1/x}^\infty u^{-1} \varphi_3 \end{pmatrix}, \quad I_{ch} = 2\pi i \begin{pmatrix} 1/\alpha + 1/(\gamma-\alpha) & 0 \\ 0 & -1/\beta + 1/(\beta-\gamma) \end{pmatrix} \quad I_h = - \begin{pmatrix} d_{12}/d_1 d_2 & 0 \\ 0 & d_{30}/d_3 d_0 \end{pmatrix},$$

$c_{jk\dots} = c_j c_k \dots, d_{jk\dots} = c_j c_k \dots - 1$
 $c_j = \exp 2\pi i \alpha_j$

$$\int_0^1 u \varphi_1 = B(\alpha, \gamma-\alpha) F(\alpha, \beta, \gamma; x),$$

$$\int_{1/x}^\infty u \varphi_1 = -(-1)^{\gamma-\alpha-\beta} x^{1-\gamma} B(\beta-\gamma+1, -\beta+1) \times F(\beta-\gamma+1, \alpha-\gamma+1, 2-\gamma; x),$$

• Riemann Twisted Period Relations

$$P^+ {}^t I_h^{-1} {}^t P^- = I_{ch}$$

(1,2)- component $F(\alpha, \beta, \gamma; x) F(1-\alpha, 1-\beta, 2-\gamma; x) = F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma; x) F(\gamma-\alpha, \gamma-\beta, \gamma; x)$

(1,1)-component $F(\alpha, \beta, \gamma; x) F(-\alpha, -\beta, -\gamma; x) - 1 = \frac{\alpha\beta(\gamma-\alpha)(\gamma-\beta)}{\gamma^2(\gamma+1)(\gamma-1)} F(\beta-\gamma+1, \alpha-\gamma+1, -\gamma+2; x) \times F(\gamma-\beta+1, \gamma-\alpha+1, \gamma+2; x).$

Elliot's Identity from Intersections Matsumoto & P.M.

The complete elliptic integrals \mathcal{K} and \mathcal{E} of the first and second kind

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - r^2 \sin^2 \phi}} \quad \mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 \phi} d\phi$$

● Legendre Identity

$$\mathcal{E}\mathcal{K}' + \mathcal{E}'\mathcal{K} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2}$$

$$\mathcal{K}'(r) = \mathcal{K}(r') \text{ and } \mathcal{E}'(r) = \mathcal{E}(r')$$
$$r^2 + r'^2 = 1$$

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● Elliot's Identity and Hypergeometric Functions

Balasubramanian, Naik, Ponnusamy, Vuorinen (2001)

$$\begin{aligned} & F\left(\frac{1}{2} + \lambda, -\frac{1}{2} - \nu, 1 + \lambda + \mu; r\right) F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \\ & + F\left(\frac{1}{2} + \lambda, \frac{1}{2} - \nu, 1 + \lambda + \mu; r\right) F\left(-\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \\ & - F\left(\frac{1}{2} + \lambda, \frac{1}{2} - \nu, 1 + \lambda + \mu; r\right) F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \\ & = \frac{\Gamma(1 + \lambda + \mu)\Gamma(1 + \mu + \nu)}{\Gamma(\lambda + \mu + \nu + \frac{3}{2})\Gamma(\mu + \frac{1}{2})}. \end{aligned}$$

the choice $\lambda = \mu = \nu = 0$ gives the Legendre relation.

Elliot's Identity from Intersections Matsumoto & P.M.

● *Hypothesis: too close to RTPR to be accidental*

● *Proof*

$$u(t) = t^{1/2+\lambda}(1-t)^{-1/2+\mu}(1-rt)^{1/2+\nu},$$

$$\varphi_1 = \frac{dt}{t}, \quad \varphi_2 = \frac{dt}{t(1-rt)} = \left(\frac{1}{t} - \frac{1}{t-1/r}\right)dt,$$

$$\psi_1 = \frac{dt}{1-t} = \frac{-dt}{t-1}, \quad \psi_2 = \frac{dt}{t(1-t)} = \left(\frac{1}{t} - \frac{1}{t-1}\right)dt.$$

$$\gamma = (0, 1) \otimes u(t) \text{ and } \delta = (-\infty, 0) \otimes 1/u(t)$$

● *Twisted Riemann Period Relation*

$${}^t\Pi_\omega {}^tH_c^{-1}\Pi_{-\omega} = H_h.$$

$$\left(\int_0^1 u(t)\varphi_1, \int_0^1 u(t)\varphi_2\right) {}^tH_c^{-1} \begin{pmatrix} \int_{-\infty}^0 \frac{1}{u(t)}\psi_1 \\ \int_{-\infty}^0 \frac{1}{u(t)}\psi_2 \end{pmatrix} = \frac{-1}{e^{2\pi\sqrt{-1}\lambda} + 1}.$$

$$\left(F\left(\frac{1}{2} + \lambda, -\frac{1}{2} - \nu, 1 + \lambda + \mu; r\right), F\left(\frac{1}{2} + \lambda, \frac{1}{2} - \nu, 1 + \lambda + \mu; r\right)\right) \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \\ F\left(-\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \end{pmatrix} = \frac{\Gamma(\lambda + \mu + 1)\Gamma(\mu + \nu + 1)}{\Gamma(\lambda + \mu + \nu + \frac{3}{2})\Gamma(\mu + \frac{1}{2})}$$

Elliot's Identity from Intersections

Matsumoto & P.M.

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- Quadratic relations for Feynman Integrals

Broadhurst, Roberts (2018)

Lee, Pomeranski (2019)

$$\mathbf{P}_k^{\text{BR}} \cdot \mathbf{D}_k^{\text{BR}} \cdot {}^t\mathbf{P}_k^{\text{BR}} = \mathbf{B}_k^{\text{BR}}$$

Fresan, Sabbah, Yu (2020)

- String-Theory Amplitudes: **KLT relations = TRPR**

Mizera (2016/17)

$$\mathcal{A}^{\text{GR}} = \sum_{\beta, \gamma} \mathcal{A}^{\text{YM}}(\beta) m^{-1}(\beta|\gamma) \mathcal{A}^{\text{YM}}(\gamma)$$

$$\mathcal{A}^{\text{closed}} = \sum_{\beta, \gamma} \mathcal{A}^{\text{open}}(\beta) m_{\alpha'}^{-1}(\beta|\gamma) \mathcal{A}^{\text{open}}(\gamma)$$