The S-Matrix and boundary correlators in flat space

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Set up

- We consider asymptotically flat space with a boundary cutoff. We impose some boundary conditions for fields at this boundary cut-off surface.

- We compute the path integral as a function of these boundary values.

We ask what information does the path integral carry?

Motivation

AdS/CFT

$$\phi(z,x) \sim z^{d-\Delta}\phi_0(x) + z^{\Delta}\phi_1(x)$$

Fix the value of growing mode (ϕ_0) at boundary $z = \epsilon$.

Path integral as a function of (ϕ_0) carries information about the bulk dynamics.

Flat Space

Fix the value of field (ϕ_0) at boundary (which can be either timelike or spacelike).

We ask what information does the path integral as a function of ϕ_0 carry?

Difference: In AdS ϕ_0 couples to a boundary operator of dimension Δ and the corresponding path integral has independent description in dual theory.

- In this work, we focussed on scalar fields but our results can be generalised to other fields as well.

[Kim,Kraus, Monten, Myers '23]

- For most of the calculations, we consider the boundary cut-off surface to be a union of two spacelike slices, one in the far past (at time -T) and another in the far future (at time +T).

- However, most of our results can be generalised to arbitrary boundary surface (We consider null boundary as another example and make some comments about relation to CCFT).

Results/Outline

- We provide a precise relationship between the flat space S-matrix and the "Path integral as a functional of boundary values".
- This approach can be extended to de Sitter space.
- S-matrix unitarity provides non-trivial constraint on this path integral.
- We conjecture that the flat space wave functional and the S-matrix are related by analytic continuation.
- We analysed the analytic structure of $G_{\rm bdry}$ in position space both for massive and massless particles.
- For massless particles, $G_{\rm bdry}$ exhibits features like bulk point singularity (and it's generalisations) whose coefficient encode the flat space S-matrix.

S-matrix as boundary observable

$$S(\{p_i\},\{q_j\}) = \prod_{i=1}^{n}\int_M d^{d+1}x_i\, f_{p_i}(x_i)(\partial_i^2 - m^2)\prod_{j=1}^{m}\int_M d^{d+1}y_j\, \bar{f}_{q_j}(y_j)(\partial_j^2 - m^2)G(\{x_i, y_i\})$$

Satisfies free EOM i.e. $f_p(x) = e^{ip.x}$

$$\int_M d^{d+1}x\, f_p(x)(\partial^2 - m^2)\phi(x) = \int_M d^{d+1}x\, \partial^\mu\Big(f_p(x)\partial_\mu\phi(x) - \phi(x)\partial_\mu f_p(x)\Big)$$

$$= \prod_{i=1}^{n}\int_B d^d x_i \sqrt{h}\, n^{\mu_i}(f_{p_i}\partial_{\mu_i} - \partial_{\mu_i} f_{p_i})\prod_{j=1}^{m}\int_B d^d y_j \sqrt{h}\, n^{\mu_j}(\bar{f}_{q_j}\partial_{\mu_j} - \partial_{\mu_j}\bar{f}_{q_j})G(\{x_i, y_j\})$$

Path Integral

We can relate S-matrix (Euclidean version) to the bulk Euclidean path-integral with specified boundary conditions on a boundary surface B.

$$Z[\beta_0] = \int_{\phi|_B=\beta_0} [D\phi]\, e^{-S[\phi]}$$

$$G_{\text{bdry}}(\{x_i\}, \{y_i\}) = \prod_{i=1}^{n} \frac{\delta}{\delta\beta_0(x_i)} \prod_{j=1}^{m} \frac{\delta}{\delta\beta_0(y_j)} Z[\beta_0] \Big|_{\beta_0=0}$$

Using Hamilton-Jacobi:

$$G_{\rm bdry} = \int_{\phi|_B=0} [D\phi] \prod_{i=1}^{n} n^{\mu_i}\partial_{\mu_i}\phi(x_i) \prod_{j=1}^{m} n^{\mu_j}\partial_{\mu_j}\phi(y_j)\, e^{-S[\phi]}$$

$$(f_{p_i} n^{\mu_i}\partial_{\mu_i}\phi(x_i) - \phi(x_i) n^{\mu_i}\partial_{\mu_i} f_{p_i})$$

$$S_E(\{p_i\},\{q_j\}) = \int \prod_{i=1}^{n} d^d x_i f_{p_i}(x_i) \prod_{j=1}^{m} d^d y_j \bar{f}_{q_j}(y_j)\, G_{\rm bdry}(\{x_i\},\{y_j\})$$

Analytic Continuation

Euclidean Dirichlet problem → Lorentzian Dirichlet problem but with a twist

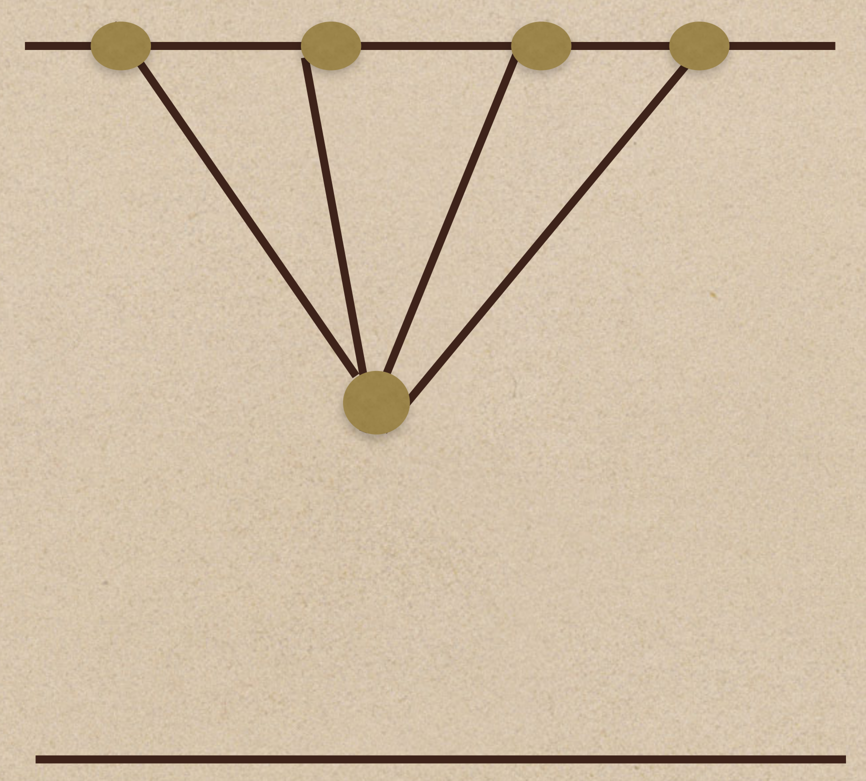
- Take the large T limit first and then analytically continue to Lorentizian space.

$$S(\{p_i\},\{q_j\}) = \int \prod_{i=1}^{n} d^d x_i f_{p_i}(x_i) \prod_{j=1}^{m} d^d y_j \bar{f}_{q_j}(y_j)\, G^L_{\text{bdry}}(\{x_i\},\{y_j\})$$

- We conclude that the Dirichlet path integral serves as a generating functional for S-matrices.

Other Contributions

- The Path integral $Z[\beta]$ carries much more information than just the S-matrix.
- It contains information about the vacuum wave-functional.



Example (Flat space)

Consider scalar field theory on a "slab" boundary

$$\mathscr{L} = \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

Boundary Conditions:

$$\beta(x) = \phi^+(-T, x) = \int d^d k \, \beta_k e^{i\vec{k}.\vec{x}}$$

$$\bar{\beta}(x) = \phi^-(T, x) = \int d^d k \, \bar{\beta}_k e^{-i\vec{k}.\vec{x}}$$

$$\mathcal{S} \propto \int \prod_i d^d k_i \, \frac{i\lambda \left(e^{-2iTE_{in}} - e^{-2iTE_{out}}\right)}{E_{in} - E_{out}} \delta^d \left(\sum_i \vec{k}_i^{\mathrm{in}} - \sum_j \vec{k}_j^{\mathrm{out}} \right) \prod_j \bar{\beta}_{k_j} \prod_i \beta_{k_i}$$

To compute S-matrix:

1. Compute G_{bdry} by taking derivatives w.r.t. source.
2. Multiply with free mode functions f_{p_i} and integrate.

$$S(\{p_i\}, \{q_j\}) = i\lambda \frac{\sin\left(T(E_{out} - E_{in})\right)}{\pi(E_{out} - E_{in})} \delta^d \left(\sum_i \vec{p}_i - \sum_j \vec{q}_j \right) \approx i\lambda \delta^{d+1} \left(\sum_i p_i - \sum_j q_j \right)$$

Wave-function & S-matrix

- We found that at tree level, the wave function contains a pole in $\sum \omega_i$.

$$\psi(\{\bar{\beta}_k\}) \approx \left(-\int \prod_i d^d k_i \, \frac{i\lambda}{\sum_i \omega_i} \delta^d \left(\sum_i \vec{k}_i \right) \prod_i \bar{\beta}_{k_i} \right)$$

Coefficient of singularity gives S-matrix.

Wavefunction $\Longrightarrow$ S-matrix

[P. Benincasa '18,...]

De Sitter S-matrix

[Melville, Pimentel '23]

LSZ formula for Bunch-Davies S-matrix

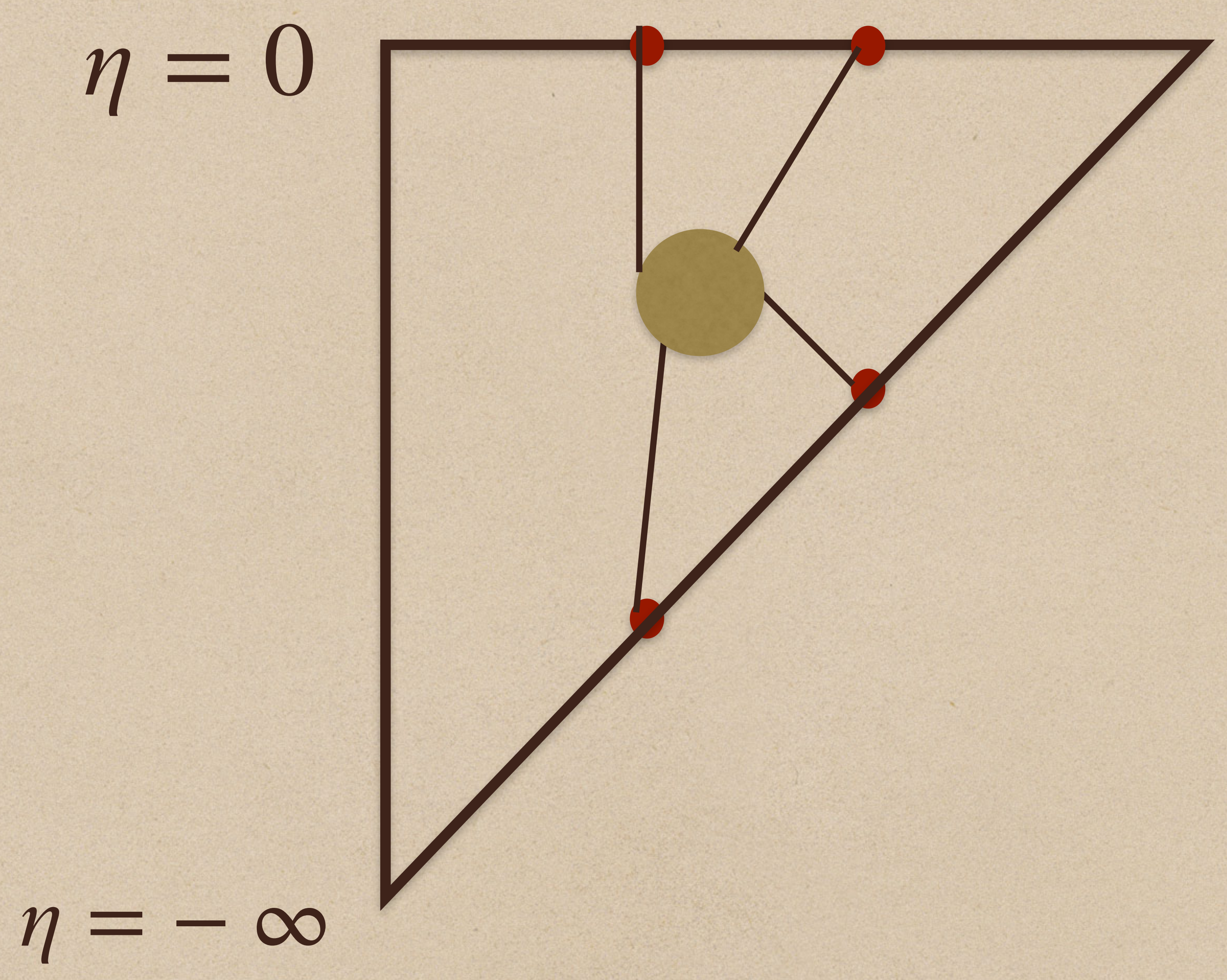
dS EOM

$$\mathcal{S} = \left[\prod_{b=1}^{n}\int_{-\infty}^{0}\frac{d\eta_b}{-\eta_b} f^{+}(k_b\eta_b) i\mathcal{E}(k_b,\eta_b)\right]\left[\prod_{b'=1}^{n'}\int_{-\infty}^{0}\frac{d\eta'_{b'}}{-\eta'_{b'}} f^{-}(k'_{b'}\eta'_{b'}) i\mathcal{E}(k'_{b'},\eta'_{b'})\right] G_{n\to n'}(k,\eta;k',\eta')$$

Hankel functions

Using manipulations similar to flat space, we see that the above S-matrix can be related to the path integral as a functional of boundary values in dS.

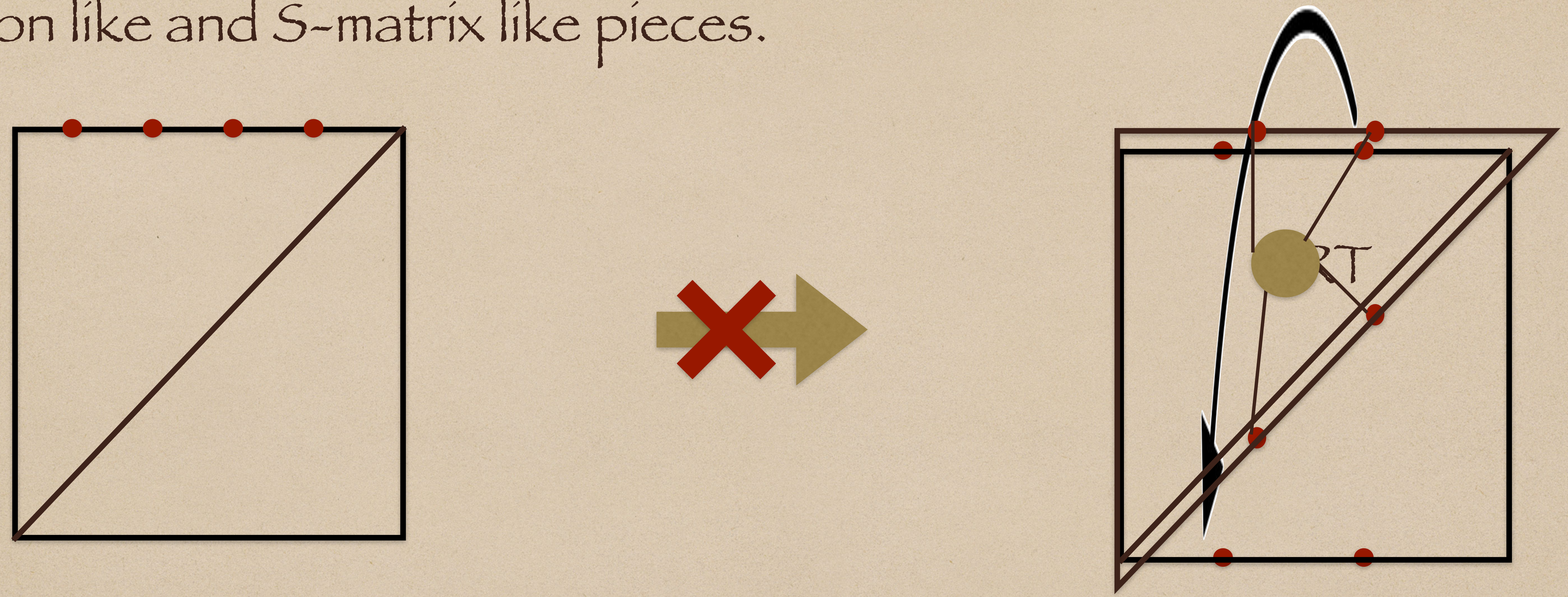
$$\mathcal{S}(\vec{k},\vec{k}') = (-1)^{d(n+n')}\prod_{i=1}^{n}\left(\int d^d x_i e^{i\vec{k}.\vec{x}_i} f^{+}(k\eta)\right)\Bigg|_{\eta=0}\prod_{j=1}^{n'}\left(\int d^d y_j e^{i\vec{k}'.\vec{y}_i} f^{-}(k'\eta)\right)\Bigg|_{\eta=-\infty} G_{\rm bdry}(\vec{x}_i,\vec{y}_j)$$



The on-shell action in de Sitter contains both S-matrix & wavefunction.

Is there a relation between dS S-matrix and the wavefunction?

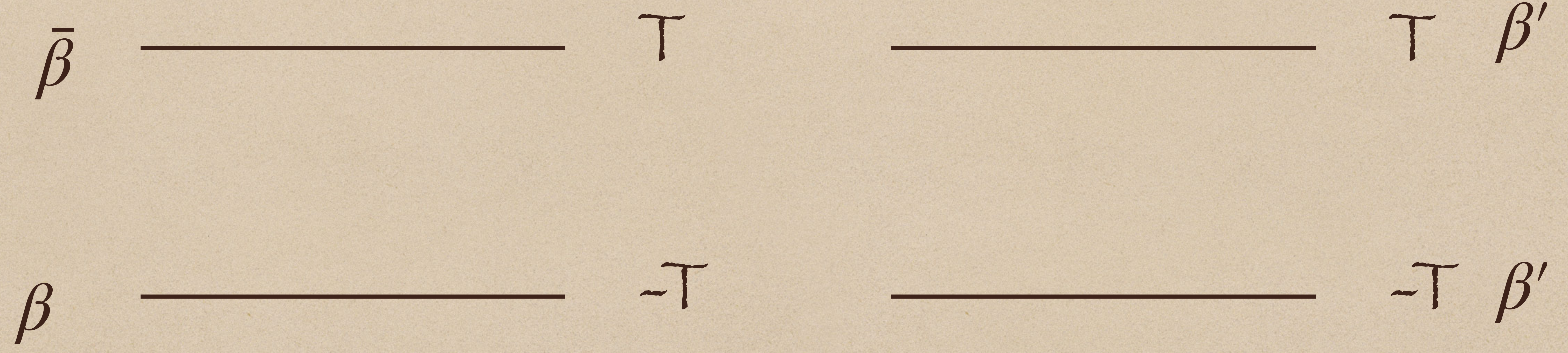
dS on-shell action (both in Poincare patch and global) also contains both wave function like and S-matrix like pieces.



It seems that the dS wave function is related to "Global S-matrix" de Sitter.

Unitarity

$$S^\dagger S = \mathbb{1}$$



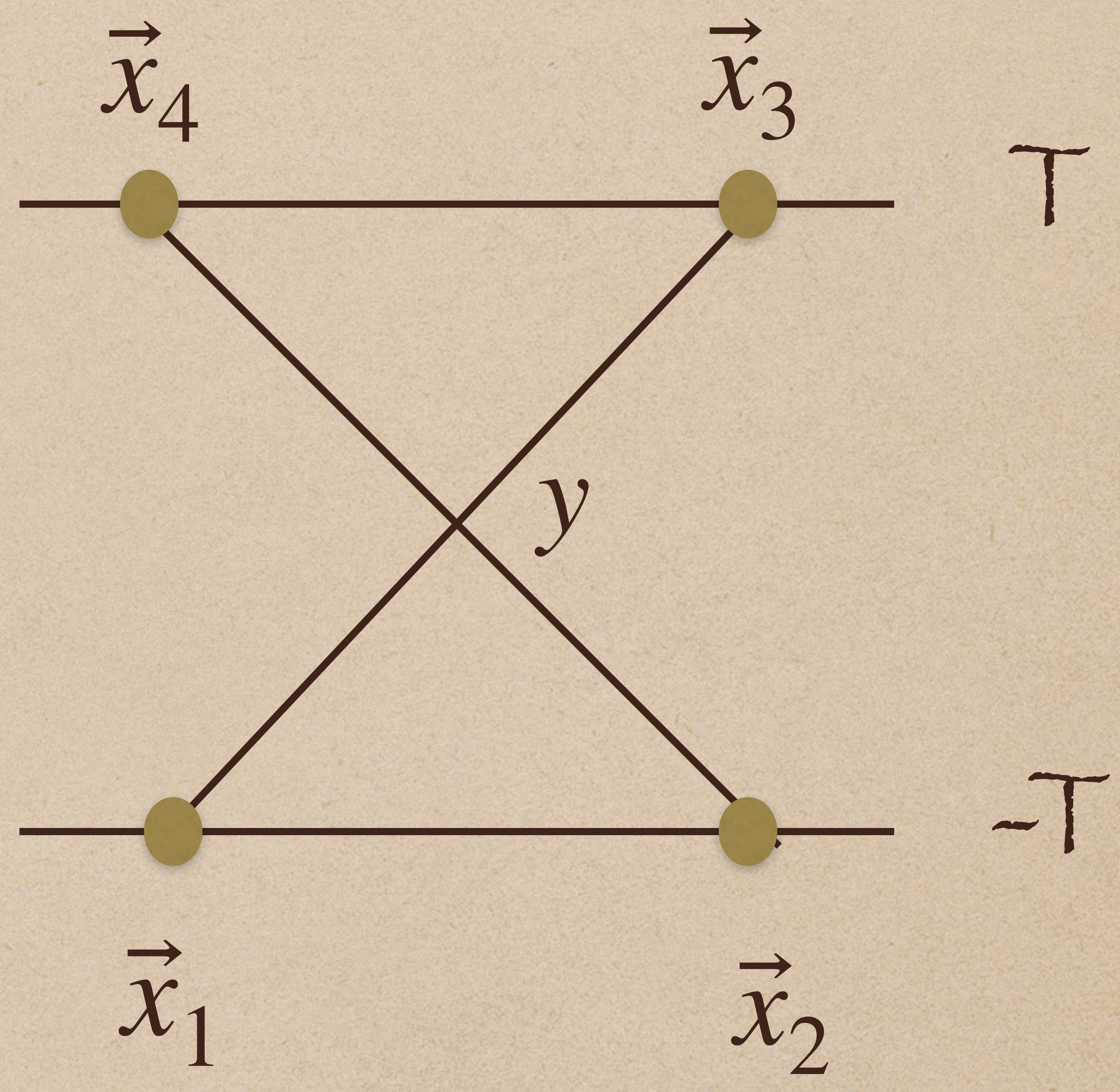
$$\int \mathscr{D}\bar\beta' \mathscr{D}\bar\beta \, \exp\left(-\int \frac{d^dk}{(2\pi)^d} 2\omega_k \bar\beta^*_{-\vec{k}} \bar\beta'_{\vec{k}} \right) Z^*[\beta, \bar\beta] Z[\beta', \bar\beta'] = \exp\left(\int \frac{d^dp}{(2\pi)^d} 2\omega_{\vec{p}} \beta^*_{\vec{p}} \beta_{-\vec{p}} \right)$$

G_{bdry} : massless particles

$$G_{\text{bdry}} = \lambda \int d^{d+1}y \prod_{i=1}^{n} G_{\partial B}(x_i, y)$$

Near Singularity

$$G_{\partial B}(x_i, y) = (2n \,.\, \nabla G(x, y))\Big|_{x \to x_i}$$



G_{bdry} : massless particles

$$G(x,y) = \int d^{d+1}y \prod_{i=1}^{n} \left(\frac{1}{\left((x_i - y)^2 - i\epsilon\right)^{\frac{D-2}{2}}} \right)$$

$G(x,y)$ has pole-type singularities whenever $(x_i - y)^2 = 0$.

Pinch off:

$$(x_i - y)^2 = 0 \qquad \text{for} \quad i > 3 \qquad\qquad \sum_{i=1}^{m} \omega_i (x_i - y) = 0 \qquad \forall \qquad \omega_i > 0$$

We show that residue at this singularity carries information about S-matrix.

[J. Maldacena, D. Simmons-Duffin and A. Zhiboedov '17]

Singularity

The equation for pinch-off for G_{bdry} can be phrased in terms of the distance matrix.

$$N_{ij} = (x_i - x_j)^2 = \Big((x_i - y) - (x_j - y)\Big)^2$$

Assuming there exists a bulk point y s.t. $(x_i - y)^2 = 0$

$$N_{ij} = 2(x_i - y)\,.\,(x_j - y)$$

Pinch-off/ Momentum conservation:

$$\sum_{i=1}^{m} \omega_i N_{ij} = 0$$

Singularity appears when N_{ij} has a zero eigenvalue with a positive eigenvector.

Co-dimension of Singularity

Q1: Given a generic set of $\{x_i\}$, how many "tunings" will one need to perform in order to obtain a singular G_{bdry}?

- Intersection of light cones.
- Momentum conservation.

$$\begin{array}{lll} c = 1 & \text{if} & \text{m} \leq \text{D} + 1 \\ c = m - D & \text{if} & \text{m} > \text{D} + 1 \end{array}$$

Q2: Given a set of boundary points $\{x_i\}$ such that G_{bdry} is singular for those insertions, does G_{bdry} receive contributions from one S-matrix or many?

Again depends on number of insertions and dimension of spacetime.

- When $m \leq D + 1$, only one S-matrix.
- When $m > D + 1$, G_{bdry} receives contributions from $m - D$, S-matrices.

Same as co-dimension of singularity.

G_{bdry} : massless particles

- We find that for massless particles, $G_{\text{bdry}}(x_i)$ (at tree level) is an analytic function in the space of boundary insertions with pole type singularities.
- These singularities exist on a co-dimension greater than or equal to one ($c \geq 1$) in the space of boundary insertions.
- The location of these singularities can be characterised in terms of zero eigenvalues of the boundary distance matrix: $N_{ij} = (x_i - x_j)^2$.
- The residue at these singularities contain the information about flat space S-matrix.

Two ways to extract S-matrix from G_{bdry}:

1. Multiply with mode functions and integrate (essentially Fourier transform).
2. The coefficient of singularity of G_{bdry} is the S- matrix.

Relation to Celestial CFT

- As a special case, we can work with Minkowski spacetime with null cutoff (boundary) surface.
- We found that G_{bdry} is an analytic function with some pole-type singularities and again the coefficient of these singularities give S-matrix.
- In the case of four point correlator, the location of the singularity in G_{bdry} is the same as the location of delta function in CCFT correlator.

[S. Banerjee '24]

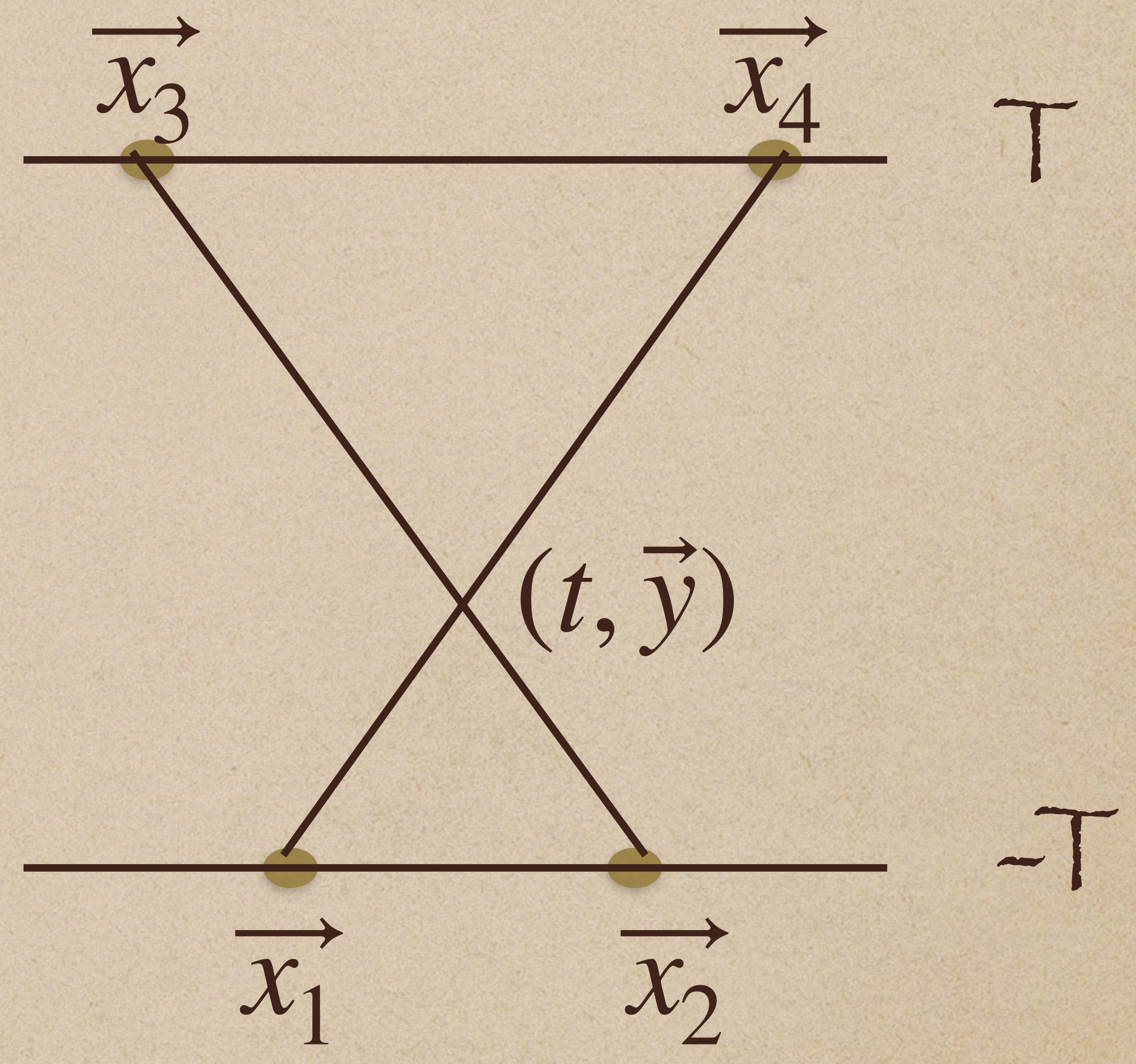
G_{bdry} : massive particles

We computed boundary correlates in position space at tree level for massive scalar fields using saddle point approximation (treating T as the large parameter).

$$G_{\text{bdry}} = \int d^{d+1}y\lambda \prod_{i=1}^{n} G_{\partial B}(x_i, y)$$

$$G_{\partial B}(x_i, y) = (2n \,.\, \nabla G(x, y))\Big|_{x \to x_i}$$

$$G(x, y) = C \frac{e^{-im(x-y)^2}}{((x-y)^2)^{\frac{D-1}{4}}}$$



G_{bdry} : massive particles

$$\sum_i m_i \frac{(x_i - y)^\mu}{d_i(x_i, y)} = 0$$

This equation gives momentum conservation at the bulk point y.

$$G_{\text{bdry}} \approx \prod_{i=1}^{n} \left(\left(\frac{m_i}{2\pi} \right)^{\frac{D-1}{2}} \frac{(-T-t)}{d_i^{in\,\frac{D+1}{2}}} e^{-im_i d_i^{in}} \right) \prod_{i=1}^{m} (\text{out})\, S \left(\frac{m_i(\vec{x}_i^{in} - \vec{y})}{d_i^{in}}, \frac{m_i(\vec{x}_i^{out} - \vec{y})}{d_i^{out}} \right)$$

G_{bdry} : massive particles

Two ways to extract S-matrix from G_{bdry}:

1. Multiply with mode functions and integrate (essentially Fourier transform)
2. Strip off extra factors from G_{bdry} and obtain the S-matrix.

Holographic Renormalization is non-local!!

Results/Outline

- S-matrix can be thought of as a boundary observable and can be computed using "Path integral as a functional of boundary values" (can be extended to de Sitter).
- S-matrix unitarity provides non-trivial constraint on this path integral.
- We argue that the flat space wave functional and the S-matrix are related by analytic continuation.
- We also analyse properties of $G_{\rm bdry}$ in position space both for massive and massless particles.
- For massless particles, $G_{\rm bdry}$ exhibits features like bulk point singularity whose coefficient encode the flat space S-matrix.

Food for Thought!!

- Is there a way to define LSZ for "Global S-matrix"?
- Can the relationship between wave-function & S-matrix be understood as crossing?
- Transition amplitudes ("Global S-matrix") must be unitary.
- Can de Sitter cutting rules be derived by considering unitarity in the Global patch and relating it to wave function?
- Can we define de Sitter S-matrix beyond tree level?

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