Critical Dynamics - Theory

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Tutorial Workshop for High Energy Heavy-Ion Collisions

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Main messages

1. Location and existence of QCD critical point is still unclear
2. QCD critical point $\sim$ Ising model
   - Why baryon number fluctuations?
   - Why higher order cumulants?
3. Non-equilibrium condition hinders critical fluctuation to equilibrate
   - No divergence even when the system passes the critical point
   - Need to be considered in experimental critical point search

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Lattice QCD has the sign problem at finite μ

Path integral for QCD partition function

$$Z(\beta,\mu) = e^{P(T,\mu)V/T} = \mathrm{Tr}\, e^{-\beta(H-\mu N)}$$
$$= \int \mathcal{D}U \, \underbrace{\det[\not{D}(U) + m_q - \mu\gamma_4]}_{\text{complex "probability" at } \mu \neq 0} \, e^{S_E[U]}$$

Monte Carlo method fails → several attempts

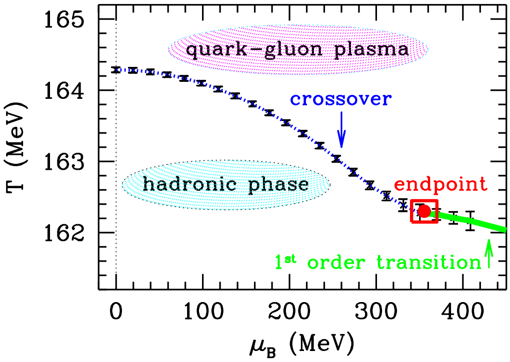
- Taylor expansion
- Canonical approach
- Reweighting method
- Imaginary chemical potential
- Lefschetz thimble
- Path optimization
- Complex Langevin method

⋮

QCD phase diagram by reweighting method

Phase quenching reweighting method

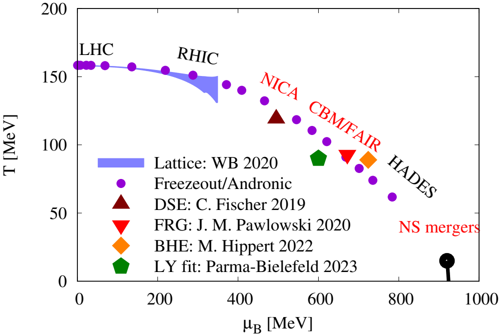
$$\langle O\rangle = \frac{\int dx O(x) P(x)}{\int dx P(x)} = \frac{\int dx O(x) e^{i\theta(x)}|P(x)|}{\int dx |P(x)|} \Big/ \underbrace{\frac{\int dx e^{i\theta(x)}|P(x)|}{\int dx |P(x)|}}_{\sim 0 \text{ at large } V,\beta,\mu}$$



[Fodor-Katz, JHEP04(2004)050]

Several approaches presented at CPOD2024

- Lattice QCD with Taylor expansions ←
- Lee-Yang edge singularities ←
- Dyson-Schwinger equations
- Functional renormalization group
- Black hole engineering ←



[Borsanyi @CPOD2024]

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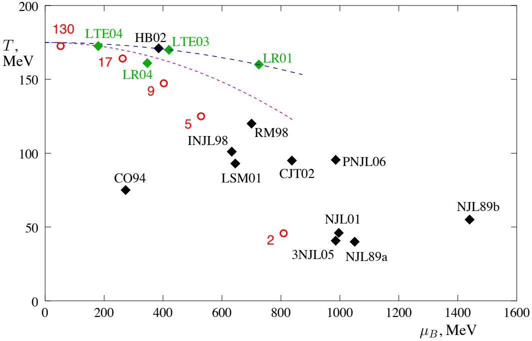


[Stephanov @Lattice2006]

Lattice QCD with Taylor expansions

Basic idea: Expand $P(T,\mu)$ in terms of μ

$$\frac{P(T,\mu)}{T^4} = \frac{P(T,0)}{T^4} + \frac{1}{2!}\frac{\mu^2}{T^2}\chi_2(T) + \frac{1}{4!}\frac{\mu^4}{T^4}\chi_4(T) + \cdots$$

$$e^{P(T,\mu)V/T} = \int \mathcal{D}U \, \det[\not{D}(U) + m_q - \mu\gamma_4] e^{S_E[U]}$$

Lattice calculation of χ_{2n} involves expansion of $\det(\cdots)$

$$\chi_{200}^{uds} = +\langle B_u\rangle + \langle A_u^2\rangle - \langle A_u\rangle^2 \tag{B.5}$$

$$\chi_{110}^{uds} = +\langle A_u^2\rangle - \langle A_u\rangle^2 \tag{B.6}$$

$$\chi_{101}^{uds} = +\langle A_u A_s\rangle - \langle A_s\rangle\langle A_u\rangle \tag{B.7}$$

$$\chi_{300}^{uds} = +\langle C_u\rangle + 3\langle A_u B_u\rangle + \langle A_u^3\rangle - 3\langle B_u\rangle\langle A_u\rangle - 3\langle A_u\rangle\langle A_u^2\rangle + 2\langle A_u\rangle^3 \tag{B.8}$$

$$\chi_{210}^{uds} = +\langle A_u B_u\rangle + \langle A_u^3\rangle - \langle B_u\rangle\langle A_u\rangle - 3\langle A_u\rangle\langle A_u^3\rangle + 2\langle A_u\rangle^3 \tag{B.9}$$

$$\chi_{120}^{uds} = +\langle A_u B_u\rangle + \langle A_u^3\rangle - \langle B_u\rangle\langle A_u\rangle - 3\langle A_u\rangle\langle A_u^2\rangle + 2\langle A_u\rangle^3 \tag{B.10}$$

$$\chi_{111}^{uds} = +\langle A_u A_u A_s\rangle - \langle A_s\rangle\langle A_u^2\rangle - 2\langle A_u\rangle\langle A_u A_s\rangle + 2\langle A_s\rangle\langle A_u\rangle^2 \tag{B.11}$$

$$\begin{aligned}\chi_{400}^{uds} = &+\langle D_u\rangle + 3\langle B_u B_u\rangle + 4\langle A_u C_u\rangle + 6\langle A_u^2 B_u\rangle + \langle A_u^4\rangle \\ &- 4\langle C_u\rangle\langle A_u\rangle - 3\langle B_u\rangle^2 - 6\langle B_u\rangle\langle A_u^2\rangle - 12\langle A_u\rangle\langle A_u B_u\rangle \\ &- 4\langle A_u\rangle\langle A_u^3\rangle - 3\langle A_u A_u\rangle\langle A_u^2\rangle + 12\langle B_u\rangle\langle A_u\rangle^2 \\ &+ 12\langle A_u\rangle^2\rangle\langle A_u^2\rangle - 6\langle A_u\rangle^4\end{aligned} \tag{B.12}$$

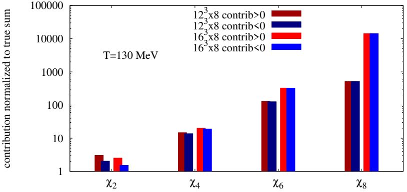
$$\begin{aligned}\chi_{310}^{uds} = &+\langle A_u C_u\rangle + 3\langle A_u^2 B_u\rangle + \langle A_u^4\rangle - \langle C_u\rangle\langle A_u\rangle - 3\langle B_u\rangle\langle A_u^2\rangle \\ &- 6\langle A_u\rangle\langle A_u B_u\rangle - 4\langle A_u\rangle\langle A_u^3\rangle - 3\langle A_u^2\rangle\langle A_u^2\rangle \\ &+ 6\langle B_u\rangle\langle A_u\rangle^2 + 12\langle A_u\rangle\langle A_u\rangle\langle A_u^2\rangle - 6\langle A_u\rangle^4\end{aligned} \tag{B.13}$$

$$\begin{aligned}\chi_{220}^{uds} = &+\langle B_u^2\rangle + 2\langle A_u^2 B_u\rangle + \langle A_u^4\rangle - \langle B_u\rangle^2 - 2\langle B_u\rangle\langle A_u^2\rangle \\ &- 4\langle A_u\rangle\langle A_u B_u\rangle - 4\langle A_u\rangle\langle A_u^3\rangle - 3\langle A_u^2\rangle\langle A_u^2\rangle \\ &+ 4\langle B_u\rangle\langle A_u\rangle\langle A_u\rangle + 12\langle A_u\rangle\langle A_u\rangle\langle A_u^2\rangle - 6\langle A_u\rangle^4\end{aligned} \tag{B.14}$$

$$\begin{aligned}\chi_{211}^{uds} = &+\langle A_u B_u A_s\rangle + \langle A_u^3 A_s\rangle - \langle A_s\rangle\langle A_u B_u\rangle - \langle A_s\rangle\langle A_u^3\rangle - \langle B_u\rangle\langle A_u A_s\rangle - \langle B_u A_s\rangle\langle A_u\rangle \\ &- 3\langle A_u\rangle\langle A_u^2 A_s\rangle - 3\langle A_u A_s\rangle\langle A_u^2\rangle + 2\langle A_s\rangle\langle B_u\rangle\langle A_u\rangle + 6\langle A_s\rangle\langle A_u\rangle\langle A_u^2\rangle \\ &+ 6\langle A_u\rangle^2\langle A_u A_s\rangle - 6\langle A_s\rangle\langle A_u\rangle^3\end{aligned} \tag{B.15}$$

[Borsanyi et al, JHEP10(2018)205]

Large cancellation in χ_{2n}

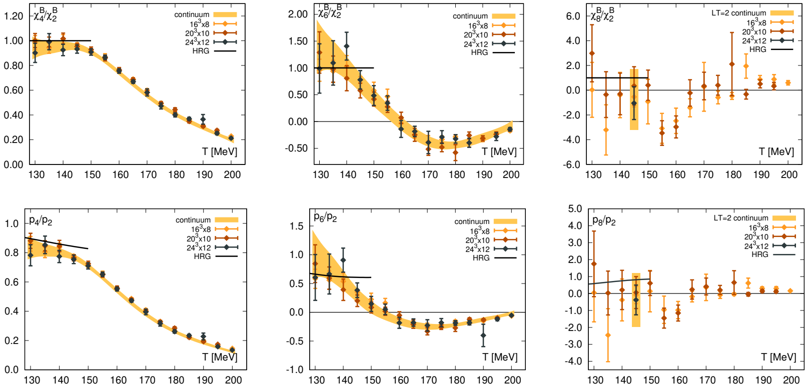


[Borsanyi @CPOD2024]

Taylor coefficients up to μ^8 (1/2)

Budapest-Wuppertal: First results of $\chi_{6,8}$ with continuum extrapolation

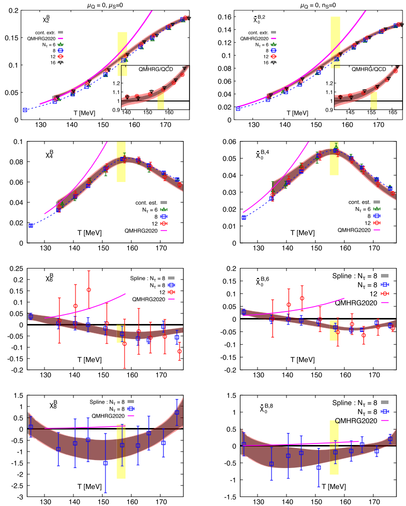
upper (lower) panels: $\mu_S = 0$ ($n_S = 0$)



[Borsanyi et al, 2312.07528]

Taylor coefficients up to μ^8 (2/2)

Hot QCD Collaboration: Continuum extrapolation up to χ_4?



[Bollweg et al, PRD105(2022)074511]

What can we learn from χ_{2-8}? – Padé approximation (1/2)

1. Taylor series up to μ^8

$$\frac{P(\mu)-P(0)}{T^4} = P_2\hat{\mu}^2 + P_4\hat{\mu}^4 + P_6\hat{\mu}^6 + P_8\hat{\mu}^8, \quad \hat{\mu} = \frac{\mu}{T}$$

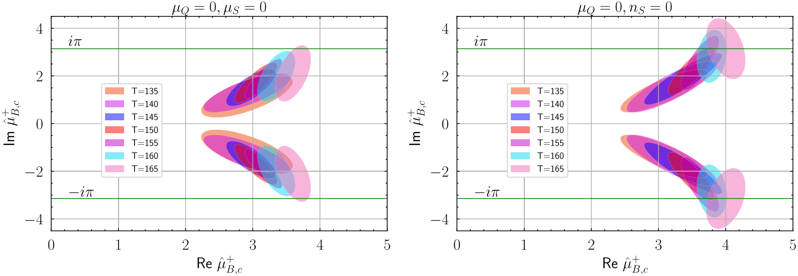
$$\frac{P_4 \Delta P(\mu)}{P_2^2 T^4} = \bar{x}^2 + \bar{x}^4 + c_6\bar{x}^6 + c_8\bar{x}^8, \quad \bar{x} = \sqrt{\frac{P_4}{P_2}}\hat{\mu}$$

2. [4,4] Padé approximation

$$\frac{P_4 \Delta P(\mu)}{P_2^2 T^4} \simeq \underbrace{\frac{(1-c_6)\bar{x}^2 + (1-2c_6+c_8)\bar{x}^4}{(1-c_6)+(c_8-c_6)\bar{x}^2+(c_6^2-c_8)\bar{x}^4}}_{\text{4 poles in complex } \bar{x} \text{ plane}} = P[4,4]$$

What can we learn from χ_{2-8}? – Padé approximation (2/2)

3. Poles in complex $\hat{\mu}$ plane (nearest to the origin)



[Bollweg et al, PRD105(2022)074511]

4. Critical point is unlikely to exist in $135 \leq T \leq 165$ MeV
   - Because poles are away from real $\hat{\mu}$
   - It may exist below $T = 135$ MeV
   - Conformal Padé approach can be more quantitative [Basar, 2112.06952]

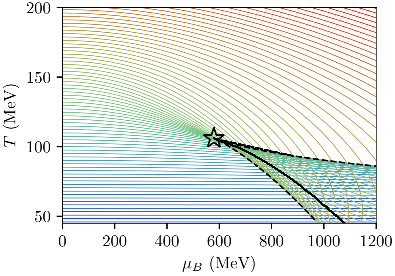
What can we learn from χ_{2-8}? – Black hole engineering (1/2)

1. Black hole solutions of Einstein-Maxwell-Dilaton equations

$$S = \frac{1}{2\kappa_5}\int_{\mathcal{M}_5} d^5x\sqrt{-g}\Big[R - \frac{(\partial\phi)^2}{2} - \underbrace{V(\phi)}_{s^{\mathrm{lat}}} - \frac{F_{\mu\nu}^2}{4}\underbrace{f(\phi)}_{\chi_2^{\mathrm{lat}}}\Big]$$

- Determine $V(\phi)$ and $f(\phi)$ by fitting $s^{\mathrm{lat}}(T,\mu=0)$ and $\chi_2^{\mathrm{lat}}(T,\mu=0)$
- Introduce dilaton ϕ for non-conformal systems

2. Calculate the phase diagram for a particular $V(\phi)$ and $f(\phi)$

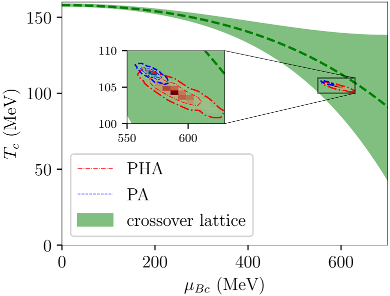


[Hippert et al, 2309.00579]

What can we learn from χ_{2-8}? – Black hole engineering (2/2)

3. Systematics with Bayesian inference

$$\underbrace{P(V,f|s,\chi_2)}_{\text{posteriori}} P(s,\chi_2) = \underbrace{P(s,\chi_2|V,f)}_{\text{likelihood}} \underbrace{P(V,f)}_{\text{prior}}$$



[Hippert et al, 2309.00579]

- PHA and PA: parametrizations of $V(\phi)$ and $f(\phi)$ with ~ 10 parameters
- No critical point in 20% of prior samples
- Predicts a critical point at $(T_c, \mu_c) \sim (105\text{ MeV}, 580\text{ MeV})$

Contents

Critical point search in heavy-ion collisions

Obstacles

- System size is finite $\leftarrow$
- System lifetime is finite
- System is non-equilibrium

Goal of this section

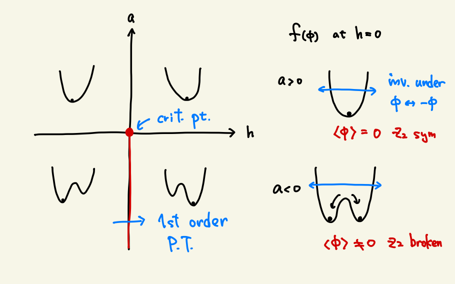
- Basics of the QCD critical point
- Why baryon number fluctuations?
- Why higher order fluctuations?

Phase transition in the mean field approximation

1. Double-well potential

$$f(\phi) = \frac{1}{2}a\phi^2 + \frac{1}{4}b\phi^4 - h\phi, \quad b > 0$$

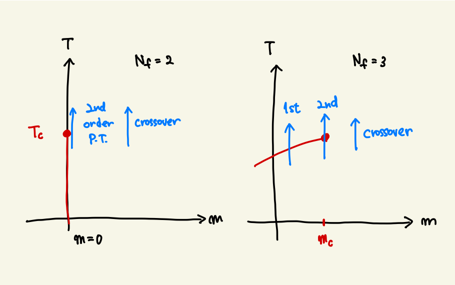
2. Phase diagram and symmetry breaking

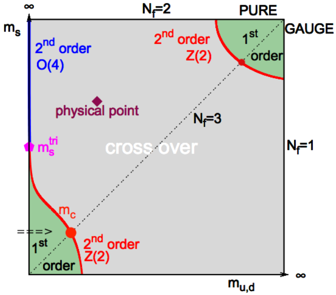


Lesson: identify the symmetry (Z_2) and find the order parameter (ϕ)

QCD phase transition in the mean field approximation

1. Symmetry for massless quarks: $\mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R$
2. Order parameter: $N_f \times N_f$ matrix $\Sigma \to$ construct $f(\Sigma)$
3. Columbia plot: Order of phase transition at finite T and $\mu = 0$

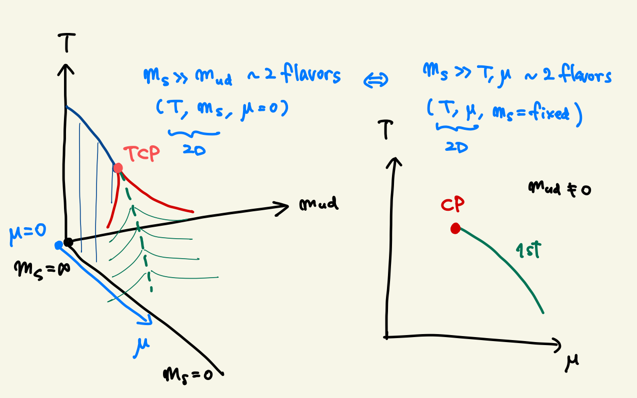




[de Forcrand and D'Elia (2017)]

QCD critical point in the mean field approximation

1. Columbia plot indicates its existence

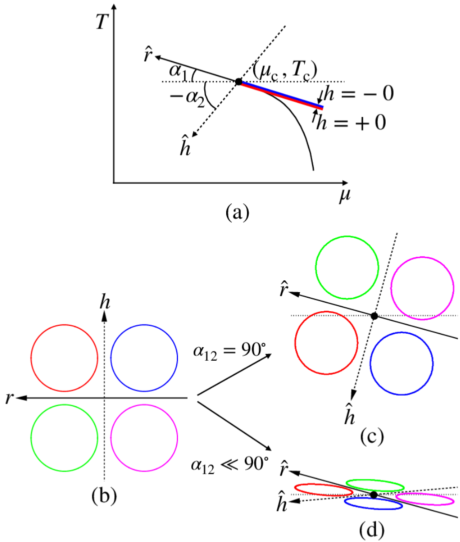


2. Z_2 universality class: same with Ising model and liquid-gas transition

We can use the critical exponents of Ising model for *static* observables

Mapping Ising model on QCD phase diagram

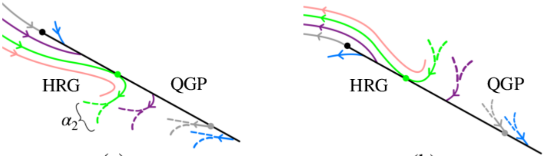
Mapping



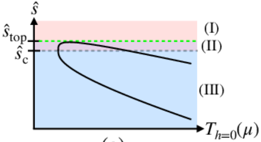
[Pradeep-Sogabe-Stephanov-Yee (24)]

Isentropic trajectories on QCD phase diagram (1/2)

Non-monotonicity of s/n on $T_{h=0}(\mu)$



(a) (b)



(c)

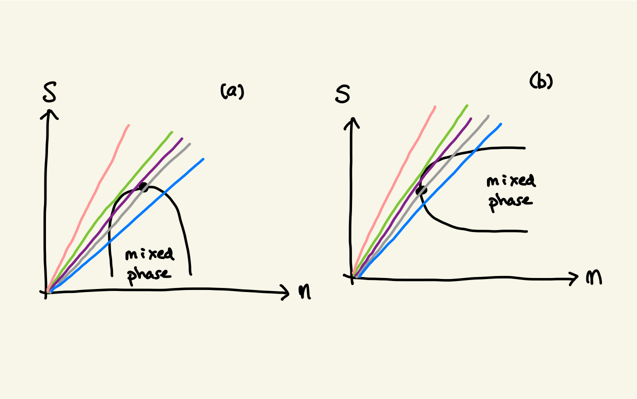
	(a)	(b)
(I)	Crossover	
(II)	HRG → HRG	QGP → QGP
(III)	QGP → HRG	

(d)

[Pradeep-Sogabe-Stephanov-Yee (24)]

Isentropic trajectories on QCD phase diagram (2/2)

Simple geometric picture on (s, n) plane



Generalization of Fig.1 of [Akamatsu-Teaney-Yan-Yin (19)]

Critical fluctuations of QCD critical point

1. Ising model: $r = a - a_c$

$$\underbrace{\frac{\partial}{\partial h} \leftrightarrow \phi,}_{\text{critical}} \quad \underbrace{\frac{\partial}{\partial r} \leftrightarrow \phi^2}_{\text{less singular}}$$

2. QCD critical point: $\varphi \equiv \langle \bar{q}q \rangle - \langle \bar{q}q \rangle_c$

$$\frac{\partial}{\partial \hat{h}} \leftrightarrow \varphi, \quad \frac{\partial}{\partial \hat{r}} \leftrightarrow \varphi^2$$

3. QCD thermodynamics

$$\frac{\partial}{\partial \hat{h}} = c_1 \frac{\partial}{\partial \beta} + c_2 \frac{\partial}{\partial (\beta\mu)} \leftrightarrow c_1 \underbrace{(e - e_c)}_{\Delta e} + c_2 \underbrace{(n - n_c)}_{\Delta n} \leftrightarrow \varphi,$$

$$\frac{\partial}{\partial \hat{r}} = c_3 \frac{\partial}{\partial \beta} + c_4 \frac{\partial}{\partial (\beta\mu)} \leftrightarrow c_3 (e - e_c) + c_4 (n - n_c) \leftrightarrow \varphi^2$$

Almost any linear combinations of Δe and Δn are critical

Correlation length

1. Mean field approximation around ground state $\phi - \langle\phi\rangle = \delta\phi \to \phi$

$$f(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{3}b_3\phi^3 + \frac{1}{4}b_4\phi^4$$

2. Naive extension to field theory

$$F[\phi] = \int_x f(\phi(x)) = \underbrace{\int_x \frac{1}{2}m^2\phi^2 + \frac{1}{3}b_3\phi^3 + \frac{1}{4}b_4\phi^4}_{\text{each point is independent !}}$$

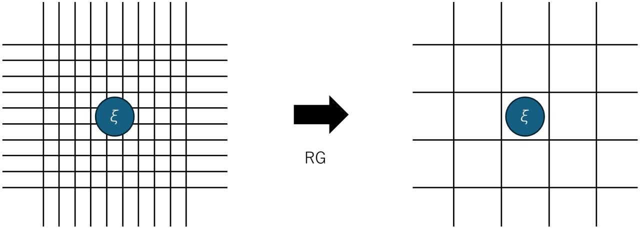
3. Smooth configuration costs less energy

$$F[\phi] = \int_x \underbrace{\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2}_{\xi = 1/m} + \frac{1}{3}b_3\phi^3 + \frac{1}{4}b_4\phi^4$$

For $\Delta x > \xi$, we can neglect the kinetic term

Non-Gaussian fluctuations (1/2)

1. Renormalization group



2. Roughly, free energy with $\Delta x \sim \xi$ is at the fixed point

$$F[\phi] \simeq \xi^3 \sum_i \left[\frac{1}{2} m^2 \phi_i^2 + \frac{1}{3} b_3 \phi_i^3 + \frac{1}{4} b_4 \phi_i^4 \right] \sim T,$$

$$\phi_i \sim \sqrt{T/\xi}, \quad b_3 \sim \bar{b}_3 / T^{1/2} \xi^{3/2}, \quad b_4 \sim \bar{b}_4 / T\xi$$

Non-Gaussian fluctuations (2/2)

3. Cumulants near the critical point

$$V\kappa_2 = \frac{1}{V}\int_{x,y}\langle\phi(x)\phi(y)\rangle_c \sim \frac{\xi^6}{V}\sum_i\langle\phi_i^2\rangle_c \sim \frac{\xi^6}{V}\frac{V}{\xi^3}\frac{T}{\xi} \sim T\xi^2,$$

$$V^{n-1}\kappa_n = \frac{1}{V}\int_{x_1,\cdots,x_n}\langle\phi(x_1)\cdots\phi(x_n)\rangle_c \sim \frac{\xi^{3n}}{V}\sum_i\langle\phi_i^n\rangle_c$$

$$\sim \frac{\xi^{3n}}{V}\frac{V}{\xi^3}\left(\frac{T}{\xi}\right)^{n/2} \sim T^{n/2}\xi^{5n/2-3}$$

4. Formula [Stephanov (06)]

$$\therefore \kappa_n \sim \frac{T^{n/2}\xi^{n(5-\eta)/2-3}}{V^{n-1}}, \quad \eta \approx 0.04$$

Higher-order cumulants are sensitive to ξ but suppressed by $1/V$

Contents

Critical point search in heavy-ion collisions

Obstacles

- System size is finite
- System lifetime is finite $\leftarrow$
- System is non-equilibrium $\leftarrow$

Goal of this section

- Dynamics of the QCD critical point
- What is Kibble-Zurek scaling?

Hydrodynamic description

Long-time and long wavelength phenomena

- Conserved densities $\rightarrow$ change only through surface
- Nambu-Goldstone modes $\rightarrow$ massless boson
- Critical amplitudes near the critical point $\rightarrow$ large correlation length
- Gauge fields $\rightarrow$ unscreened magnetic field

Modes with $\lim_{k\rightarrow 0} \omega(\boldsymbol{k}) = 0$ are relevant

Soft modes near the QCD critical point

1. Recall that almost any linear combinations of Δe and Δn are critical

$$\frac{\partial}{\partial \hat{h}} = c_1 \frac{\partial}{\partial \beta} + c_2 \frac{\partial}{\partial(\beta\mu)} \leftrightarrow c_1 \underbrace{(e - e_c)}_{\Delta e} + c_2 \underbrace{(n - n_c)}_{\Delta n} \leftrightarrow \varphi$$

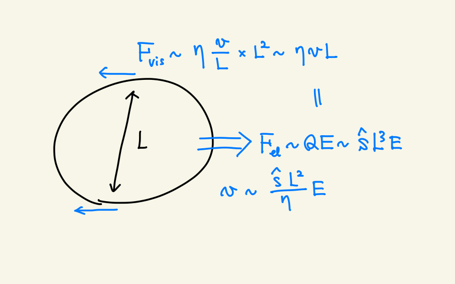
2. QCD critical point is Z_2 symmetry breaking $\rightarrow$ no NG modes
3. Color magnetic screening at long distance
4. Mixed with momentum density $\boldsymbol{g}$ $\rightarrow$ Hydrodynamics $\rightarrow$ Model H

Candidate: Hydrodynamics with $\Delta e, \Delta n, \boldsymbol{g}$

Keep relevant modes: Model H with $\hat{s} = \Delta(s/n), \boldsymbol{g}_T$

Conductivity $\sigma \propto \xi$, intuitively [Hohenberg-Halperin (77)]

1. A lump of $\hat{s}$ with linear dimension L in an electric field E



2. Electric current $j \sim \hat{s}v$ in the equilibrium

$$j \sim \hat{s}v = \underbrace{\frac{\hat{s}^2}{\eta}}_{\sim \sigma} L^2 E, \quad \hat{s}^2 \sim \underbrace{L^{-3}\frac{1}{L^{-2}+\xi^{-2}}}_{d^3k/(k^2+\xi^{-2})}, \quad \sigma \sim \frac{1}{\eta L}\frac{1}{L^{-2}+\xi^{-2}} \lesssim \frac{\xi}{\eta}$$

Conductivity scales with $\sigma \propto \xi$

Dynamical critical exponent $z \simeq 3$

1. We can think of electric field generated by chemical potential slope

$$\partial_t \hat{s} = -\nabla \cdot j = -\sigma \nabla \cdot E = \sigma \nabla \cdot \underbrace{\nabla \mu}_{=-E} = \underbrace{\frac{\sigma}{\chi}}_{=D} \nabla^2 \hat{s}$$

2. Diffusion constant scales with

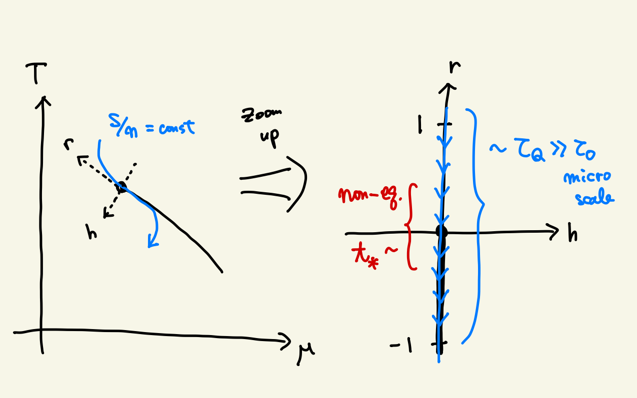
$$D = \frac{\sigma}{\chi} \sim \frac{\xi}{\xi^2} = \frac{1}{\xi}$$

3. Time scale of diffusion for wavelength ξ

$$\frac{1}{t} \sim D\nabla^2 \sim \frac{1}{\xi} \cdot \frac{1}{\xi^2} \sim \frac{1}{\xi^3}, \quad t \sim \xi^3 (=: \xi^z) \quad \therefore z \simeq 3$$

Relaxation time diverges $\propto \xi^z$ (critical slowing down)

QCD critical point in an expanding system



Trajectory on the Ising phase diagram

$$r(t) = t/\tau_Q, \quad h(t) = 0, \quad \xi(t) \sim \ell_o |r(t)|^{-\nu} \quad (\nu \approx 0.5)$$

Kibble-Zurek scaling (1/2) [Chandran-Erez-Gubser-Sondhi (12)]

1. Longest wavelength ℓ_Q equilibrated at $t \sim \tau_Q$

$$\underbrace{\tau_Q = \tau_o \left(\frac{\ell_Q}{\ell_o}\right)^2}_{\text{expansion time = relaxation time}} \quad \rightarrow \quad \ell_Q = \ell_o \left(\frac{\tau_Q}{\tau_o}\right)^{\frac{1}{2}}$$

2. Effective time scales near the critical point

$$\text{Power-laws of } r(t) \quad \rightarrow \quad \frac{\dot{r}(t)}{r(t)} = \frac{1}{t}$$

3. Scales when the critical mode ξ starts to get out of equilibrium

$$\underbrace{t_* = \tau_o \left(\frac{\xi(t_*)}{\ell_o}\right)^z \sim \tau_o \left(\frac{t_*}{\tau_Q}\right)^{-\nu z}}_{\text{effective changing time = relaxation time}} \rightarrow t_* = \tau_o \left(\frac{\tau_Q}{\tau_o}\right)^{\frac{\nu z}{1+\nu z}}, \quad \ell_* = \xi(t_*)$$

$$\tau_o \ll t_* \ll \tau_Q, \quad \ell_o \ll \ell_* = \underbrace{\ell_o \left(\frac{\tau_Q}{\tau_o}\right)^{\frac{\nu}{1+\nu z}} \ll \ell_Q}_{\because z \simeq 3 > 2}$$

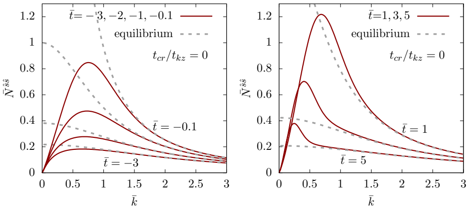
Kibble-Zurek scaling (2/2) [Chandran-Erez-Gubser-Sondhi (12)]

4. In the slow passing limit $\tau_Q/t_o \gg 1$, scaling with t_* and ℓ_*

$$\langle \phi(t_1,x_1)\phi(t_2,x_2)\rangle = \left(\frac{1}{\ell_*}\right)^{2\Delta} \mathcal{G}\left(\frac{t_1}{t_*}, \frac{t_2}{t_*}, \frac{x_1-x_2}{\ell_*}\right) \quad : \quad \text{KZ scaling}$$

5. Mean field approx. of model B & H [Akamatsu-Teaney-Yan-Yin (19)]

$$\bar{N}_{\hat{s}\hat{s}}(t,k) = N_{\hat{s}\hat{s}}(t,k)/C_p(t_*), \quad \text{Note: } \hat{s} = n\Delta(s/n) \text{ in this paper}$$



Baryon number correlation enhances to $(n/s)^2 C_p(t_*)$ for $k \sim 1/\ell_*$

Baryon fluctuation $\propto C_p(t_* \neq 0)$ is finite even in the luckiest case

Contents

Main messages again

1. Location and existence of QCD critical point is still unclear
2. QCD critical point $\sim$ Ising model
   - Why baryon number fluctuations? $\rightarrow$ how Ising h axis is embedded
   - Why higher order cumulants? $\rightarrow$ basically from $\phi(x \sim \xi) \sim \sqrt{T/\xi}$
3. Non-equilibrium condition hinders critical fluctuation to equilibrate
   - No divergence even when the system passes the critical point

$$\langle \phi(t_1, x_1)\phi(t_2, x_2)\rangle_{\mathrm{KZ}} = \underbrace{\left(\frac{1}{\ell_*}\right)^{2\Delta}}_{\text{finite}} \mathcal{G}\left(\frac{t_1}{\ell_*^z}, \frac{t_2}{\ell_*^z}, \frac{x_1 - x_2}{\ell_*}\right)$$

c.f. $$\langle \phi(t_1, x_1)\phi(t_2, x_2)\rangle_{\mathrm{eq}} = \underbrace{\left(\frac{1}{\xi}\right)^{2\Delta}}_{\text{singular at CP}} \mathcal{F}\left(\frac{t_1 - t_2}{\xi^z}, \frac{x_1 - x_2}{\xi}\right)$$

   - Need to be considered in experimental critical point search