Cosmological correlators in slow-roll violating inflation

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Work with Jun'ichi Yokoyama

References:
- Phys. Rev. Lett 132, 221003 (2024) [2211.03395]
- Phys. Rev. D 109, 103541 (2024) [2303.00341]
- JCAP 10 (2024) 036 [2405.12145]
- Springer textbook on PBH (invited chapter) [2405.12149]

Canonical inflation

The most minimal model of inflation.

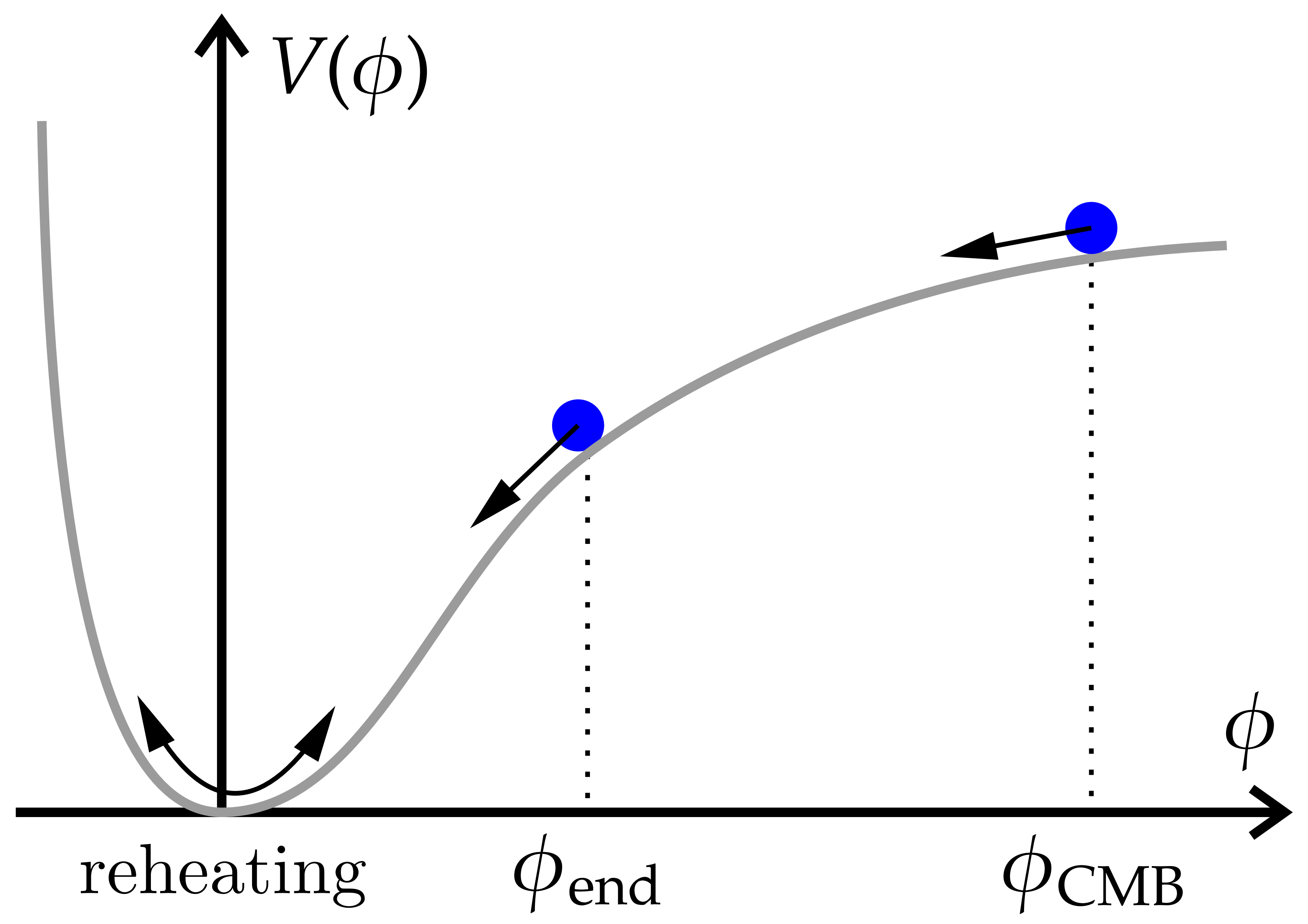
Action: $S = \dfrac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[M_\mathrm{P}^2 R - (\partial_\mu\phi)^2 - 2V(\phi)\right].$

Background: $\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\ \mathrm{d}\mathbf{x}^2 = a^2(\tau)(-\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}^2).$

Equation of motion:

Friedmann equation: $\dot{H} = -\dfrac{\dot{\phi}^2}{2M_\mathrm{P}^2}$ and $H^2 = \dfrac{1}{3M_\mathrm{P}^2}\left(\dfrac{1}{2}\dot{\phi}^2 + V(\phi)\right).$

Klein-Gordon equation: $\ddot{\phi} + 3H\dot{\phi} + \dfrac{\mathrm{d}V}{\mathrm{d}\phi} = 0.$



Slow-roll inflation

SR approximation: $\left|\frac{\ddot{\phi}}{\dot{\phi}H}\right| \ll 1$ and $\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2M_P^2H^2} \ll 1.$

Performing SR approximation, the equations of motion become

$$H^2 \approx \frac{V(\phi)}{3M_P^2} \approx \text{const} \;\text{ and }\; \dot{\phi} \approx -\frac{V_{,\phi}}{3HM_P} \longrightarrow \epsilon \approx \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2.$$

Solution: $a(t) \simeq -\frac{1}{H\tau} \propto e^{Ht}$ (quasi-dS), with domain of conformal time $\tau < 0$.

Slow-roll inflation

More systematically, define n-th SR parameter: $\epsilon_{n+1} = \frac{\dot{\epsilon}_n}{\epsilon_n H}$ and $\epsilon_1 = -\frac{\dot{H}}{H^2}$.

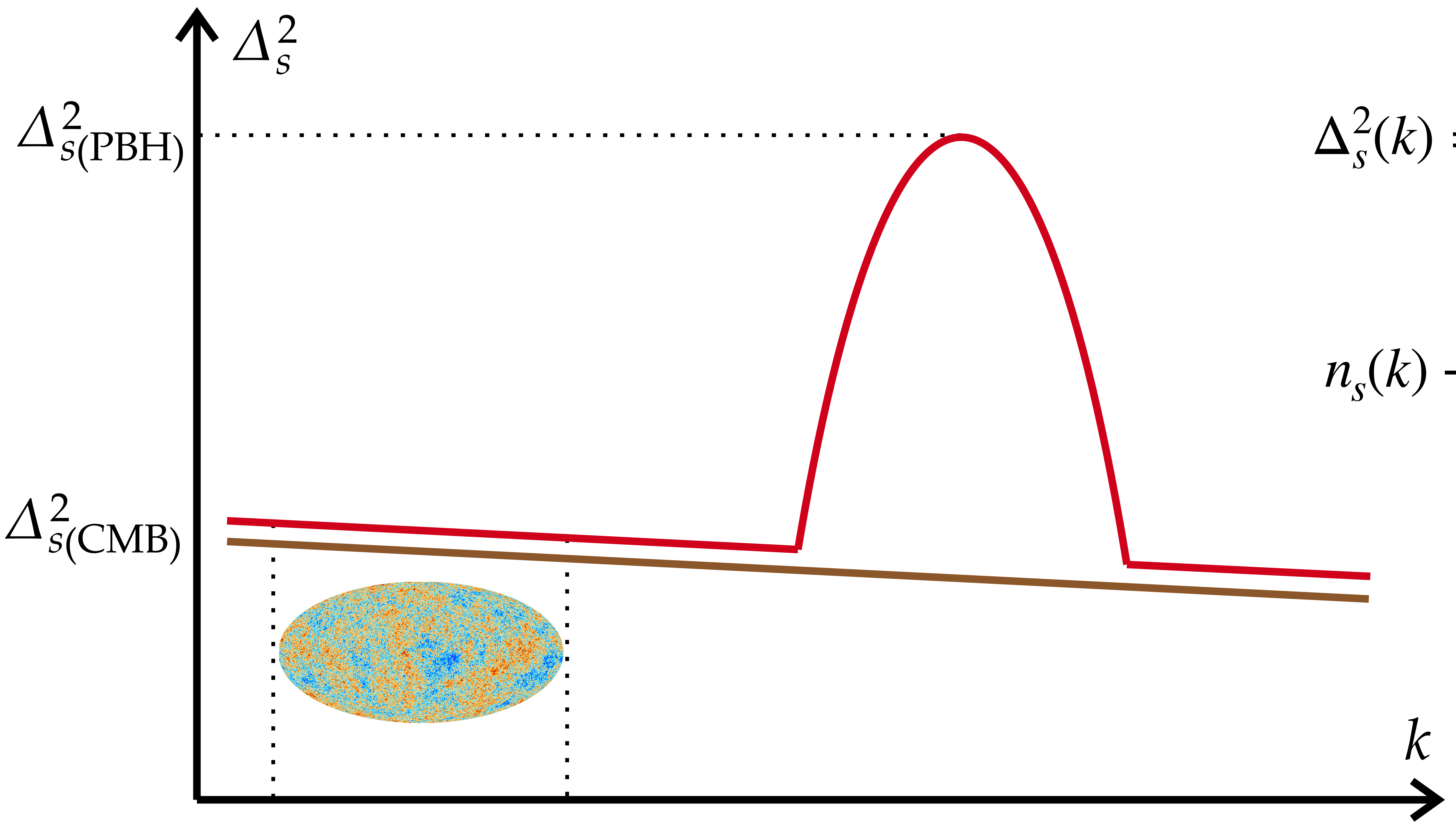
Substituting equation of motions:

$$\epsilon_1 = \frac{\dot{\phi}^2}{2M_{\rm P}^2 H^2}\,,\ \epsilon_2 = 2\epsilon_1 + 2\frac{\ddot{\phi}}{\dot{\phi}H}\,,\ \ldots$$

SR approximation: $|\epsilon_n| \ll 1.$

SR approximation implies quasi-dS, however converse statement is not true.

Power spectrum



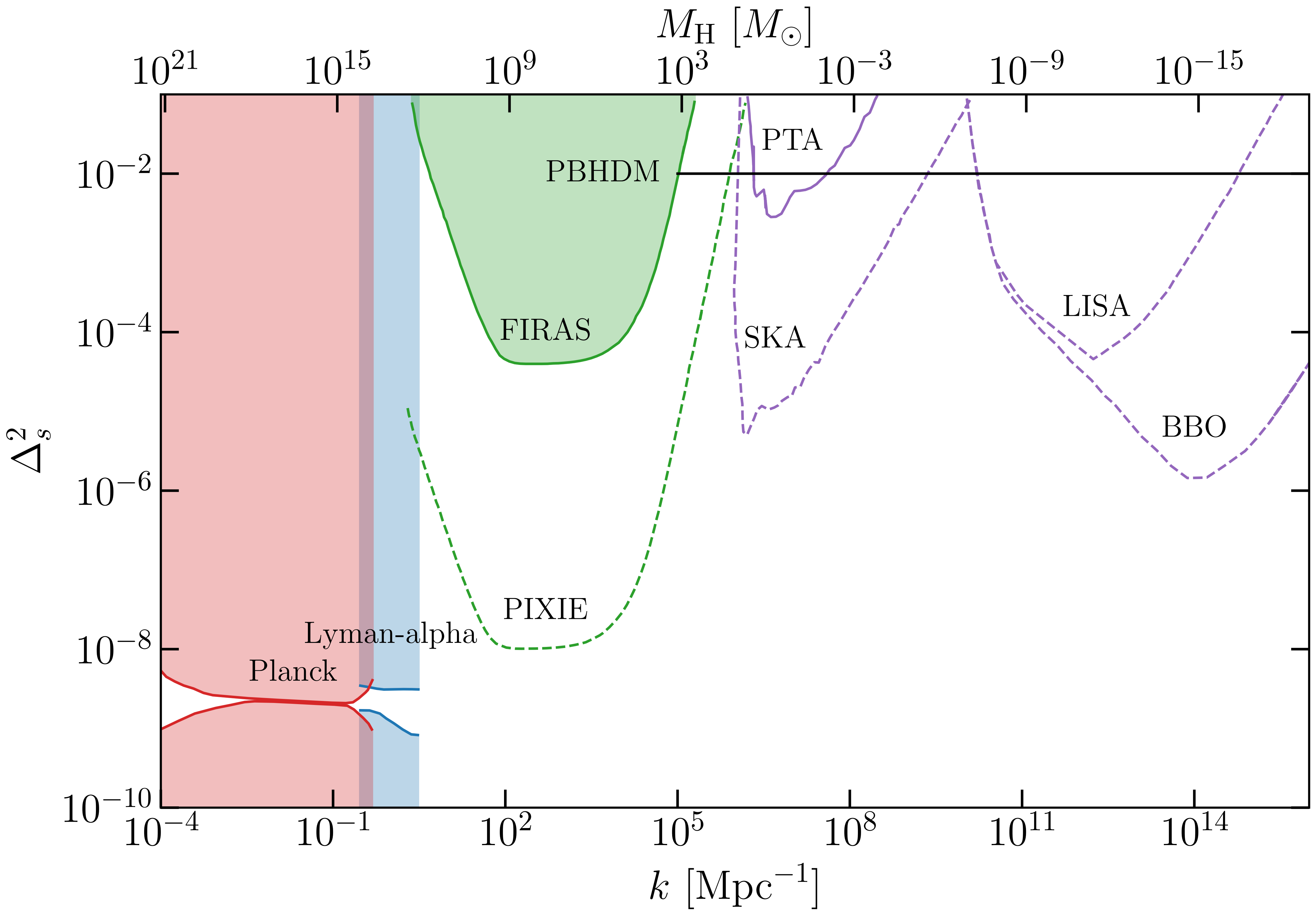
$$\Delta_s^2(k) = \left[\frac{H^2(\tau)}{8\pi^2 M_{\mathrm{P}}^2 \epsilon_1(\tau)}\right]_{\tau=-1/k}$$

$$n_s(k) - 1 = \frac{\mathrm{d}\log \Delta_s^2}{\mathrm{d}\log k} \sim \mathcal{O}(\epsilon)$$

Decrease $\epsilon_1(\tau)$ at late time to amplify the power spectrum on small scales. How to achieve that?

Constraints on power spectrum

Power spectrum is tightly constrained on large scale. However, constraints are very loose on small scale.



Green and Kavanagh (2007.10722)

Violation of SR approximation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0$$

SR approximation: $3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} \approx 0$

$$\epsilon_1 = \frac{\dot{\phi}^2}{2M_\mathrm{P}^2 H^2} \approx \frac{M_\mathrm{P}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \ll 1$$

$$\epsilon_2 = 2\epsilon_1 + 2\frac{\ddot{\phi}}{\dot{\phi}H} \ll 1$$

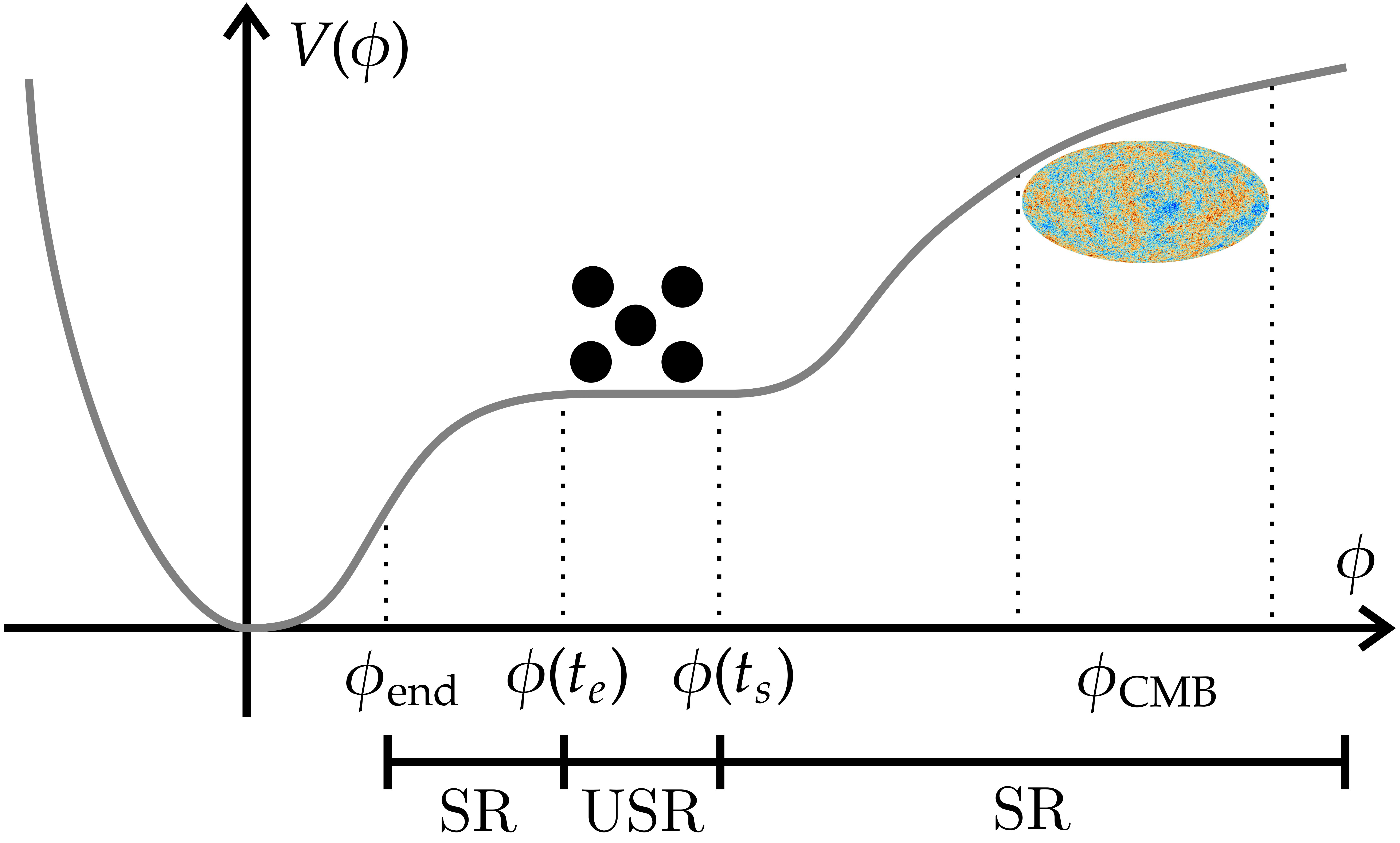
USR condition: $\ddot{\phi} + 3H\dot{\phi} = 0 \longrightarrow \dot{\phi} \propto a^{-3}$

$$\epsilon_1 = \frac{\dot{\phi}^2}{2M_\mathrm{P}^2 H^2} \propto a^{-6} \ll 1$$

$$\epsilon_2 = 2\epsilon_1 + 2\frac{\ddot{\phi}}{\dot{\phi}H} \simeq -6$$

Inoue and Yokoyama (hep-ph/0104083), Kinney (gr-qc/0503017)

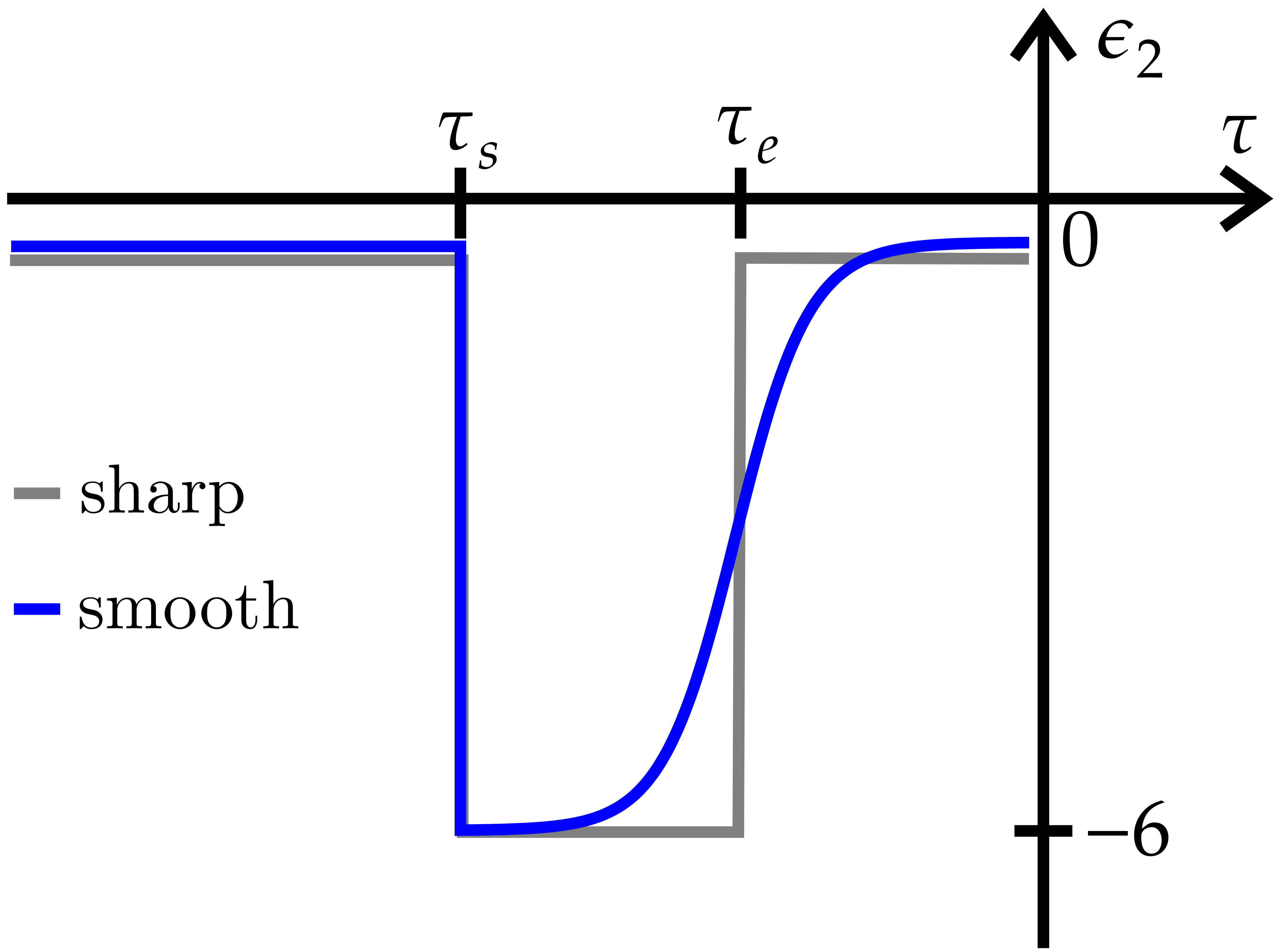
Potential of the inflaton



Ivanov et. al. (PRD 1994)

Evolution of the second SR parameter

- Sharp: step function at both $\tau = \tau_s$ and $\tau = \tau_e$.
- Smooth: continuous function at $\tau > \tau_s$.



Cosmological perturbations

Small perturbations:

- Inflaton: $\phi(\mathbf{x}, t) = \bar{\phi}(t) + \delta\phi(\mathbf{x}, t)$
- Spacetime: $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t)$

Gauge fixing condition:

- Comoving: $\delta\phi = 0$ and $\gamma_{ij} = a^2 e^{2\zeta}\delta_{ij}$
- Flat-slicing: $\delta\phi \neq 0$ and $\gamma_{ij} = a^2\delta_{ij}$

(Non-linear) gauge transformation: $\zeta = \zeta_n + \frac{1}{4}\epsilon_2\zeta_n^2 + \frac{1}{H}\dot{\zeta}_n\zeta_n + \mathcal{O}(\zeta_n^3)$, $\zeta_n = -\frac{\delta\phi}{M_{\mathrm{P}}\sqrt{2\epsilon_1}}$

Compute correlation function of ζ_n , then obtain correlation function of ζ.

Second-order action

Second-order action: $S^{(2)} = M_{\mathrm{P}}^2 \int \mathrm{d}t\ \mathrm{d}^3x\ \epsilon_1 a^3 \left[\dot{\zeta}^2 - \frac{1}{a^2}(\partial_i \zeta)^2\right]$

Mukhanov-Sasaki (MS) variable: $v = z\zeta M_{\mathrm{P}}$, $z = a\sqrt{2\epsilon_1}$

$$S^{(2)} = \frac{1}{2}\int \mathrm{d}\tau\ \mathrm{d}^3x \left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z}v^2\right]$$

Equation of motion: $v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0,\ \frac{z''}{z} = (aH)^2\left(2 - \epsilon_1 + \frac{3}{2}\epsilon_2 - \frac{1}{2}\epsilon_1\epsilon_2 + \frac{1}{4}\epsilon_2^2 + \frac{1}{2}\epsilon_2\epsilon_3\right)$

- SR ($\epsilon_1, |\epsilon_2|, |\epsilon_3| \ll 1$)
- USR ($\epsilon_1, |\epsilon_3| \ll 1$, $\epsilon_2 = -6$)

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right)v_k = 0$$

Curvature perturbation

Pure USR inflation ($V(\phi) = \text{constant}$) corresponds to $\mathscr{A}_k = 1$ and $\mathscr{B}_k = 0$

$$\lim_{k\to 0} \zeta_k(\tau) = \frac{iH}{2M_\mathrm{P}\sqrt{k^3\epsilon_1(\tau)}} \longrightarrow \Delta_s^2(k\to 0,\tau) = \frac{H^2}{8\pi^2 M_\mathrm{P}^2\epsilon_1(\tau)} \propto a^6(\tau)$$

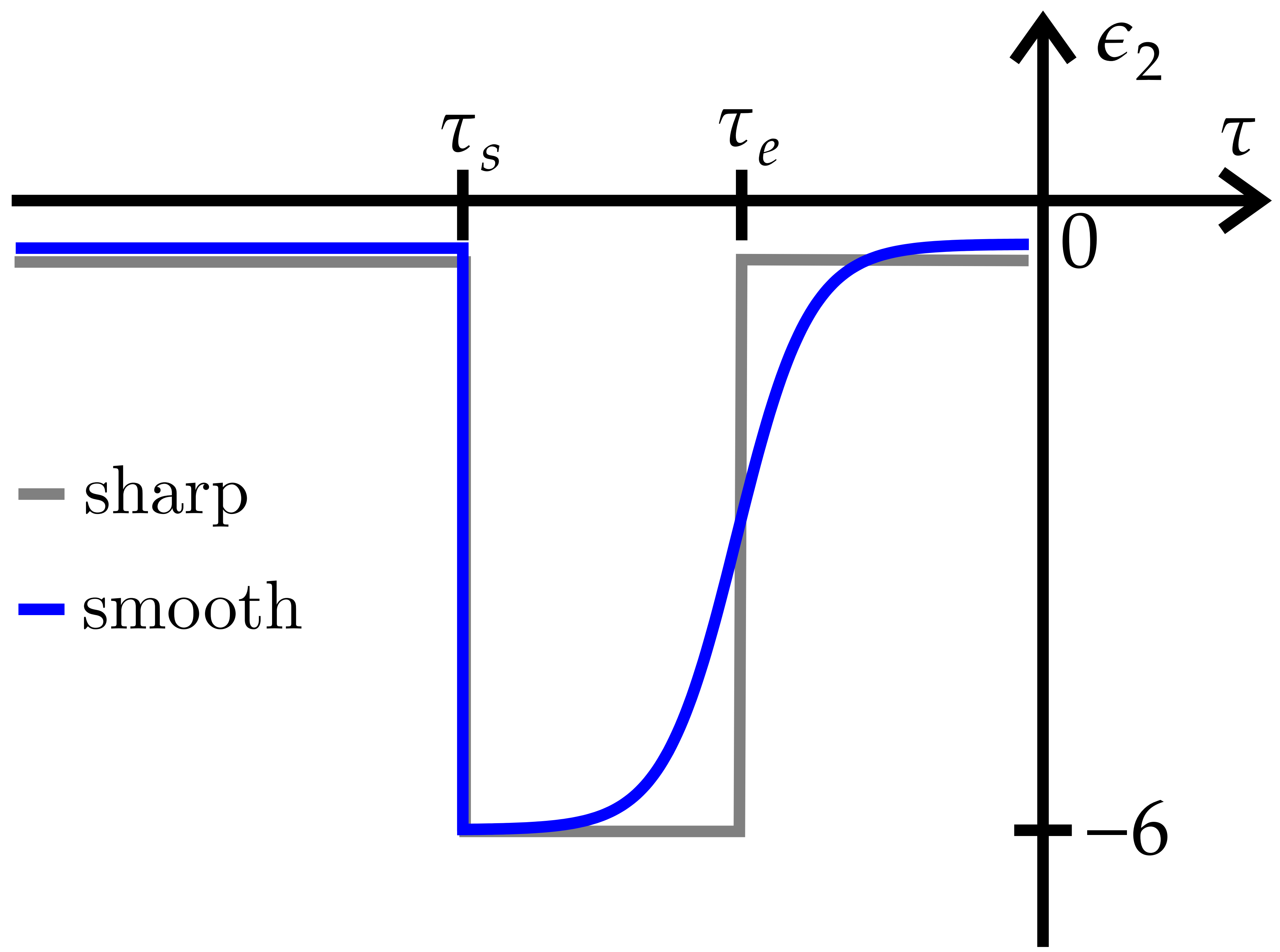
Superhorizon evolution of scale-invariant perturbation even at tree-level.

Transition makes initial condition of the USR period deviates from Bunch-Davies.

$$\zeta_k(\tau) = \frac{iH}{2M_\mathrm{P}\sqrt{k^3\epsilon_1(\tau)}} \left[\mathscr{A}_k e^{-ik\tau}(1+ik\tau) - \mathscr{B}_k e^{ik\tau}(1-ik\tau)\right]$$

Coefficients $\mathscr{A}_k$ and $\mathscr{B}_k$ are obtained by requiring continuity of $\zeta_k(\tau)$ and $\zeta_k'(\tau)$ at the transition.

Sharp transition



Two-point functions

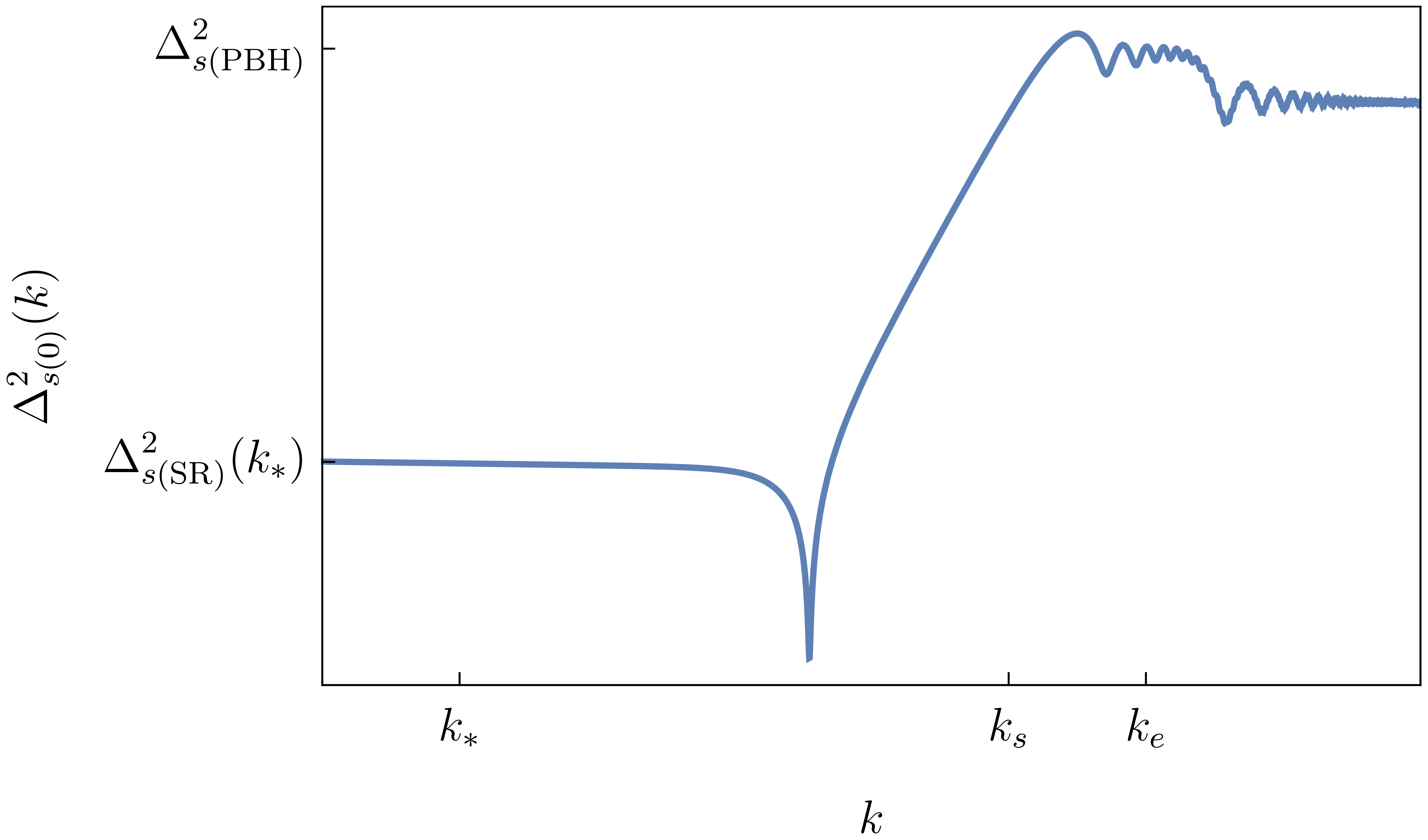
Requiring continuity of $\zeta_k(\tau)$ and $\zeta_k'(\tau)$ at transition $\tau = \tau_s$:

$$\mathscr{A}_k = 1 - \frac{3(1+k^2\tau_s^2)}{2ik^3\tau_s^3} \text{ and } \mathscr{B}_k = -\frac{3(1+ik\tau_s)^2}{2ik^3\tau_s^3}e^{-2ik\tau_s}$$

Power spectrum at the end of inflation: $\Delta^2_{s(0)}(k) = \frac{k^3}{2\pi^2}|\zeta_k(\tau \to 0)|^2$

Large scale: $\Delta^2_{s(\mathrm{SR})}(k) \equiv \Delta^2_{s(0)}(k \ll k_s) = \frac{H^2}{8\pi^2 M_\mathrm{P}^2 \epsilon_1(\tau_s)}$

Small scale: $\Delta^2_{s(\mathrm{PBH})} \approx \Delta^2_{s(\mathrm{SR})}(k_s)\left(\frac{k_e}{k_s}\right)^6$



Higher-point interactions

Taylor expansion of the potential: $S_{\delta\phi}^{(n)} = -\int \mathrm{d}^4x \ \frac{V_n}{n!}(\delta\phi)^n$, $V_n \equiv \frac{\mathrm{d}^n V}{\mathrm{d}\phi^n}$, $V_{n+1} = \dot{V}_n/\dot{\bar{\phi}}$

In decoupling limit ($\epsilon_1 \to 0$):

$$V_1 = \frac{H^2 M_\mathrm{P}}{\sqrt{2}}\sqrt{\epsilon_1}(6+\epsilon_2)$$

$$V_2 = -\frac{H^2}{4}\epsilon_2(6+\epsilon_2+2\epsilon_3)$$

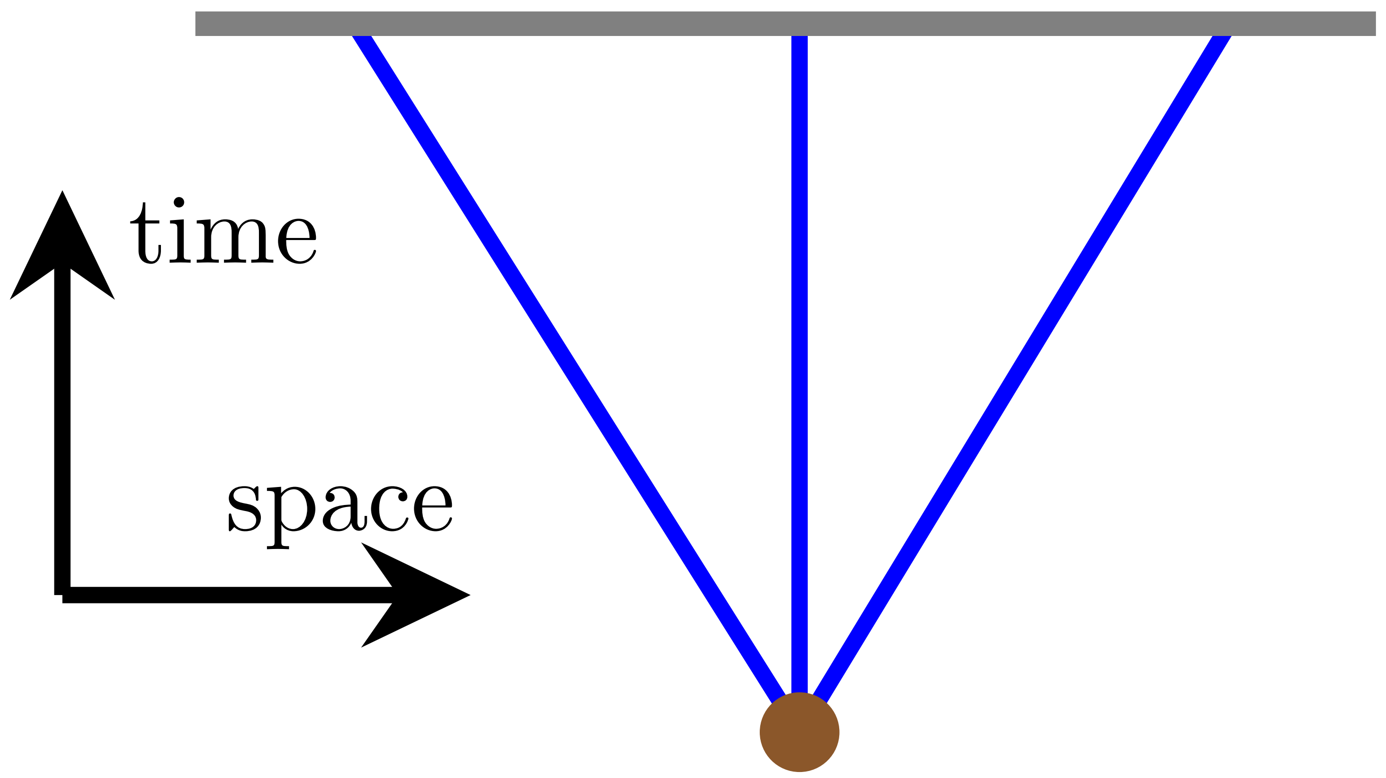
$$V_3 = -\frac{H^2}{2M_\mathrm{P}\sqrt{2\epsilon_1}}\epsilon_2\epsilon_3(3+\epsilon_2+\epsilon_3+\epsilon_4) = -\frac{\partial_t(a^3\epsilon_1\dot{\epsilon}_2)}{M_\mathrm{P}(a\sqrt{2\epsilon_1})^3}$$

Gravitational effects are suppressed by ϵ_1 .

Bispectrum

Leading interaction: $H^{(3)}_{\delta\phi} = -\frac{1}{2}M_{\rm P}^2 \int {\rm d}^3x\ \epsilon_1 \epsilon_2' a^2 \zeta_n' \zeta_n^2$

Time integral: $\int_{-\infty}^{0} {\rm d}\tau\ \epsilon_2'(\tau) f(\tau) = \Delta\epsilon_2 [f(\tau_e) - f(\tau_s)]$



Bispectrum (in-in perturbation theory):

$$\langle\langle \zeta_{\mathbf{k}_1}(\tau_0)\zeta_{\mathbf{k}_2}(\tau_0)\zeta_{\mathbf{k}_3}(\tau_0)\rangle\rangle = -2M_{\rm P}^2 \int_{-\infty}^{\tau_0} {\rm d}\tau\ \epsilon_1(\tau)\epsilon_2'(\tau) a^2(\tau) {\rm Im}\left[\zeta_{k_1}(\tau_0)\zeta_{k_2}(\tau_0)\zeta_{k_3}(\tau_0)\zeta^*_{k_1}(\tau)\zeta^*_{k_2}(\tau)\zeta'^*_{k_3}(\tau)\right] + \text{perm.}$$

Perturbativity: $\frac{\langle\zeta\zeta\zeta\rangle}{\langle\zeta\zeta\rangle^{3/2}} \ll 1 \longrightarrow \left[\Delta^2_{s({\rm PBH})}\right]^{1/2} \ll \frac{1}{|\Delta\epsilon_2|}$

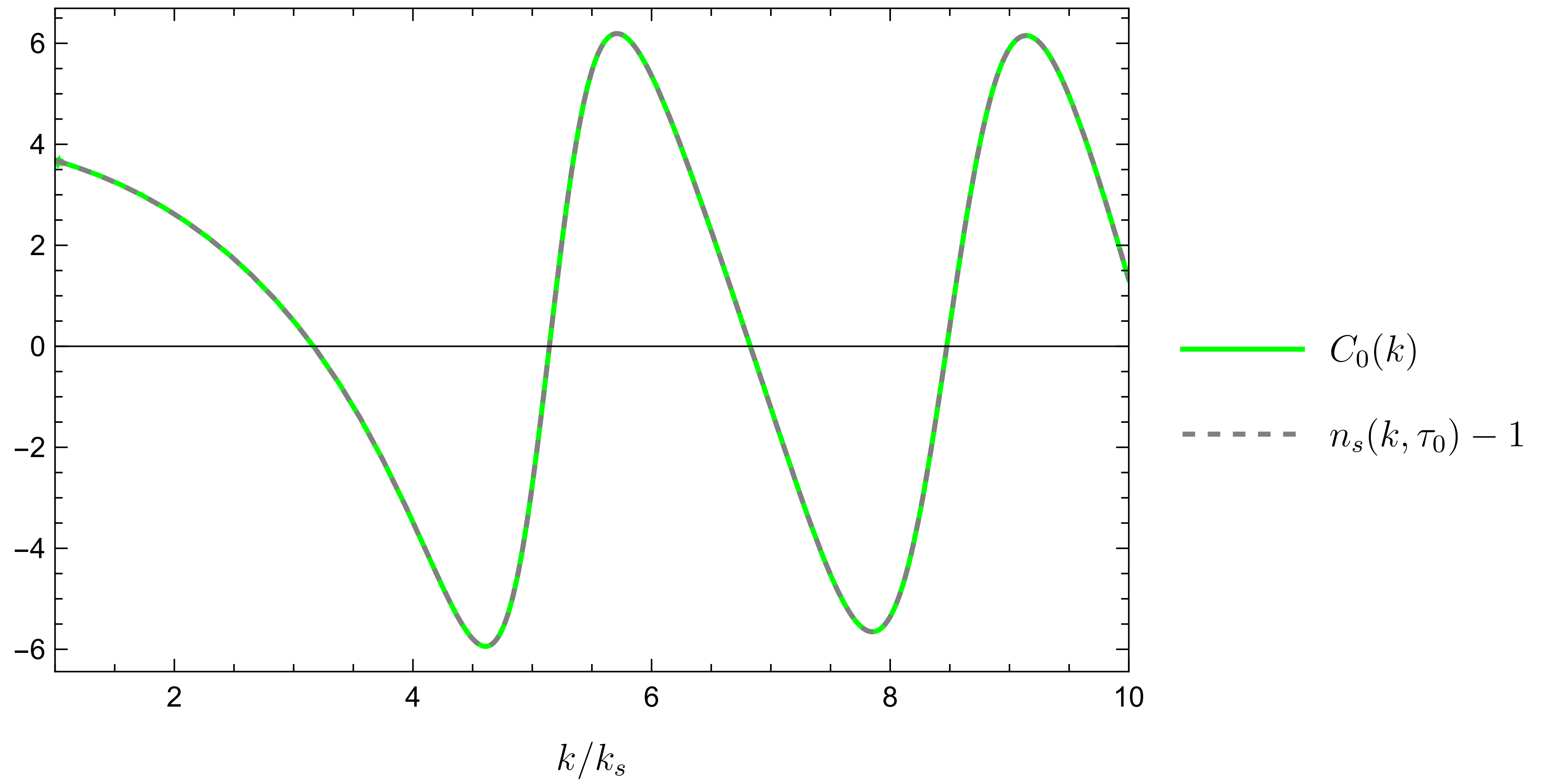
Bispectrum

Squezeed limit: $\langle\langle \zeta_{\mathbf{k}_1}(\tau_0)\zeta_{\mathbf{k}_2}(\tau_0)\zeta_{-\mathbf{k}_2}(\tau_0)\rangle\rangle = -\, C_0(k_2)\,|\,\zeta_{k_1}(\tau_0)\,|^2\,|\,\zeta_{k_2}(\tau_0)\,|^2,$

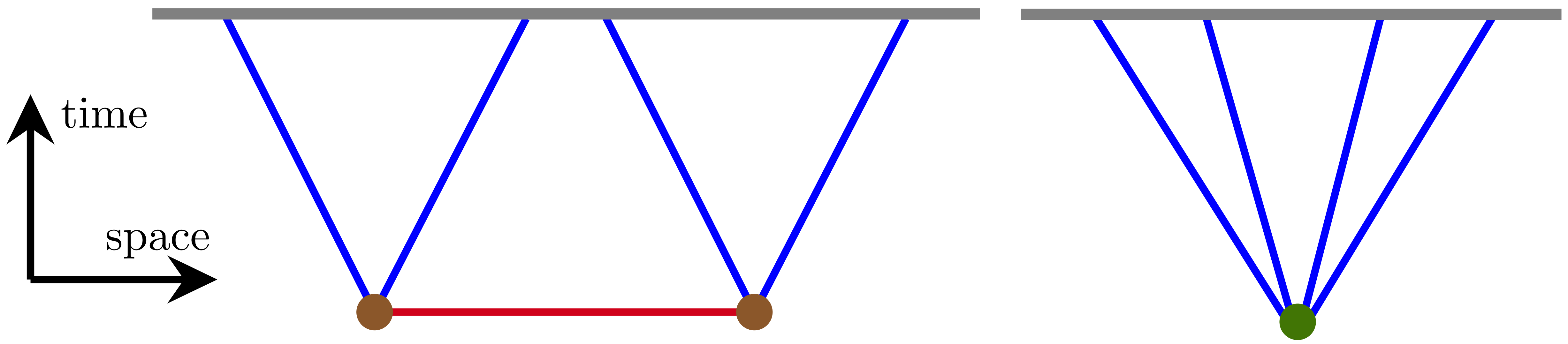
$$C_0(k) = 4M_{\mathrm{P}}^2\Delta\epsilon_2 \mathrm{Im}\left\{\frac{\zeta_k^2(\tau_0)}{|\,\zeta_k(\tau_0)\,|^2}\left[\epsilon_1(\tau_e)a^2(\tau_e)\zeta_k^*(\tau_e)\zeta_k'^*(\tau_e) - \epsilon_1(\tau_s)a^2(\tau_s)\zeta_k^*(\tau_s)\zeta_k'^*(\tau_s)\right]\right\}.$$

Maldacena’s theorem on squeezed limit of the bispectrum:

$$\lim_{k_L\to 0}\langle\langle \zeta_{\mathbf{k}_L}(\tau)\zeta_{\mathbf{k}_S}(\tau)\zeta_{-\mathbf{k}_S}(\tau)\rangle\rangle = -\,(n_s(k_S,\tau)-1)\,|\,\zeta_{k_S}(\tau)\,|^2\,|\,\zeta_{k_L}(\tau)\,|^2,\; n_s(k,\tau)-1 = \frac{\mathrm{d}\log\Delta_s^2(k,\tau)}{\mathrm{d}\log k}\,.$$



Trispectrum



Total contributions to the trispectrum:

- Exchange diagram with two $H^{(3)}_{\delta\phi}$ vertices
  - s-channel: $s = |\mathbf{k}_1 + \mathbf{k}_2|$
  - t-channel: $t = |\mathbf{k}_1 + \mathbf{k}_3|$
  - u-channel: $u = |\mathbf{k}_1 + \mathbf{k}_4|$
- Contact diagram with $H^{(4)}_{\delta\phi}$ vertex

$$\langle \mathcal{O}(\tau)\rangle^{(3)} = \langle \mathcal{O}(\tau)\rangle^{\dagger}_{(0,2)} + \langle \mathcal{O}(\tau)\rangle_{(1,1)} + \langle \mathcal{O}(\tau)\rangle_{(0,2)} \,,$$

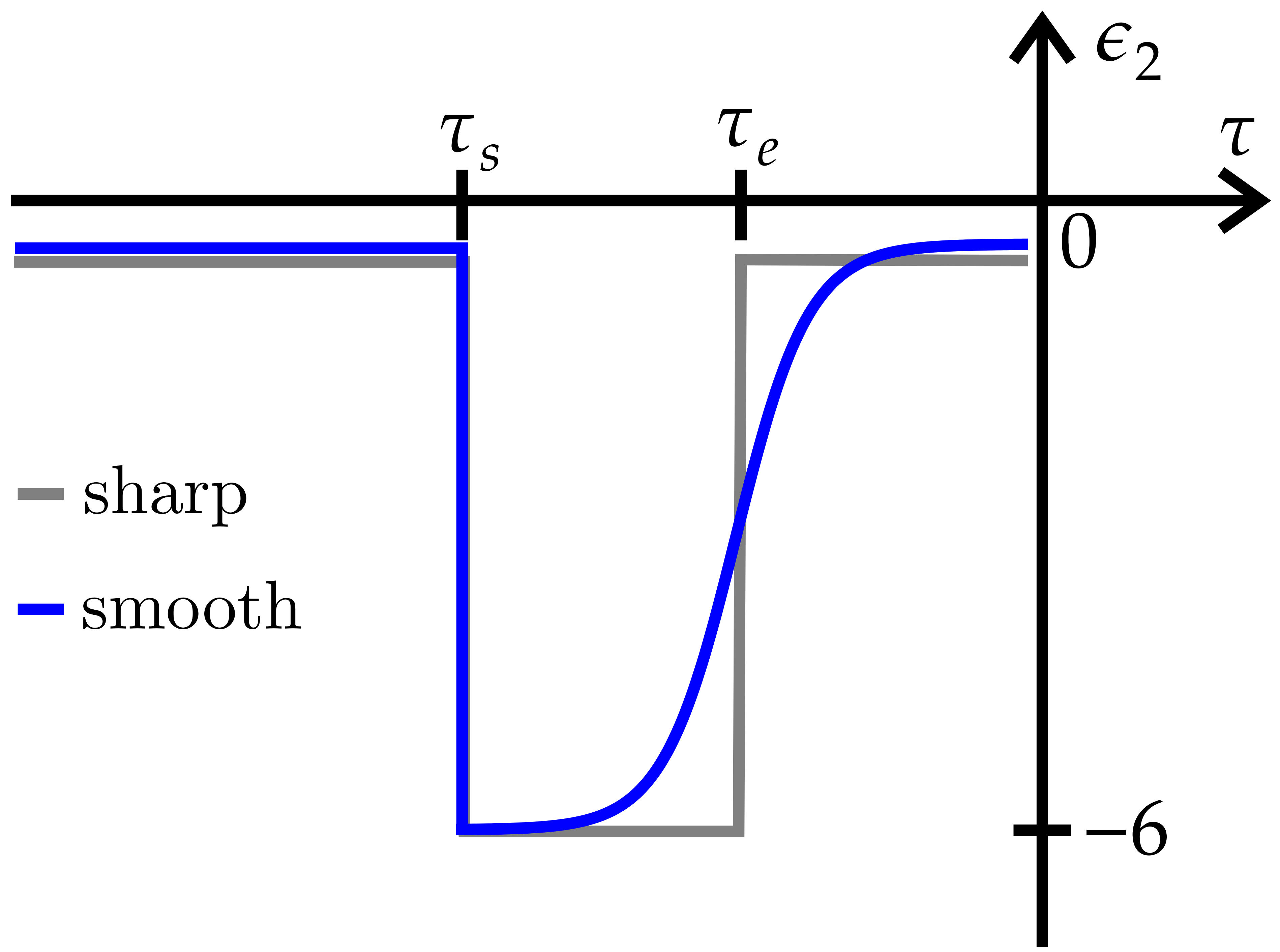
$$\langle \mathcal{O}(\tau)\rangle_{(1,1)} = \int_{-\infty}^{\tau} \mathrm{d}\tau_1 \int_{-\infty}^{\tau} \mathrm{d}\tau_2 \left\langle H^{(3)}(\tau_1)\hat{\mathcal{O}}(\tau)H^{(3)}(\tau_2)\right\rangle \,,$$

$$\langle \mathcal{O}(\tau)\rangle_{(0,2)} = -\int_{-\infty}^{\tau} \mathrm{d}\tau_1 \int_{-\infty}^{\tau_1} \mathrm{d}\tau_2 \left\langle \hat{\mathcal{O}}(\tau)H^{(3)}(\tau_1)H^{(3)}(\tau_2)\right\rangle \,.$$

$$\langle \mathcal{O}(\tau)\rangle^{(4)} = 2\int_{-\infty}^{\tau} \mathrm{d}\tau_1 \, \mathrm{Im}\left\langle \hat{\mathcal{O}}(\tau)H^{(4)}(\tau_1)\right\rangle$$

Kristiano and Yokoyama (in preparation)

Smooth transition



Wands duality

$$\frac{z''}{z} = (aH)^2\left(2 - \epsilon_1 + \frac{3}{2}\epsilon_2 - \frac{1}{2}\epsilon_1\epsilon_2 + \frac{1}{4}\epsilon_2^2 + \frac{1}{2}\epsilon_2\epsilon_3\right)$$

Almost constant $\epsilon_2 \longrightarrow |\epsilon_3| \ll 1$

$$\frac{z''}{z} \simeq \frac{2}{\tau^2}$$

SR: $\epsilon_1, |\epsilon_2| \ll 1$

USR: $\epsilon_1 \ll 1, \epsilon_2 \simeq -6$

Dynamical $\epsilon_2 \longrightarrow |\epsilon_3| \sim \mathcal{O}(1)$

$$\frac{z''}{z} \simeq \frac{2}{\tau^2}$$

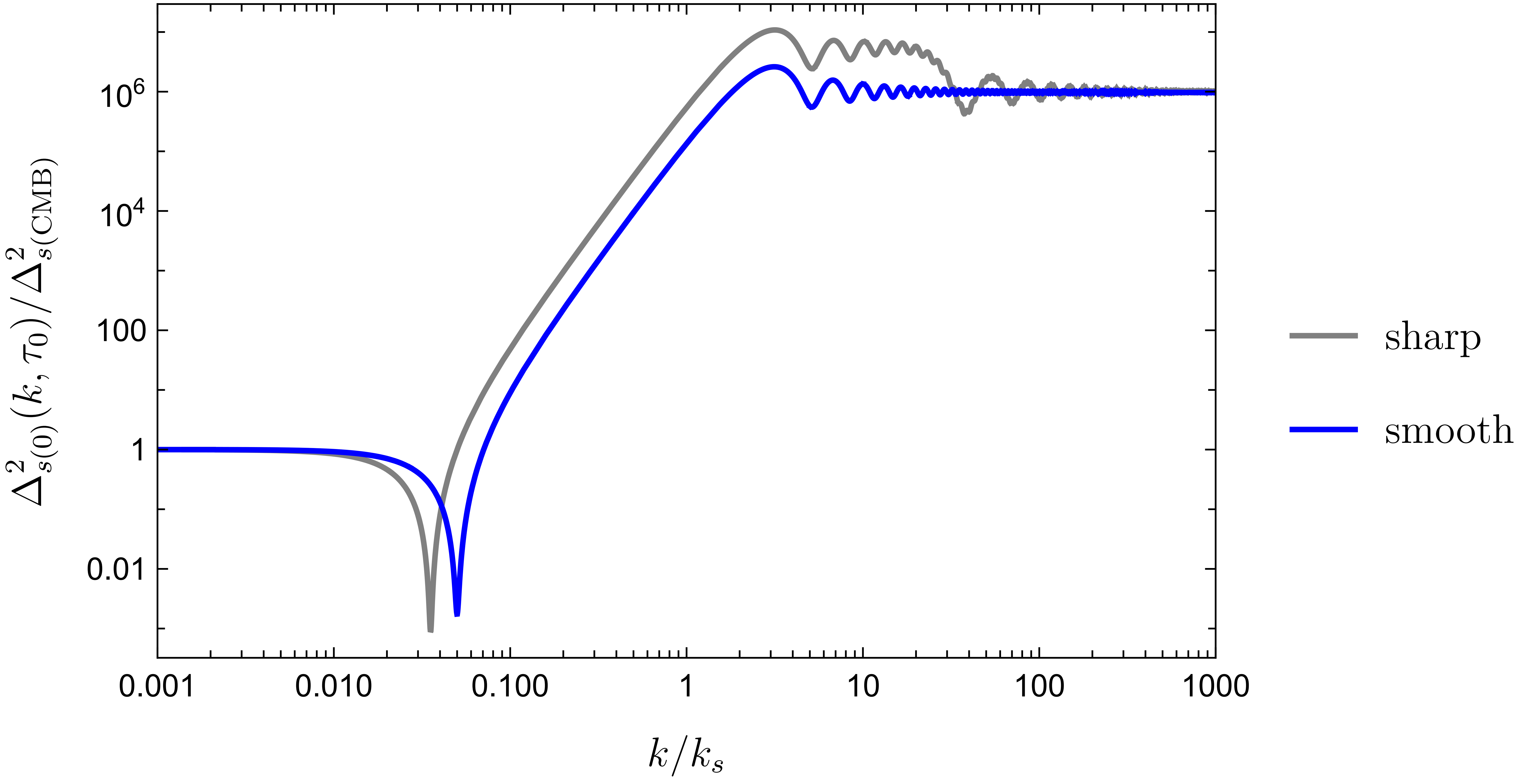
SR: $\epsilon_1, |\epsilon_2| \ll 1$

USR: $\epsilon_1 \ll 1, \epsilon_2 \simeq -6$

Transition: $\frac{3}{2}\epsilon_2 + \frac{1}{4}\epsilon_2^2 + \frac{\dot{\epsilon}_2}{2H} \simeq 0$

Two-point functions

Comparing power spectrum:



More on Wands duality

Differential equation: $\dfrac{3}{2}\epsilon_2 + \dfrac{1}{4}\epsilon_2^2 + \dfrac{\dot{\epsilon}_2}{2H} = \text{constant.}$

Taking time derivative: $0 = 2\epsilon_2' + \epsilon_2'\epsilon_2 - \epsilon_2''\tau$.

Prove that $\epsilon_1(\tau)a^2(\tau)\epsilon_2'(\tau) = \text{constant}$:

$$(\epsilon_1 a^2 \epsilon_2')' = \epsilon_1 a^3 H \left(2\epsilon_2' + \epsilon_2\epsilon_2' - \epsilon_2''\tau\right) = 0.$$

Therefore in this setup: $H^{(3)}_{\delta\phi} = \dfrac{1}{6}M_{\rm P}^2 \displaystyle\int \mathrm{d}^3x \left(a^2\epsilon_1\epsilon_2'\right)' \zeta_n^3 = 0$

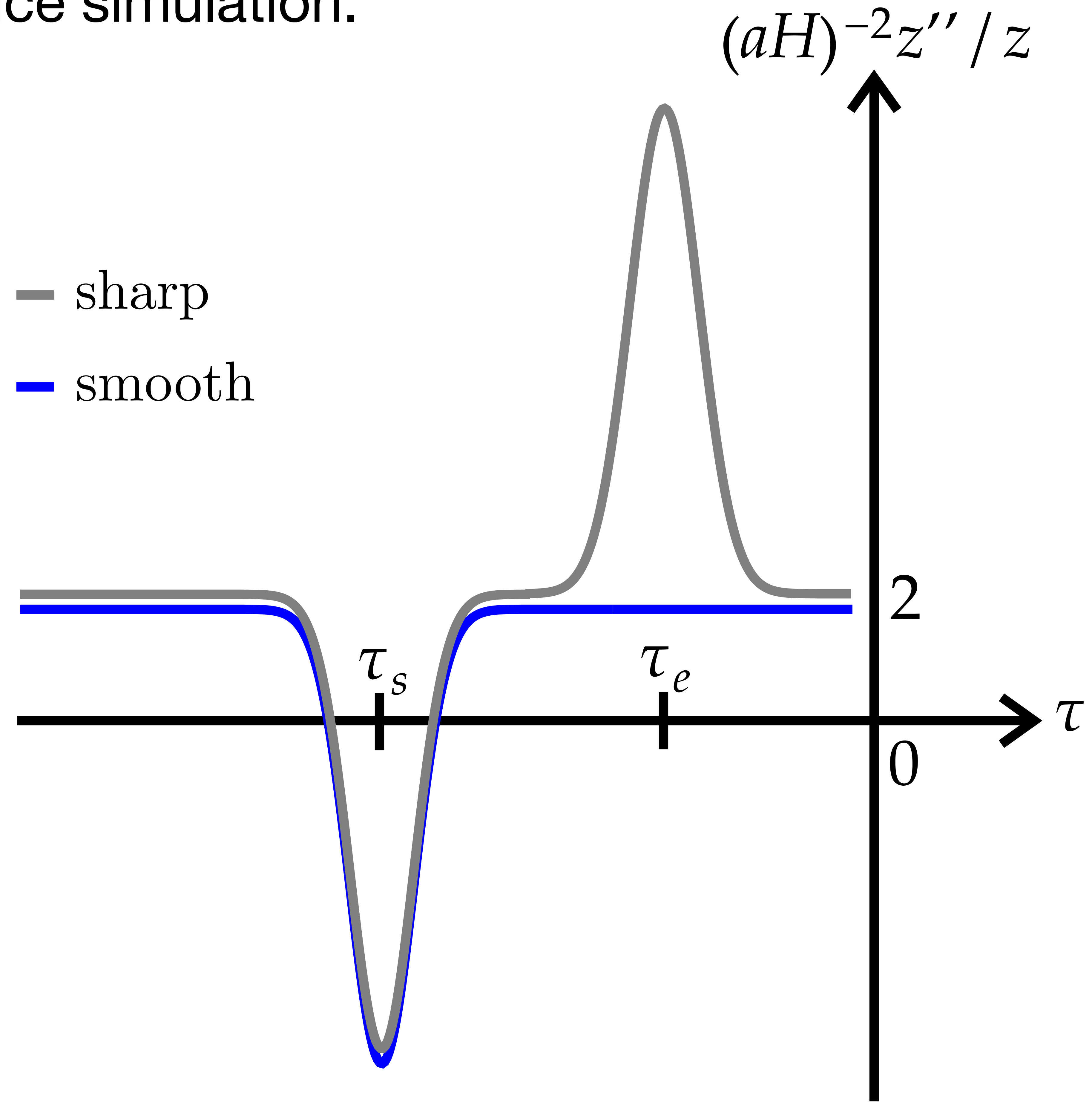
$$H^{(4)}_{\delta\phi} = -\frac{1}{24}M_{\rm P}^2 \int \mathrm{d}^3x \left[\frac{1}{aH}\left(\epsilon_1 a^2 \epsilon_2'\right)'' - \left(4 + \frac{3}{2}\epsilon_2\right)\left(\epsilon_1 a^2 \epsilon_2'\right)'\right] \zeta_n^4 = 0$$

Bigger picture

Deviation from Wands duality condition generates higher-order correction to the correlation functions.

Confirmed by non-perturbative lattice simulation.

Possible guidance for bootstrap?



Caravano et. al. (2410.23942)

Conclusion and Future Direction

Take home messages:

- Precision cosmology for inflation model with large fluctuations has just begun!
- Leading interactions at decoupling limit come from Taylor expansion of the inflationary potential, which yield correlation function that satisfies Maldacena's theorem.
- Most minimal model: SR $\rightarrow$ Wands duality phase.

Future directions:

- Bootstrapping USR correlators: perturbation in USR grows as $\zeta \sim \tau^{-3}$, can we obtain USR correlators from dS correlators? Weight shifting operator to $\Delta = -3$?
- Bootstrapping cosmology with transition: SR $\rightarrow$ USR transition makes the mode function during USR does not start from Bunch-Davies vacuum. Bootstrapping correlation function with deviation from Bunch-Davies initial condition?