Landscape of Yang Mills Theory Vacuum and Condensation of Magnetic Fluxes in QCD

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1. How large is the space of covariantly constant gauge fields
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2. Covariantly constant Yang Mills vacuum fields and condensation of magnetic fluxes
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3. Landscape of QCD Vacuum
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How large is the class of covariantly constant gauge fields

how large is the class of covariantly constant gauge fields defined by the equation

$$\nabla^{ab}_{\rho} G^{b}_{\mu\nu} = 0.$$

$\nabla^{ab}_{\mu} G^{b}_{\mu\nu} = 0.$ By taking covariant derivative ∇^{ca}_{λ} $\quad [\nabla_{\lambda}, \nabla_{\rho}]^{ab} G^{b}_{\mu\nu} = 0.$

$$[G_{\lambda\rho}, G_{\mu\nu}] = 0.$$

the field strength tensor factorises into the product of Lorentz tensor $G_{\mu\nu}(x)$ and colour unit vector $n^a(x)$,

$$G^{a}_{\mu\nu}(x) = G_{\mu\nu}(x) n^{a}(x).$$

The solution has the following form

$$A^{a}_{\mu} = -\frac{1}{2} F_{\mu\nu} x_{\nu} n^{a},$$

Let us consider the Cho Ansatz

$$A^a_\mu = B_\mu n^a + \frac{1}{g}\varepsilon^{abc} n^b \partial_\mu n^c,$$

$$n^a n^a = 1,\ n^a \partial_\mu n^a = 0.$$

$$\nabla^{ab}_\mu n^b = \partial_\mu n^a - g\varepsilon^{abc} A^b_\mu n^c = 0,$$

and therefore $[\nabla_\mu, \nabla_\nu]^{ab} n^b = 0.$ $\qquad [\nabla_\mu, \nabla_\nu]^{ab} n^b = -g\varepsilon^{acb} G^c_{\mu\nu} n^b = 0.$

It follows that the field strength tensor factorises

$$G^a_{\mu\nu} = (F_{\mu\nu} + \frac{1}{g} S_{\mu\nu})\ n^a,$$

Cho Ansatz

It follows that the field strength tensor factorises

$$G^a_{\mu\nu} = (F_{\mu\nu} + \frac{1}{g}S_{\mu\nu})\ n^a,$$

where

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \qquad S_{\mu\nu} = \varepsilon^{abc} n^a \partial_\mu n^b \partial_\nu n^c.$$

$$n^a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

$$S_{\mu\nu} = \sin\theta(\partial_\mu\theta\partial_\nu\phi - \partial_\nu\theta\partial_\mu\phi).$$

In both cases the field strength tensor factorises !

The important property of the ansatz is that the field strength tensor factorises into the Lorentz and colour structures. This factorisation is identical to the factorisation of covariantly constant gauge fields. It is therefore natural to search solutions in this form.

$$G^a_{\mu\nu}(x) = G_{\mu\nu}(x) n^a(x).$$

$$G^a_{\mu\nu} = \left(F_{\mu\nu} + \frac{1}{g} S_{\mu\nu}\right) n^a,$$

Covariantly constant gauge fields should fulfil the following equation

$$n^a \partial_\rho (F_{\mu\nu} + \frac{1}{g} S_{\mu\nu}) = 0,$$

meaning that the sum of terms in the brackets should be a constant tensor:

$$G_{\mu\nu} = F_{\mu\nu} + \frac{1}{g} S_{\mu\nu},$$

One can guess that the moduli space of covariantly constant gauge fields is much larger and can be obtained by solving the following system of partial differential equations:

$$S_{12} = \sin\theta(\partial_1\theta\partial_2\phi - \partial_2\theta\partial_1\phi)$$

$$S_{23} = \sin\theta(\partial_2\theta\partial_3\phi - \partial_3\theta\partial_2\phi)$$

$$S_{13} = \sin\theta(\partial_1\theta\partial_3\phi - \partial_3\theta\partial_1\phi),$$

General solution

$$X = a_1 x + a_2 y + a_3 z + a_0 t, \qquad Y = b_1 x + b_2 y + b_3 z + b_0 t,$$

$$S_{ij} = a_i \wedge b_j \sin\theta(X)\ \theta(X)'_X\ \phi(Y)'_Y,$$

solutions with a constant tensor S_{ij}, then the following condition should be fulfilled:

$$\sin\theta(X)\ \theta(X)'_X\ \phi(Y)'_Y = 1,$$

so that

$$S_{ij} = a_i \wedge b_j.$$

$$n^a(\vec{x}) = \{\sin f(X) \cos\left(\frac{Y}{f'(X)\sin(f(X))}\right),\ \sin(f(X))\sin\left(\frac{Y}{f'(X)\sin(f(X))}\right),\ \cos(f(X))\}.$$

General solution

$$A^a_\mu = B_\mu(x) n^a(x) + \frac{1}{g} \varepsilon^{abc} n^b(x) \partial_\mu n^c(x).$$

$$B_\mu = -\frac{1}{2} F_{\mu\nu} x_\nu$$

$$n^a(\vec{x}) = \{\sin f(X) \cos\left(\frac{Y}{f'(X)\sin(f(X))}\right), \ \sin(f(X)) \sin\left(\frac{Y}{f'(X)\sin(f(X))}\right), \ \cos(f(X))\}.$$

$$X = a_1 x + a_2 y + a_3 z + a_0 t, \qquad Y = b_1 x + b_2 y + b_3 z + b_0 t,$$

We conclude that the moduli space of covariantly constant gauge fields is infinite-dimensional because of the presence of an arbitrary function $f(X)$.

In comparison, the moduli space $\mathcal{I}_{k,N}$ of the YM self-duality equation in the Euclidean space has the dimension $dim \mathcal{I}_{k,N} = 4kN$ for the $SU(N)$ group.

Properties of General solution

The square of the field strength tensor is

$$\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} = \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{a_\mu F_{\mu\nu} b_\nu}{g} + \frac{a^2 b^2 - (a\cdot b)^2}{2g^2}.$$

where a_μ and b_ν are arbitrary constant Lorentz vectors

The magnetic energy density can be represented in the following form:

$$\epsilon = \frac{\vec{H}^2}{2} - \frac{1}{g}\vec{H}\cdot(\vec{a}\times\vec{b}) + \frac{1}{2g^2}(\vec{a}\times\vec{b})^2.$$

Let us consider the solution when $B_\mu = F_{\mu\nu} = 0$, so that

$$A^a_\mu = \frac{1}{g}\varepsilon^{abc} n^b \partial_\mu n^c, \qquad\qquad \epsilon = \frac{1}{2g^2}(\vec{a}\times\vec{b})^2.$$

Properties of General solution

Let us considering the vectors $a_\mu = (0, a, 0, 0)$ and $b_\nu = (0, 0, b, 0)$, so that $\theta(x) = f(ax)$, $\phi(x, y) = by/f^{'}(ax) \sin f(ax)$. The gauge field will take the following form:

$$
A^a_\mu(x,y) = \frac{1}{g}
\left\{
\begin{array}{c}
(0,0,0) \\
a\Big(by\frac{\cos^2 f}{\sin f}\cos(\frac{by}{f'\sin f}) - f^{'}\sin(\frac{by}{f'\sin f}) + by\frac{f^{''}}{f'^2}\cos(f)\cos(\frac{by}{f'\sin f}), \\
by\frac{\cos^2}{\sin f}\sin(\frac{by}{f'\sin f}) + f^{'}\cos(\frac{by}{f'\sin f}) + by\frac{f^{''}}{f'^2}\cos(f)\sin(\frac{by}{f'\sin f}), \\
-by(\cos(f) + \frac{f^{''}}{f'^2}\sin(f))\Big) \\
\frac{b}{f'}\Big(-\cos(f)\cos(\frac{by}{f'\sin f}), -\cos(f)\sin(\frac{by}{f'\sin f}),\ \sin f\Big), \\
(0,0,0)
\end{array}
\right.
$$

where the derivatives are over the whole argument ax. One can verify explicitly that it is a solution of the Yang Mills equation.

the energy density of the chromomagnetic field is a space time constant

$$\epsilon = \frac{1}{4} G^a_{ij} G^a_{ij} = \frac{a^2 b^2}{2g^2}.$$

Properties of General solution

The non-vanishing components of the conserved current $J^a_\mu = g\epsilon^{abc} A^b_\nu G^c_{\nu\mu}$ are[5]

$$J^a_1 = \frac{ab^2}{gf'}\Big(\sin(\frac{by}{\sin f}), -\cos(\frac{by}{\sin f}), 0\Big);$$

$$J^1_2 = \frac{a^2 b}{g}\Big(f' \cos f \cos(\frac{by}{\sin f}) + by \cot f \sin(\frac{by}{\sin f}) + by \frac{f''}{f'^2} \sin(\frac{bz}{\sin f})\Big),$$

$$J^2_2 = \frac{a^2 b}{g}\Big(f' \cos f \sin(\frac{by}{\sin f}) - by \cot f \cos(\frac{by}{\sin f}) - by \frac{f''}{f'^2} \cos(\frac{by}{\sin f})\Big),$$

$$J^3_2 = -\frac{a^2 b}{g} f' \sin f.$$

$$\partial_\mu J^a_\mu = \partial_x J^a_1 + \partial_y J^a_2 = 0.$$

[5] This current is conserved on the solutions of the Yang Mills equation $\nabla^{ab}_\mu G^b_{\mu\nu} = 0$.

Properties of General solution

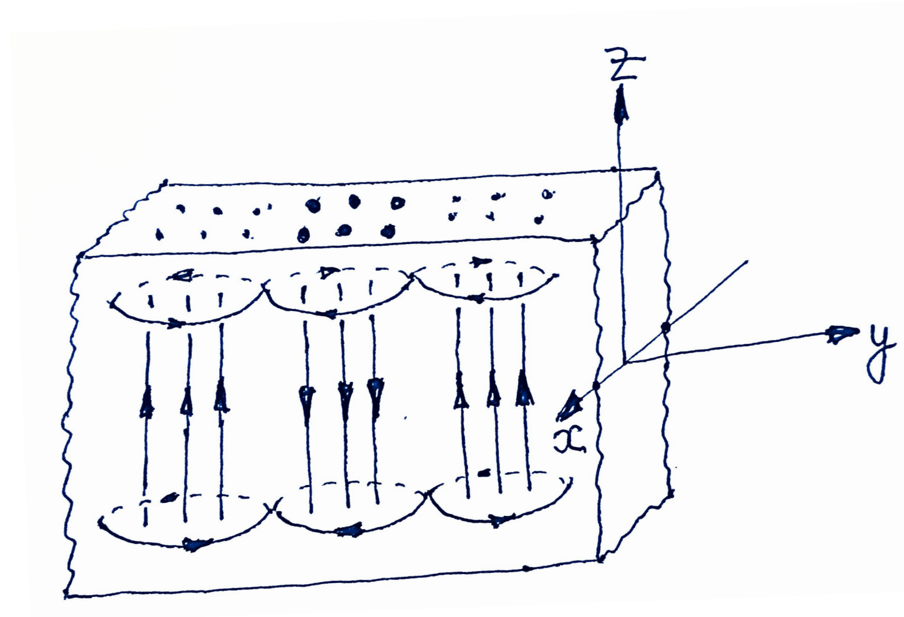


Figure 1: The figure demonstrates a finite part of an infinite sheet of finite thickness $\frac{2}{a}$ in the direction of the x axis of the solution [illegible]. It is filled by parallel chromomagnetic flux tubes. Each tube of the square area $\frac{2}{a}\frac{\pi}{b}$ carries the magnetic flux $\frac{2\pi}{g}$. The circuits show the flow of the conserved current $J^a_\mu = g\epsilon^{abc}A^b_\nu G^c_{\nu\mu}$.

the solution describes a condensate of superposed Nielsen-Olesen vortices of opposite magnetic fluxes and is a dual analog of the Cooper pairs condensate in a superconductor.

Flux Lines in a Superconductor

Properties of General solution — Magnetic Fluxes

The conserved topological current and the corresponding magnetic charge can be defined in terms of the Abelian field strength $G_{\mu\nu}$

$$K_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\partial_\nu G_{\lambda\rho} = \frac{1}{2g}\epsilon_{\mu\nu\lambda\rho}\partial_\nu S_{\lambda\rho}, \qquad \partial_\mu K_\mu = 0, \qquad Q_m = \int_V K_0 d^3x,$$

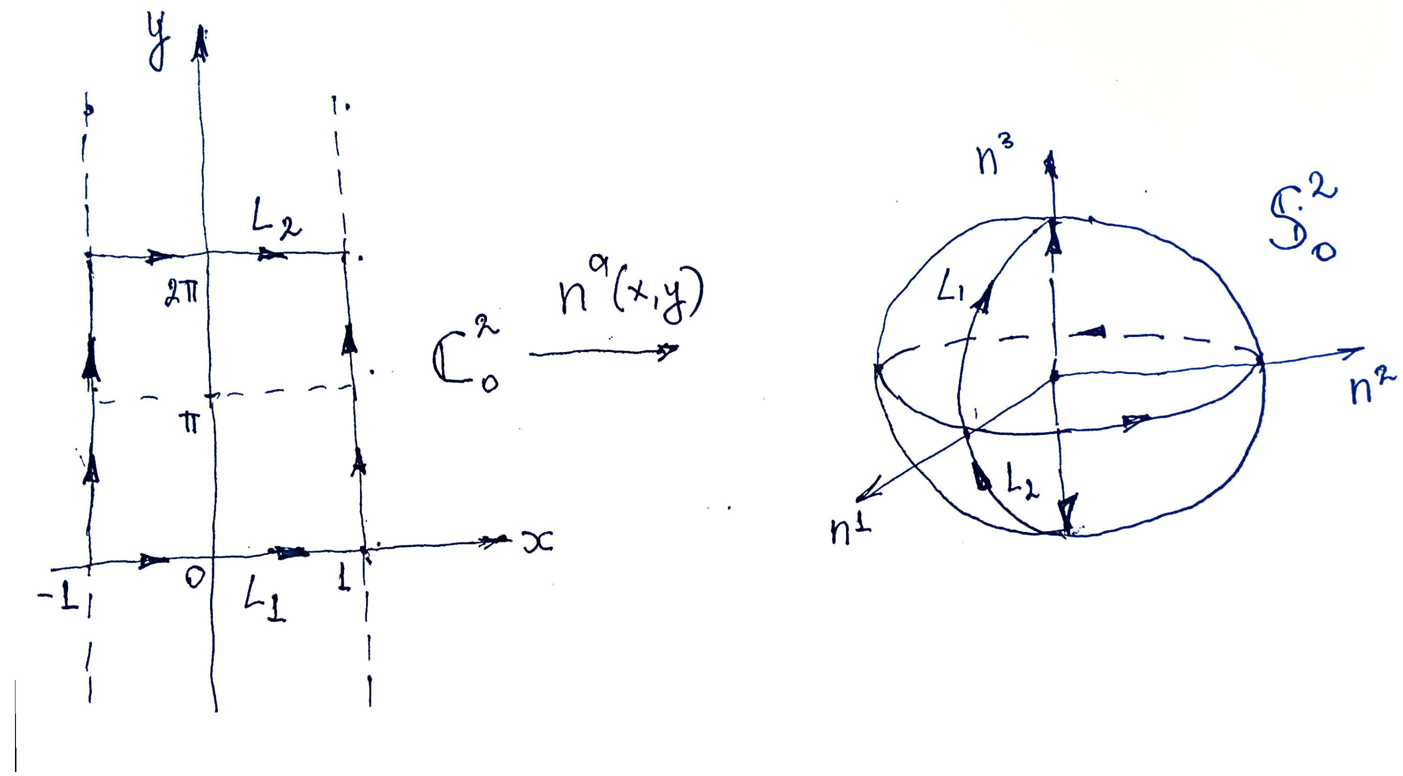
where $F_{\mu\nu} = 0$. In terms of the tensor $S_{\mu\nu}$ and of the colour unit vector $n^a(x,y)$, the topological charge will take the following equivalent forms:

$$K_0 = \frac{1}{2g}\epsilon_{ijk}\partial_i S_{jk} = \frac{1}{2g}\epsilon_{ijk}\partial_i(\epsilon^{abc}n^a\partial_j n^b\partial_k n^c),$$

$$Q_m = \frac{1}{2g}\int_V \epsilon_{ijk}\epsilon^{abc}\partial_i n^a\partial_j n^b\partial_k n^c d^3x = \frac{1}{2g}\int_{\partial V}\epsilon_{ijk}\epsilon^{abc}n^a\partial_j n^b\partial_k n^c d\sigma_i = \frac{1}{2g}\int_{\partial V} d\sigma_i\ \epsilon_{ijk}S_{jk}.$$

As far as the solution is homogeneous in z direction, we have to consider a topological charge within the space volume V that is a rectangular box with its two boundaries being parallel to the (x,y) plane at the distance L from each other and the other four boundaries will be defined for each particular solution individually.

Properties of General solution - Magnetic Fluxes



we can define the invariant magnetic flux in terms of the surface

$$q_m = \frac{1}{g}\int_{(x,y,0)} ab\ dxdy.$$

$$g_m(k) = \frac{1}{g}\int_{-\frac{1}{a}}^{\frac{1}{a}}\ dax \int_{\frac{2\pi}{b}k}^{\frac{2\pi}{b}(k+1)} dby = \frac{4\pi}{g}.$$

Properties of General solution — Magnetic Fluxes

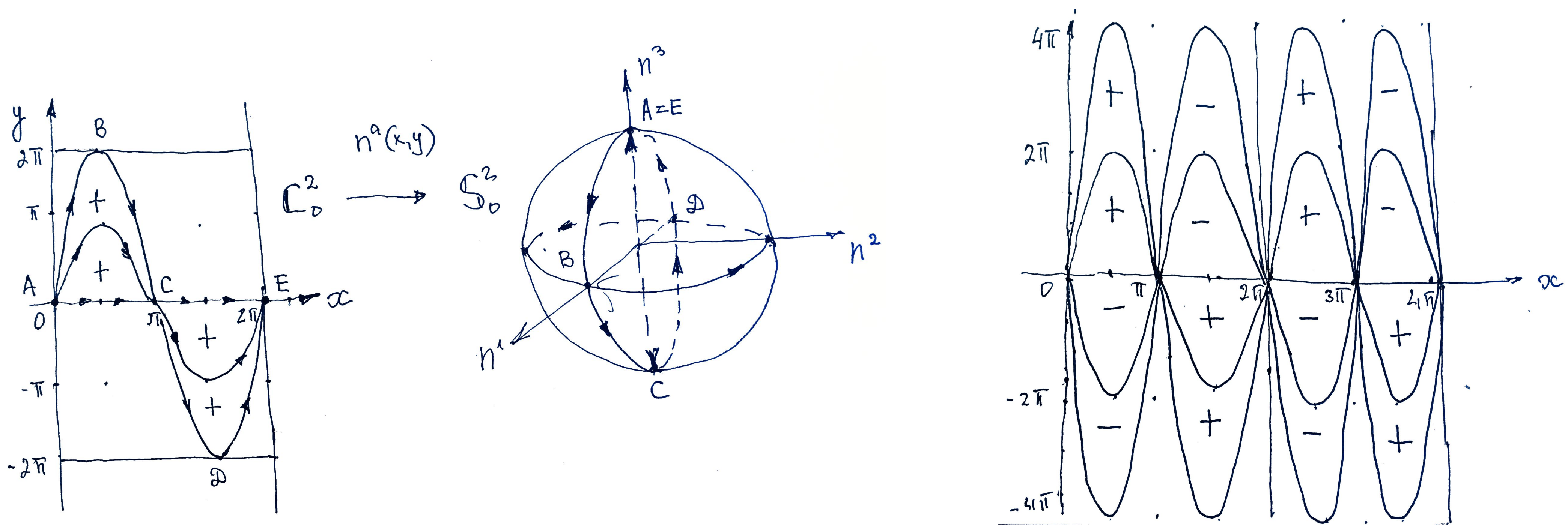


Figure 3: The l.h.s figure shows the mapping defined by the vector $n^a(x, y) = \{\sin x \cos(y/\sin x),\ \sin x \sin(y/\sin x),\ \cos x\}$ from the cylinder cells C_k^2 to the spheres S_k^2. The boundaries of the cylinders are defined by the equation $y = 0, \pm\alpha \sin x$. The positive topological charges have the mapping of the cylinders $y = 0, \alpha \sin x$, $\alpha \in [0, 2\pi k]$: $(n(0, \alpha) = (0, 0, 1)$, $n(\pi, \alpha) = (0, 0, -1)$, $n(\pi/2, \alpha) = (\cos\alpha, \sin\alpha, 0)$, $n(3\pi/2, \alpha) = (-\cos\alpha, -\sin\alpha, 0)$, $n(2\pi, \alpha) = (0, 0, 1))$. The negative topological charges have the mapping of the cylinders $y = 0, -\alpha \sin x$, $\alpha \in [0, 2\pi k])$. The part of the full structure is shown on the r.h.s of the figure and reminds the Abrikosov lattice of parallel Nielsen-Olesen magnetic vortices that are normal to the plane (x, y) and have alternating magnetic charges.

Landscape of Yang Mills theory vacuum

$$\hat{n}' = U^{-}\hat{n}U, \qquad \hat{n} = n^a\sigma^a.$$

The $SU(2)$ matrix of the corresponding singular gauge transformation has the following form:

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \cos(\frac{f}{2})e^{\frac{i}{2}(\frac{\pi}{2}-\frac{by}{f'\sin f})} & i\sin(\frac{f}{2})e^{\frac{i}{2}(\frac{\pi}{2}-\frac{by}{f'\sin f})} \\ i\sin(\frac{f}{2})e^{-\frac{i}{2}(\frac{\pi}{2}-\frac{by}{f'\sin f})} & \cos(\frac{f}{2})e^{-\frac{i}{2}(\frac{\pi}{2}-\frac{by}{f'\sin f})} \end{pmatrix}.$$

$$A_1'^3 = -\frac{ab}{g}y, \qquad G_{12}'^3 = \frac{ab}{g}, \qquad \epsilon = \frac{a^2b^2}{2g^2}.$$

$$\begin{aligned} \hat{A}_0^a &= w(\tfrac{1}{2}-\alpha)A_0^a \\ \hat{A}_1^a &= w(\tfrac{1}{2}-\alpha)A_1^a \quad + w(\tfrac{1}{2}+\alpha)A_1'^a \\ \hat{A}_2^a &= w(\tfrac{1}{2}-\alpha)A_2^a \\ \hat{A}_3^a &= w(\tfrac{1}{2}-\alpha)A_3^a \quad + w(\tfrac{1}{2}+\alpha)A_3'^a. \end{aligned}$$

$$\epsilon(x,\alpha) = \frac{a^2b^2}{2g^2}\Big((2-w_-)^2w_-^2 + w_+^2 + 2(2-w_-)w_-(1+w_-)w_+\cos f(ax) + \frac{w_-^2w_+^2}{\sin^2 f(ax)}\Big),$$

where $w_- \equiv w(\frac{1}{2}-\alpha)$ and $w_+ \equiv w(\frac{1}{2}+\alpha)$.

Landscape of Yang Mills theory vacuum

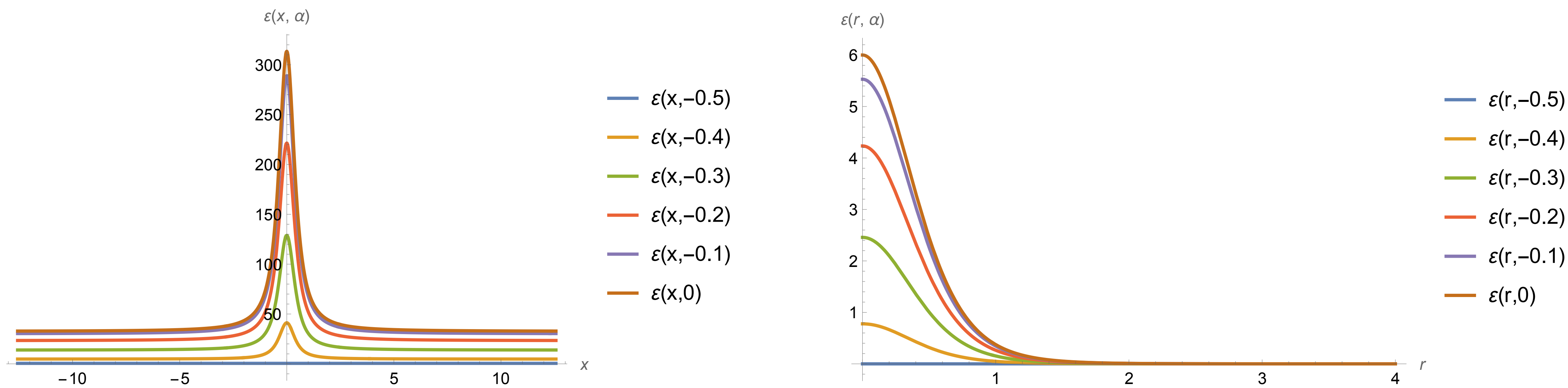


Figure 4: The l.h.s. graph shows the shape of the barrier $\epsilon(x,\alpha)$ when α parameter changes in the interval $[-\frac{1}{2},0]$. At $\alpha=-\frac{1}{2}$ the energy density is equal to $\epsilon=1/2$ $(a=b=g=1)$. As α increases, the hight of the barrier increases and reaches its maximum at $\alpha=0$, then it symmetrically decreases until $\alpha=\frac{1}{2}$, where it again is equal to $\epsilon=1/2$. The r.h.s graph shows the shape of the potential barrier between the Chern-Pontryagin vacua

$$\vec{A}_n(\vec{x})=\frac{i}{g}U_n^-(\vec{x})\nabla U_n(\vec{x}),\qquad U_1(\vec{x})=\frac{\vec{x}^2-\lambda^2-2i\lambda\vec{\sigma}\vec{x}}{\vec{x}^2+\lambda^2},\qquad U_n=U_1^n.$$

$$\epsilon(r,\alpha)=\frac{1}{4}G^a_{ij}G^a_{ij}=\frac{6\lambda^4(1-4\alpha^2)}{g^2(r^2+\lambda^2)^4}$$

Landscape of Yang Mills theory vacuum

The existence of an even larger class of covariantly constant gauge fields described above pointed out to the fact that the Yang-Mills vacuum has even higher degeneracy of vacuum field configutations. Each covariantly constant gauge field configuration on its own contains a rich diversity of emergent nonperturbative structures, and it is a challenging problem to investigate possible tunneling transitions between these highly degenerate states and to calculate the vacuum polarisation induced by the new class of covariantly constant gauge fields.

Comparison with the 't Hooft Polyakov monopole solution

The electromagnetic field strength is defined by 't Hooft as

$$G_{\mu\nu} = n^a G^a_{\mu\nu} + \frac{1}{g}\epsilon^{abc} n^a \nabla_\mu n^b \nabla_\nu n^c \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + \underline{\frac{1}{g}\epsilon^{abc} n^a \partial_\mu n^b \partial_\nu n^c}, \qquad n^a = \frac{\phi^a}{|\phi|}, \qquad (1.1)$$

where $\nabla_\mu n^a = \partial_\mu n^a - g\epsilon^{abc} A^b_\mu n^c$, $A_\mu = A^a_\mu n^a$ and n^a is a unit colour vector. It reduces to $G_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$ in the space regions where the scalar field is in the third direction $n_a = (0,0,1)$ and the Abelian field A_μ does not have Dirac string singularities.

the expression of the topologically conserved current is

$$K_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\partial_\nu G_{\lambda\rho} = \frac{1}{2g}\epsilon_{\mu\nu\lambda\rho}\underline{\epsilon^{abc}\partial_\nu n^a \partial_\lambda n^b \partial_\rho n^c}\ , \qquad \partial_\mu K_\mu = 0.$$

The 't Hooft-Polyakov solution has the following form:

$$\phi^a = u(r) n^a, \qquad A^a_i = \epsilon^{aij} n^j a(r)$$

and has the following asymptotic properties

$$u(0) = 0, \qquad a(0) = 0, \quad u(r) \underset{r\to\infty}{\rightarrow} \frac{m}{\lambda}, \qquad a(r) \underset{r\to\infty}{\rightarrow} -\frac{1}{gr}$$

The scalar field ϕ^a vanishes at $x^a = 0$ and the corresponding topological density $K_0(x)$ vanishes everywhere expect for $x^a = 0$ where it has singularity $K_0 = \frac{4\pi}{g}\delta^3(\vec{x})$, which contributes to the topological charge and is equal to the winding number of the map $n^a(x)$:

$$g_m = \int_{R^3} d^3x K_0 = \frac{1}{2g}\int_{S^2} d^2\sigma_i \underline{\epsilon_{ijk}\epsilon^{abc}n^a\partial_j n^b \partial_k n^c} = \frac{4\pi}{g}.$$

induces a magnetic flux of a single monopole:

$$H_i = \frac{x_i}{gr^3}, \qquad g_m = \int H_i dS_i = \frac{4\pi}{g}.$$

Thank you

Korean Japchae prepared by me !