Analytic Continuation of Five-Point Two-Loop Master Integrals

based on work with C. Papadopoulos, G. Bevilacqua, D. Canko and A. Spourdalakis,
in extension of [arXiv:2201.07509 [hep-ph]]

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Outline

- Hexa-Box Integral Families through the SDE Approach
- Mathematica Implementation
- Analytic Continuation

Hexa-Box Integral Families through the SDE Approach

Brief Review of the Simplified Differential Equations (SDE) Approach

In the SDE approach for Master Integrals (Papadopoulos 2014), the ordinary external momenta q_i are parametrized by introducing a dimensionless variable x, as follows

$$q_1 \to p_{123} - x\, p_{12},\ q_2 \to p_4,\ q_3 \to -p_{1234},\ q_4 \to x\, p_1$$

where the new momenta p_i, $i = 1 \ldots 5$, now satisfy $\sum_1^5 p_i = 0$, $p_i^2 = 0$, $i = 1 \ldots 5$, whereas $p_{i \ldots j} := p_i + \ldots + p_j$. The set of independent invariants is given by $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}$, with $S_{ij} := (p_i + p_j)^2$.

Brief Review of the Simplified Differential Equations (SDE) Approach

For pure bases with rational alphabet letters or algebraic alphabet letters with rationalizable square roots, the differential equation for Master Integrals can be cast into its canonical form, easily solvable in terms of Goncharov Polylogarithms (GPLs):

$$\partial_x \mathbf{g} = \epsilon \left(\sum_a \frac{\mathbf{M}_a}{x - \ell_a} \right) \mathbf{g} \tag{1}$$

However, for some topologies there are alphabet letters with square roots that cannot be rationalized simultaneously, leading to the more general form:

$$\partial_x \mathbf{g} = \epsilon \left(\sum_a \frac{d \log L_a}{dx} \mathbf{M}_a \right) \mathbf{g} \tag{2}$$

5-Point 2-Loop 1-Mass Non-Planar Topologies

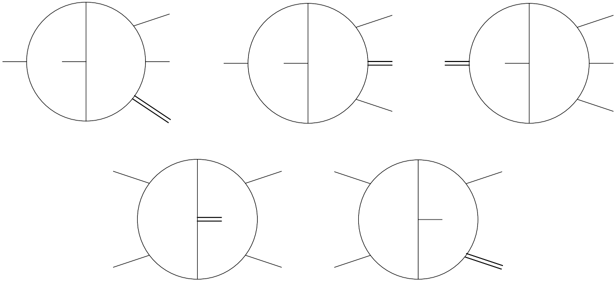

Figure: The five non-planar families with one external massive leg. The first row corresponds to the so-called hexabox topologies, whereas the diagrams of the second row are known as double-pentagons. We label them as follows: N_1 (top left), N_2 (top middle), N_3 (top right), N_4 (bottom left), N_5 (bottom right).

Hexa-Box Topologies: Analytic up to Weight 2

For all hexa-box integral families, up to transcendental weight 1, all square roots appearing in the corresponding alphabet letters can be rationalized, and analytic expressions can thus be obtained up to weight 2, using the iterated integral definition of Goncharov Polylogarithms (GPLs):

$$\mathcal{G}\left(a_n, \ldots, a_1; x\right) = \int_0^x dt \frac{1}{t - a_n} \mathcal{G}\left(a_{n-1}, \ldots, a_1; t\right) \quad (3)$$

However, for topologies N_2 and N_3 non-rationalizable square roots break this procedure above weight 2.

Hexa-Box Topologies: Semi-analytic above Weight 2

Above weight 2, we resort to a semi-numerical one-dimensional integral representation (Kardos, Papadopoulos, Smirnov, Syrrakos, Wever 2022). At weight 3,

$$\partial_x g_I^{(3)} = \sum_a (\partial_x \log L_a) \sum_J c_{IJ}^a g_J^{(2)} \tag{4}$$

$$g_I^{(3)} = g_{I,\mathcal{G}}^{(3)} + b_I^{(3)} + \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a (\partial_x \log L_a) \sum_J c_{IJ}^a g_J^{(2)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} \right) \tag{5}$$

with $b_I^{(3)}$ being the boundary terms at $\mathcal{O}(\epsilon^3)$ and $g_{I,\mathcal{G}}^{(3)} = \int_0^{\bar{x}} \mathrm{d}x \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} \Big|_{\mathcal{G}}$, with the subscript $\mathcal{G}$ indicating that the integral is represented in terms of GPLs.

Hexa-Box Topologies: Semi-analytic above Weight 2

At weight 4,

$$\begin{aligned} g_I^{(4)} =& g_{I,\mathcal{G}}^{(4)} + b_I^{(4)} + \left(\sum_a \log L_a \sum_J c_{IJ}^a g_J^{(3)} \right) - \left(\sum_a LL_a \sum_J c_{IJ}^a g_{J,0}^{(3)} \right) \\ & - \int_0^{\bar{x}} \mathrm{d}x \sum_a (\log L_a - LL_a) \sum_J c_{IJ}^a \sum_b \frac{l_b}{x} \sum_K c_{JK}^b g_{K,0}^{(2)} \\ & - \int_0^{\bar{x}} \mathrm{d}x \sum_a \log L_a \sum_J c_{IJ}^a \left(\sum_b (\partial_x \log L_b) \sum_K c_{JK}^b g_K^{(2)} - \sum_b \frac{l_b}{x} \sum_K c_{JK}^b g_{K,0}^{(2)} \right) \end{aligned} \quad (6)$$

Mathematica Implementation

Mathematica Implementation

$$\boxed{g_I^{(n+1)}} = g_{I,\mathcal{G}}^{(n+1)} + b_I^{(n+1)} + \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a \, g_J^{(n)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a \, g_{J,0}^{(n)} \right)$$

PureBasisElementNumerical[*ElementIndex*, *TranscendentalWeight*, *PhaseSpacePoint*] : numerical evaluation of pure basis elements at weight $n+1$ by integration of the analytic form of pure basis elements at weight n

Mathematica Implementation

$$g_I^{(n+1)} = g_{I,\mathcal{G}}^{(n+1)} + b_I^{(n+1)} + \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a (\partial_x \log L_a) \sum_J c_{IJ}^a \boxed{g_J^{(n)}} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a \boxed{g_{J,0}^{(n)}} \right)$$

PureBasisElementAnalytic[*ElementIndex*, *TranscendentalWeight*] : analytic form of pure basis elements, up to weight 2 for N_2 and N_3 hexa-box families

PureBasisElementAnalytic[*ElementIndex*, *TranscendentalWeight*, x → 0] : $x \to 0$ limit of analytic form of pure basis elements, e.g. $g_{J,0}^{(2)}$ are obtained by expanding $g_J^{(2)}$ around $x = 0$ and keeping terms up to order $\mathcal{O}(\log(x)^2)$

Mathematica Implementation

$$g_I^{(n+1)} = g_{I,\mathcal{G}}^{(n+1)} + \boxed{b_I^{(n+1)}} + \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a \, g_J^{(n)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a \, g_{J,0}^{(n)} \right)$$

BoundaryTerm[*ElementIndex*, *TranscendentalWeight*] : analytic form of boundary terms at $\mathcal{O}(\epsilon^{(n+1)})$

Mathematica Implementation

$$g_I^{(n+1)} = g_{I,\mathcal{G}}^{(n+1)} + b_I^{(n+1)} + \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a \boxed{(\partial_x \log L_a)} \sum_J c_{IJ}^a \, g_J^{(n)} - \sum_a \boxed{\frac{l_a}{x}} \sum_J c_{IJ}^a \, g_{J,0}^{(n)} \right)$$

DLogAlphabetLetter[*LetterIndex*] : derivatives of logs of alphabet letters with respect to parameter x

DLogAlphabetLetter[*LetterIndex*, x → 0] : $x \to 0$ limit of $\partial_x \log L_a$, obtained by expanding $\partial_x \log L_a$ around $x = 0$ and keeping terms up to order $\mathcal{O}(x^{-1})$

LogAlphabetLetter[*LetterIndex*] : logs of alphabet letters

LogAlphabetLetter[*LetterIndex*, x → 0] : $x \to 0$ limit of $\log L_a$, obtained by expanding $\log L_a$ around $x = 0$ and keeping terms up to order $\mathcal{O}(\log(x))$

Mathematica Implementation

$$g_I^{(n+1)} = g_{I,\mathcal{G}}^{(n+1)} + b_I^{(n+1)} + \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a (\partial_x \log L_a) \sum_J c_{IJ}^a \, g_J^{(n)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a \, g_{J,0}^{(n)} \right)$$

DifferentialEquationRHS[*ElementIndex*, *TranscendentalWeight*]
DifferentialEquationRHS[*ElementIndex*, *TranscendentalWeight*, x → 0]

Mathematica Implementation

$$g_I^{(n+1)} = \boxed{g_{I,\mathcal{G}}^{(n+1)}} + b_I^{(n+1)} + \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a \, g_J^{(n)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a \, g_{J,0}^{(n)} \right)$$

$$\boxed{g_{I,\mathcal{G}}^{(n+1)} = \int_0^{\bar{x}} \mathrm{d}x \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(n)} \bigg|_{\mathcal{G}}}$$

ClassicalToGoncharovPolyLogs[*expr*] and **GoncharovToClassicalPolyLogs[*expr*]** : converter between Goncharov Polylogarithms and Classical Logs and PolyLogs, making use of (Frellesvig et al. 2016)

Analytic integration performed using the function **GIntegrate** of the `PolyLogTools` package (Duhr, Dulat 2019).

Mathematica Implementation

Other auxiliary functions:

- **LettersToInvariants[*expr*]** and **InvariantsToLetters[*expr*]**
- **NumericalEvaluationMathematica[*expr*, *PhaseSpacePoint*]** and **NumericalEvaluationGiNaC[*expr*, *PhaseSpacePoint*]**
- **AssignImaginaryParts[*expr*, *PhaseSpacePoint*]** $\rightarrow$ Analytic Continuation

Analytic Continuation

Need for Analytic Continuation

- Need to analytically continue the integrand of the numerical integration, i.e. the differential equation at weight n and its $x \to 0$ limit
- Types of singularities encountered:
  - $g_J^{(n\leq 2)}$ and $g_{J,0}^{(n\leq 2)}$: logarithmic and polylogarithmic branch points/cuts
  - $\partial_x \log L_a$: poles at points $x = \ell_i$
  - $\log L_a$ and LL_a (the $x \to 0$ limits of $\log L_a$) : logarithmic branch points/cuts
  - square-root branch points/cuts

Imaginary Parts for S_{ij} Invariants from $\mathcal{F}$ Symanzik Polynomial

The ordinary momentum-space representation of Feynman integrals:

$$G(\nu_1,\ldots,\nu_n) = \int \prod_{l=1}^{L} \frac{\mathrm{d}^d k_l}{i\pi^{d/2}} \prod_{j=1}^{n} \frac{1}{D_j^{\nu_j}\left(\{k\},\{p\},m_j^2\right)}$$

The Feynman parameter representation of Feynman integrals:

$$G(\nu_1,\ldots,\nu_n) = \Gamma\left(\nu - \frac{Ld}{2}\right) \prod_{j=1}^{n} \left[\int_0^\infty \mathrm{d}x_j \frac{x_j^{\nu_j-1}}{\Gamma(\nu_j)}\right] \delta\left(1 - \sum_{j=1}^{n} x_j\right) \frac{\mathcal{U}^{\nu-(L+1)d/2}}{\mathcal{F}^{\nu-Ld/2}}$$

where the polynomials $\mathcal{U}, \mathcal{F}$ are known as first and second Symanzik polynomials.

Imaginary Parts for S_{ij} Invariants from $\mathcal{F}$ Symanzik Polynomial

Since the $\mathcal{F}$ Symanzik polynomial maintains the sign of the $i0$ prescription of Feynman propagators with all original invariants (s_{ij}), assuming $s_{ij}\left(p_{1s}\right) \to s_{ij}\left(p_{1s}\right) + i\,\eta$,

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Imaginary Parts for S_{ij} Invariants from $\mathcal{F}$ Symanzik Polynomial

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$$\delta_{45} + \delta_{23}x \geq \delta_{45}x, \quad \delta_{51}x \geq 0, \quad (x-1)(\delta_{12}x - \delta_{45}) \geq 0,$$
$$x(\delta_{34} + \delta_{12}(x-1)) \geq 0, \quad \delta_{12}x^2 \geq 0, \quad \delta_{45} \geq 0,$$
$$x(\delta_{12} - \delta_{34} + \delta_{51}) \leq 0, \quad \delta_{23}x \geq x(\delta_{45} + \delta_{51})$$

Analytic Continuation of $\partial_x \log L_a$

As mentioned earlier, $\partial_x \log L_a$ have poles at points $x = \ell_i$. To control the numerical integration over the locations of these poles, we make this pole structure manifest

$$\frac{d \log L_a}{dx} \to \frac{f(x)}{\prod_i (x - \ell_i)} ,$$

where $f(x) \in \mathbb{R}$, except for factors of $\sqrt{\Delta} = i\sqrt{|\Delta|}$ for $\Delta < 0$, and we assign imaginary parts to all ℓ_i's using those we assigned to the S_{ij} invariants

$$\ell_i(S_{ij}) \to \ell_i(S_{ij} + i\,\delta_{ij}\,\eta) \equiv \ell_i + i\,\delta_i\,\eta \ .$$

Analytic Continuation of $g_J^{(n\leq 2)}$

Concerning $g_J^{(n\leq 2)}$, their logarithmic and polylogarithmic branch points/cuts appear through expressions of the form:

- $\log(x)$, $\log(\ell_a)$, $\log(1-\ell_a)$,
- $\mathcal{G}(\ell_a;x)$, $\mathcal{G}(0;x)$, $\mathcal{G}(\ell_a,\ell_b;x)$, $\mathcal{G}(0,\ell_a;x)$, $\mathcal{G}(0,0;x)$,

In some elements of the N_2 hexa-box family we also find:

- $\mathcal{G}\left(0,1;\tilde{\ell}_a(x)\right)$ and $\mathcal{G}\left(1;\tilde{\ell}_a(x)\right)$,

where $\tilde{\ell}_a(x)$ are **algebraic expressions of** x. To control the numerical integration over the locations of all branch points, we assign imaginary parts to all ℓ_i's similarly to before, and also to the $\tilde{\ell}_a(x)$'s, such that

$$\tilde{\ell}_a(\ell_i) \to \tilde{\ell}_a(\ell_i + i\,\delta_i\,\eta) \equiv \tilde{\ell}_a(\ell_i) + i\,\delta_a\,\eta\ .$$

Analytic Continuation of $g_J^{(n\leq 2)}$

$$\tilde{\ell}_a(x) \in \left\{ \frac{x(S_{12}x - S_{23} - S_{34} + 2S_{51}) + S_{45} \pm \sqrt{\Delta(1,x)}}{2x(S_{12} - S_{34} + S_{51})}, \right.$$
$$\left. \frac{S_{12}S_{51}x^2 + x(-S_{12}S_{23} + 2S_{12}S_{45} + S_{23}S_{34} - S_{34}S_{45}) + S_{45}S_{51} \pm \sqrt{\Delta(2,x)}}{2S_{45}(S_{12} - S_{34} + S_{51})} \right\}$$

Analytic Continuation of $\log L_a$

As for the logs of the alphabet letters, we isolate their logarithmic branch points in the following way:

$$\log L_a \to \log\left(L_a \frac{\prod_{i_D}(x-\ell_{i_D})}{\prod_{j_N}(x-\ell_{j_N})}\right) - \log\left(\prod_{i_D}(x-\ell_{i_D})\right) + \log\left(\prod_{j_N}(x-\ell_{j_N})\right)$$

where $\left(L_a \frac{\prod_{i_D}(x-\ell_{i_D})}{\prod_{j_N}(x-\ell_{j_N})}\right) \in \mathbb{R}$, except for factors of $\sqrt{\Delta} = i\sqrt{|\Delta|}$ for $\Delta < 0$, and has no zeroes or poles in $x \in (0, \bar{x})$, and then we assign imaginary parts in the ordinary way to all ℓ_i's.

Summary and Outlook

- Need for semi-analytic approach above transcendental weight 2 for some of the non-planar 5-point topologies $\rightarrow$ analytic continuation of the integrands
- Successful check for N_2 integrals family on a specific physical phase-space point, against the numerical results obtained from the literature (Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2024), up to weight 4
- Need to extend our Mathematica implementation to the other non-planar topologies
- Need to further generalize and automate our analytic continuation methods

Thank you for your attention!

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