Lattice QCD Study of Heavy Quark Diffusion

Hai-Tao Shu (舒海涛)

Central China Normal University

Nuclear Science Computing Center at CCNU

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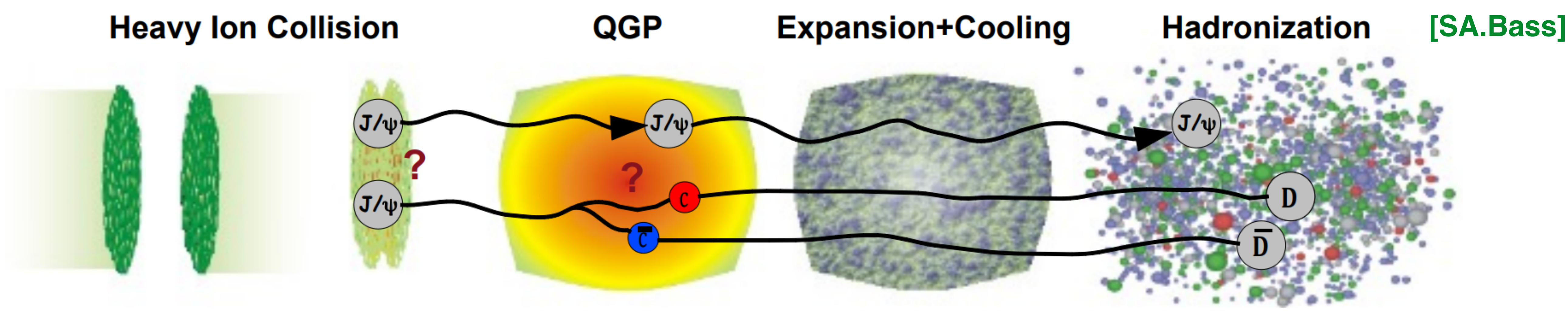
[PRD 103 (2021) 1, 014511]
[PRL 130 (2023) 23, 231902]
[PRL 132 (2024) 5, 051902]
[PRD 109 (2024) 11, 114505]

Dec. 6 - Dec. 11, 2024, Guangzhou, China

Heavy quark diffusion in HICs

See Jing Wang's talk for an experimental review



Release constituents equilibrate via diffusion process

how fast do heavy quarks equilibrate?

- Perturbative estimates: [G. Moore and D. Teaney, PRC.71.064904]

$$\tau_{\mathrm{kin,charm}} \sim 6\ \mathrm{fm/c} \gg \tau_{\mathrm{kin,light}} \sim 1\ \mathrm{fm/c}$$

- Experimental estimates (RHIC): [STAR Collaboration, PRL,106 (2011) 159902]

$$\tau_{\mathrm{kin,charm}} \approx \tau_{\mathrm{kin,light}}$$

Need non-perturbative ab-initio determination for equilibration time!

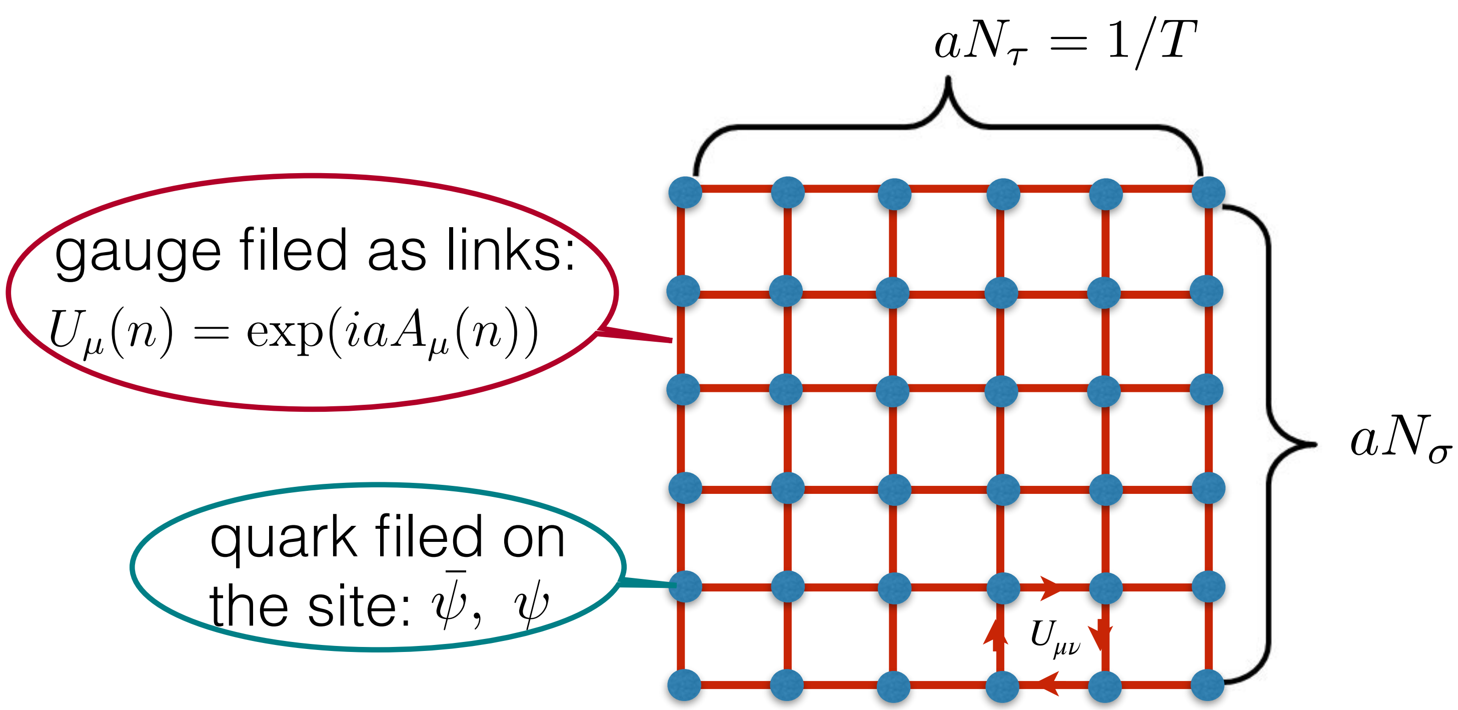
$$\tau_{\mathrm{kin}} = \frac{1}{\eta_D} = \frac{D M_{\mathrm{kin}}}{T} = \frac{2M_{\mathrm{kin}}}{T^2}\frac{1}{\kappa/T^3}$$

Introduction to Lattice QCD

LQCD is designed for non-perturbative physics: hadron structure, QCD vacuum, hadron spectrum, thermal physics, …

$$S_F = \int \mathrm{d}^4x\ \bar{\psi}((\partial + iA_\mu)\gamma^\mu + m)\psi, \qquad S_G[C] = \frac{\beta}{3}\sum_{n\in\Lambda}\sum_{\mu\le\nu} Re\ \mathrm{Tr}[\mathbb{1} - U_{\mu\nu}(n)]$$

- Basic quantities in LQCD:
  - ✳ lattice volume
  - ✳ lattice spacing
  - ✳ gauge field SU(3) matrix
  - ✳ quark field Dirac spinor (vector)
  - ✳ quark mass



Discretization of space

$a \to 0$
$N_\sigma \to \infty$

Discrete space ⟶ Continuum space

Heavy quark diffusion under HQEFT

- Langevin equations of heavy quark motion

$$\partial_t p_i = -\eta_D p_i + \xi_i(t) \qquad \langle \xi_i(t)\xi_j(t')\rangle = \kappa \delta_{ij}\delta(t-t')$$

- Mass dependent **momentum** diffusion coefficient

$$\kappa^{(M)} \equiv \frac{M^2\omega^2}{3T\chi_q}\sum_i \frac{2T\rho_V^{ii}(\omega)}{\omega}\Big|^{\eta\ll|\omega|\lesssim\omega_{\rm UV}}$$

- Large quark mass limit in HQ effective field theory

$$\kappa \equiv \frac{\beta}{3}\sum_{i=1}^{3}\lim_{\omega\to 0}\left[\lim_{M\to\infty}\frac{M^2}{\chi_q}\int_{-\infty}^{\infty}{\rm d}t\; e^{i\omega(t-t')}\int {\rm d}^3\vec{x}\,\left\langle \frac{1}{2}\{\mathcal{F}^i(t,\vec{x}),\mathcal{F}^i(0,\vec{0})\}\right\rangle\right]$$

J. Casalderrey-Solana and D. Teaney, PRD 74, 085012

S. Caron-Huot et al., JHEP 0904 (2009) 053

A. Bouttefeux, M. Laine, JHEP 12 (2020) 150

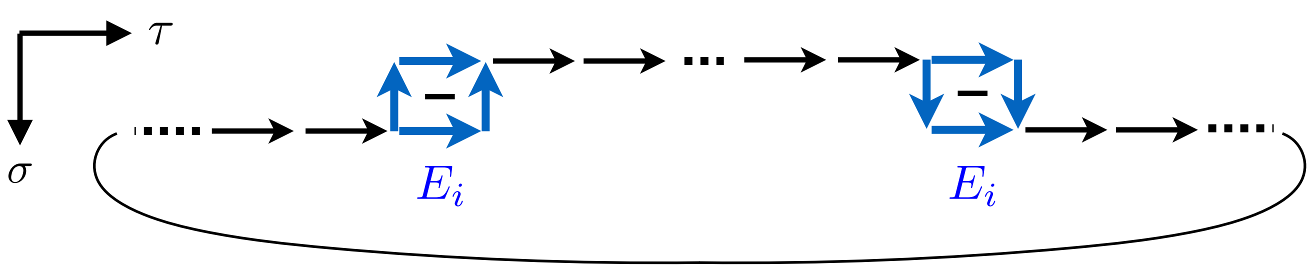
$$\partial_t \mathbf{p} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = \mathbf{F}$$

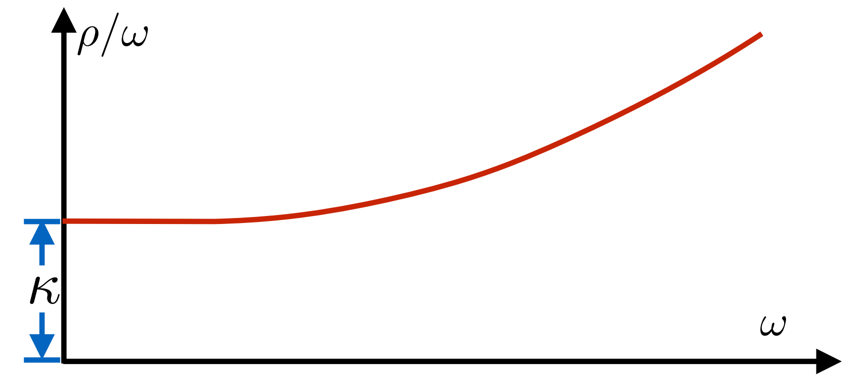
HQ momentum diffusion on the lattice

$$\kappa = \kappa_E + \frac{2}{3}\langle \mathbf{v^2} \rangle \kappa_{\mathbf{B}}$$

$$\frac{1}{2\pi T D} = \frac{\kappa}{4\pi T^3} = \frac{1}{2\pi T^2} \lim_{\omega \to 0} \frac{\rho(\omega)}{\omega}$$

Color-electric field correlation function

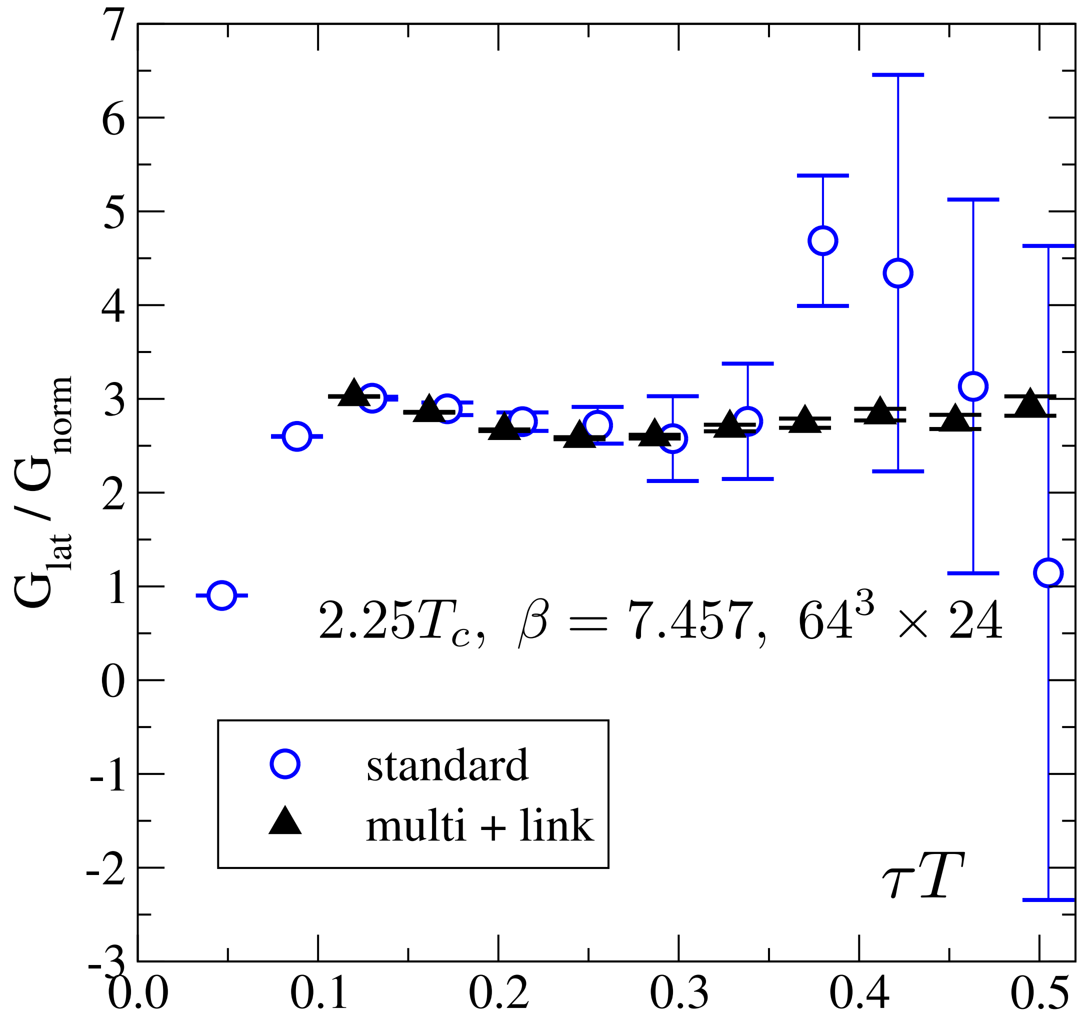




$$G(\tau, T) = \int \frac{d\omega}{\pi} \frac{\cosh(\omega(\tau T - 1/2)/T)}{\sinh(\omega/(2T))} \rho(\omega, T)$$

- Gluoinc correlators: cheaper to measure on the lattice
- Simple spectral function: absence of shape transport peak and resonances

Previous studies using multi-level algorithm



[A. Francis, et al, PRD92 (2015)116003]

ML: independent updates in each sub-lattice

[M. Luscher and P. Weisz, JHEP 09 (2001) 010]

A long history of lattice calculations, but all using multi-level:

[S. Caron-Huot, et al., JHEP 04 (2009) 053]
[D. Banerjee, et al., P.R.D 85 (2012) 014510]
[A. Francis, et al., PRD92 (2015)116003]
[D. Banerjee, et al., JHEP 08 (2022) 128]
[D. Banerjee, et al., Nucl.Phys.A.2023.122721]
[N. Brambilla, et al., PRD107 (2023) 054508]

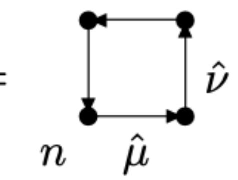
- Multi-level algorithm reduces noise in correlators
- But only applicable in quenched approximation

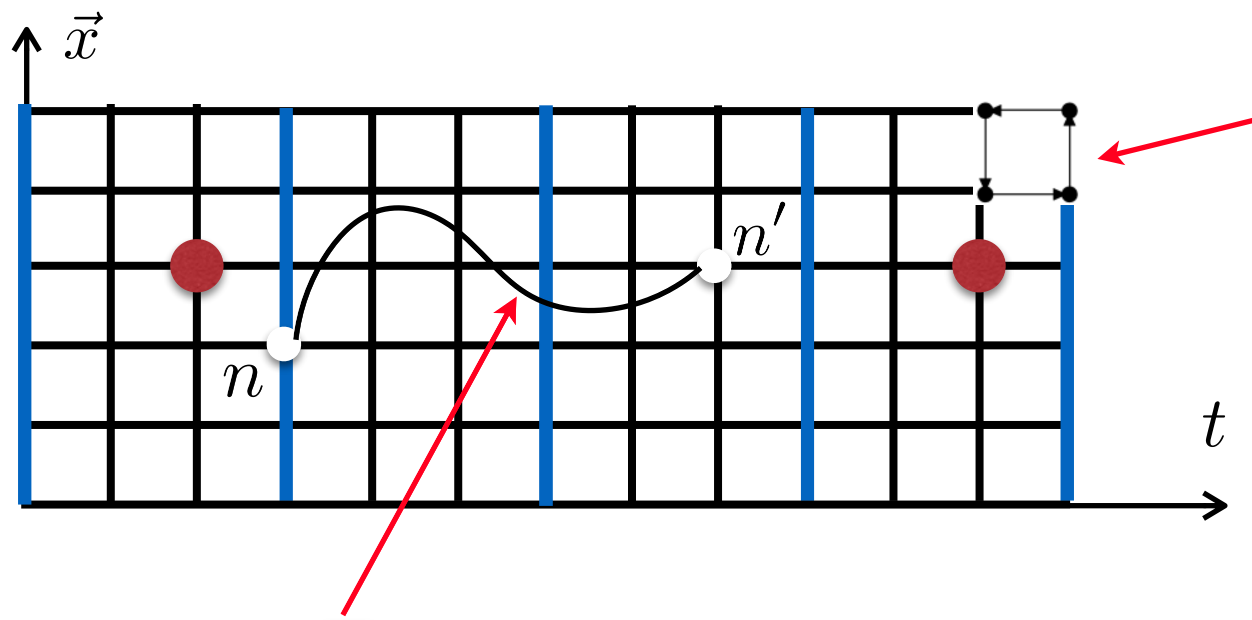
Breaking down of Multi-level algorithm in QCD

$$\mathcal{Z}(V,T) = \int [dU][d\psi][d\bar{\psi}]e^{-S_G[U]-S_F[U,\psi,\bar{\psi}]}$$

$$S_G[U] = \frac{1}{2g^2}\sum_{n,\mu,\nu} 2\mathrm{Tr}[1 - P_{\mu\nu}(n)] \qquad S_F[U,\psi,\bar{\psi}] = \bar{\psi}M_q[U]\psi$$

Action **local in quenched QCD** (sum of plaq.): $P_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U^\dagger_\mu(n+\hat{\nu})U^\dagger_\nu(n) =$





Action **non-local in full QCD** (connection between any two sites): $M_q(n,n';i,j)[U] = \hat{m}_q\delta_{n,n'}\delta_{ij} + \frac{1}{2}\sum_\mu \eta_\mu(n)\left((U_\mu(n))_{i,j}\delta_{n',n+\hat{\mu}} - (U^\dagger_\mu(n))_{i,j}\delta_{n,n'+\hat{\mu}}\right)$

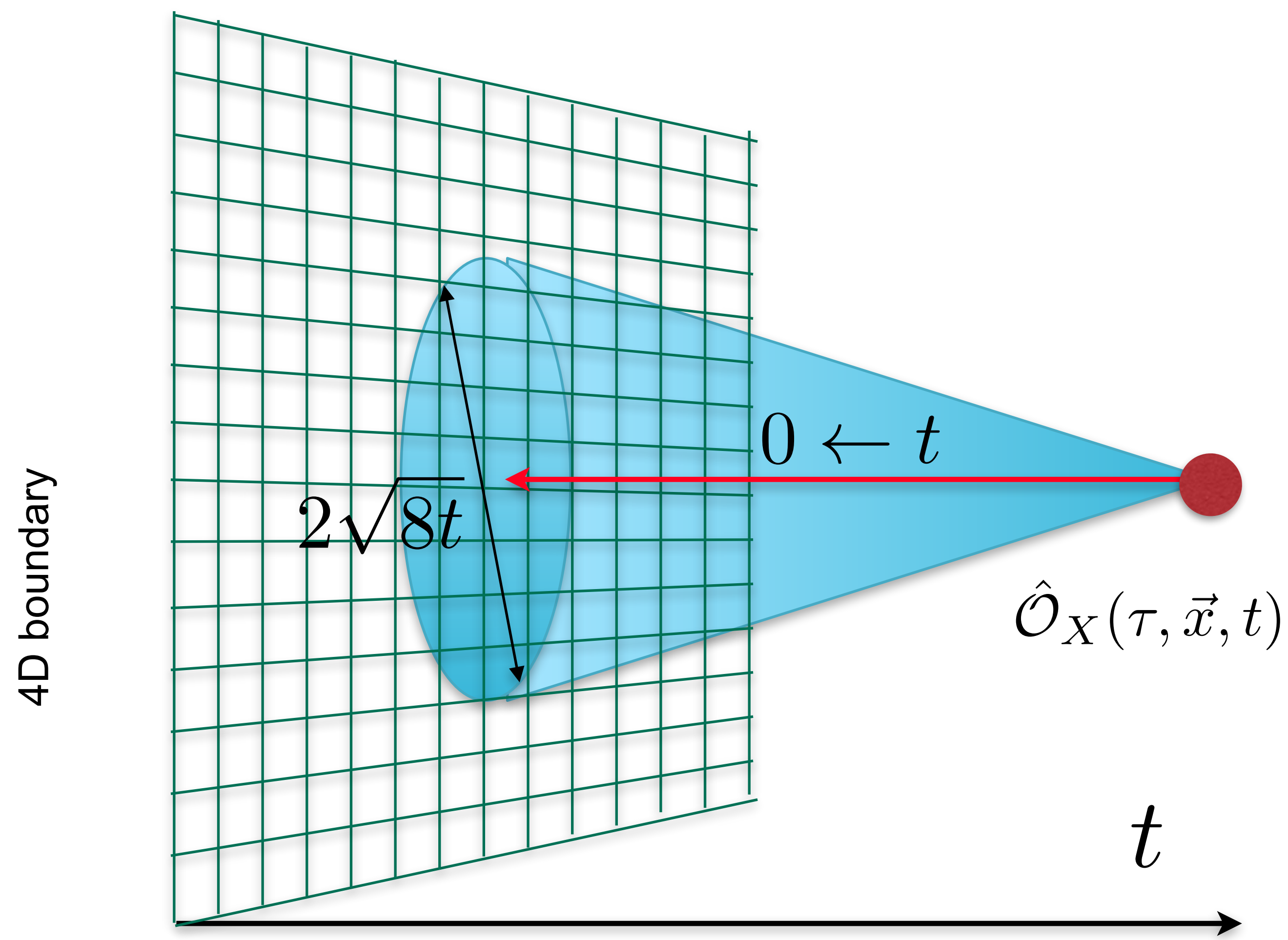
Gradient flow — the only way towards QCD

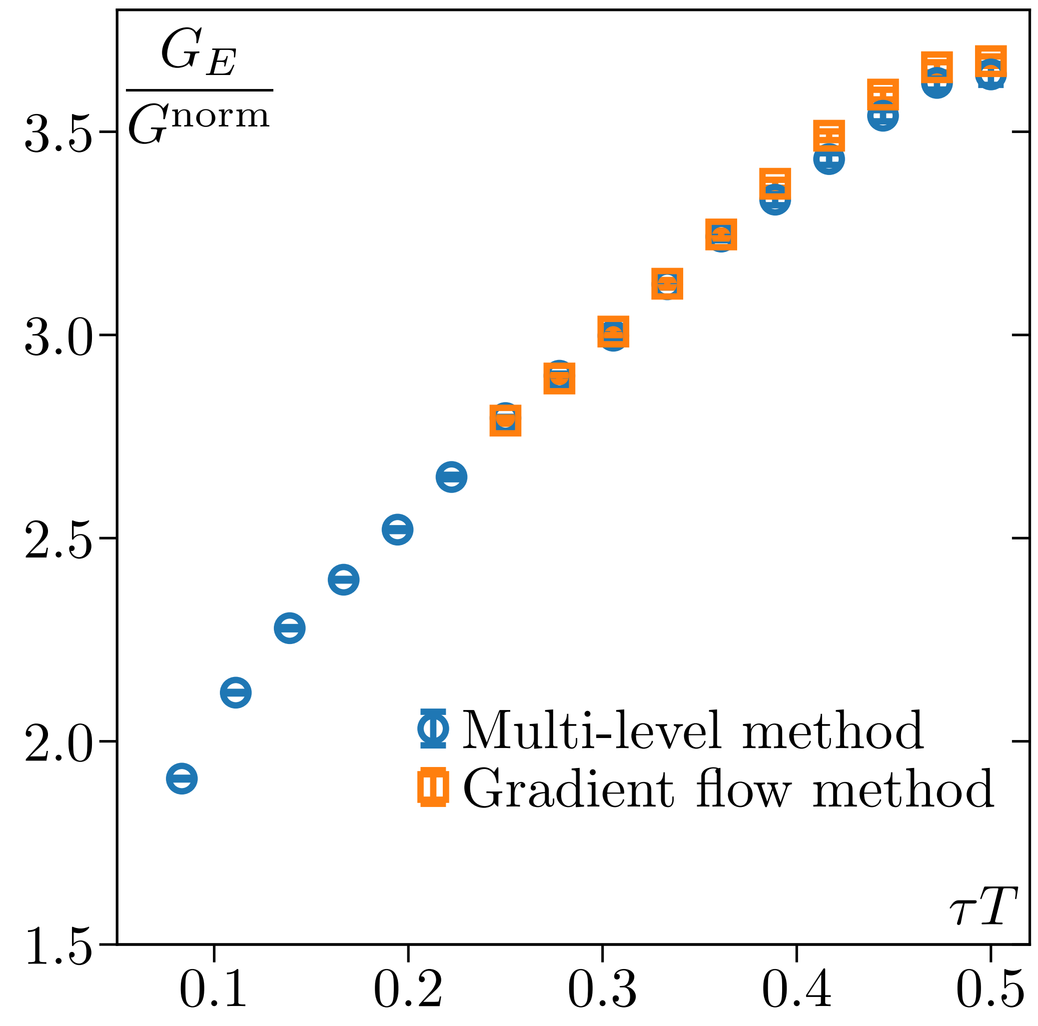
Evolve fields according to diffusion equations:

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} \sim -\frac{\delta S_G[B_\mu(x,t)]}{\delta B_\mu(x,t)} \sim D_\nu G_{\nu\mu}(x,t)$$

Luscher & Weisz, JHEP1102(2011)051
Narayanan & Neuberger, JHEP0603(2006)064

[LA, AME, OK, LM, GDM, HTS, PRD 103 (2021) 1, 014511]

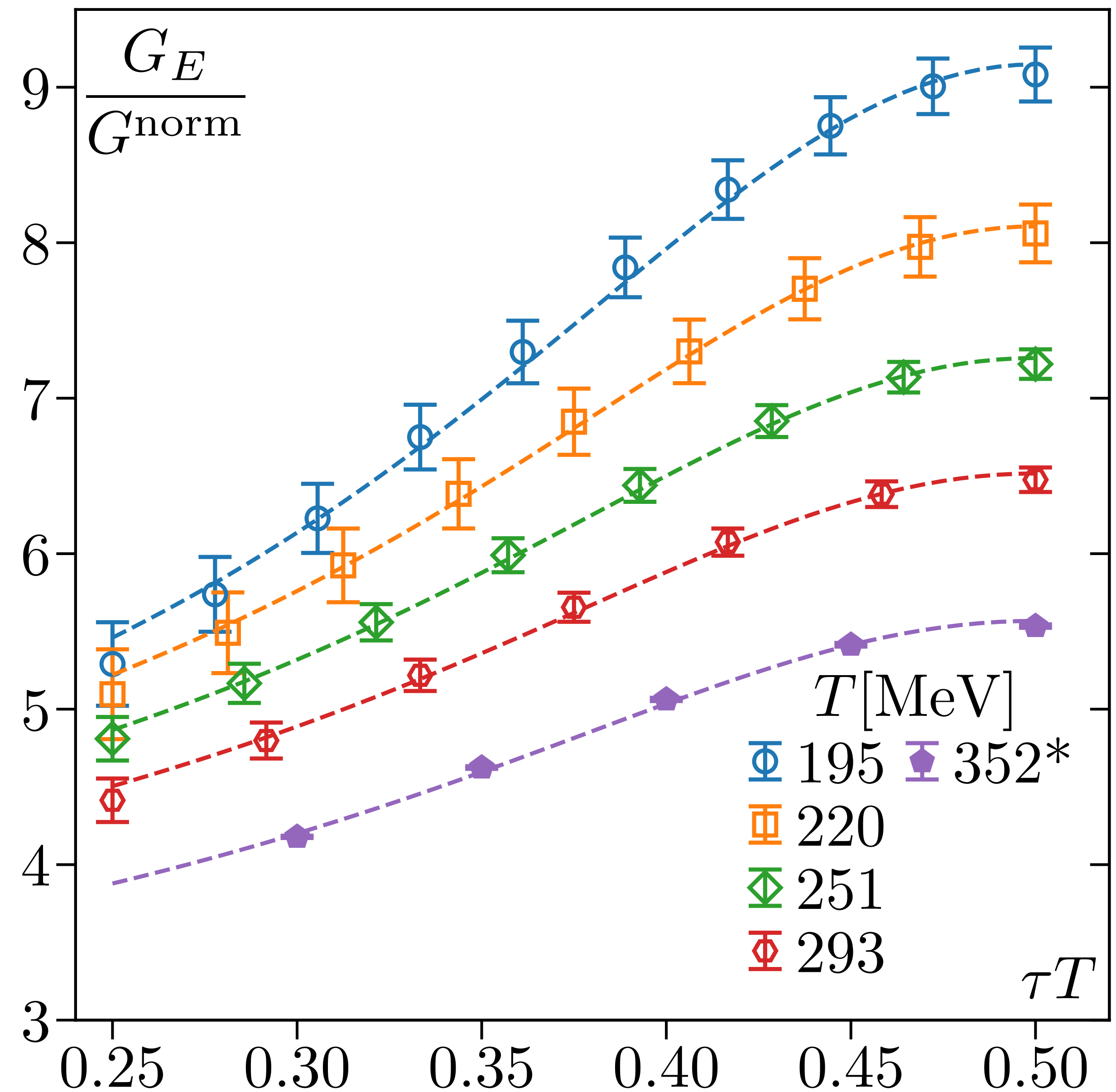




- Smear gauge field + quark field simultaneously
- Well-defined renormalization
- Consistent quenched results from ML & GF

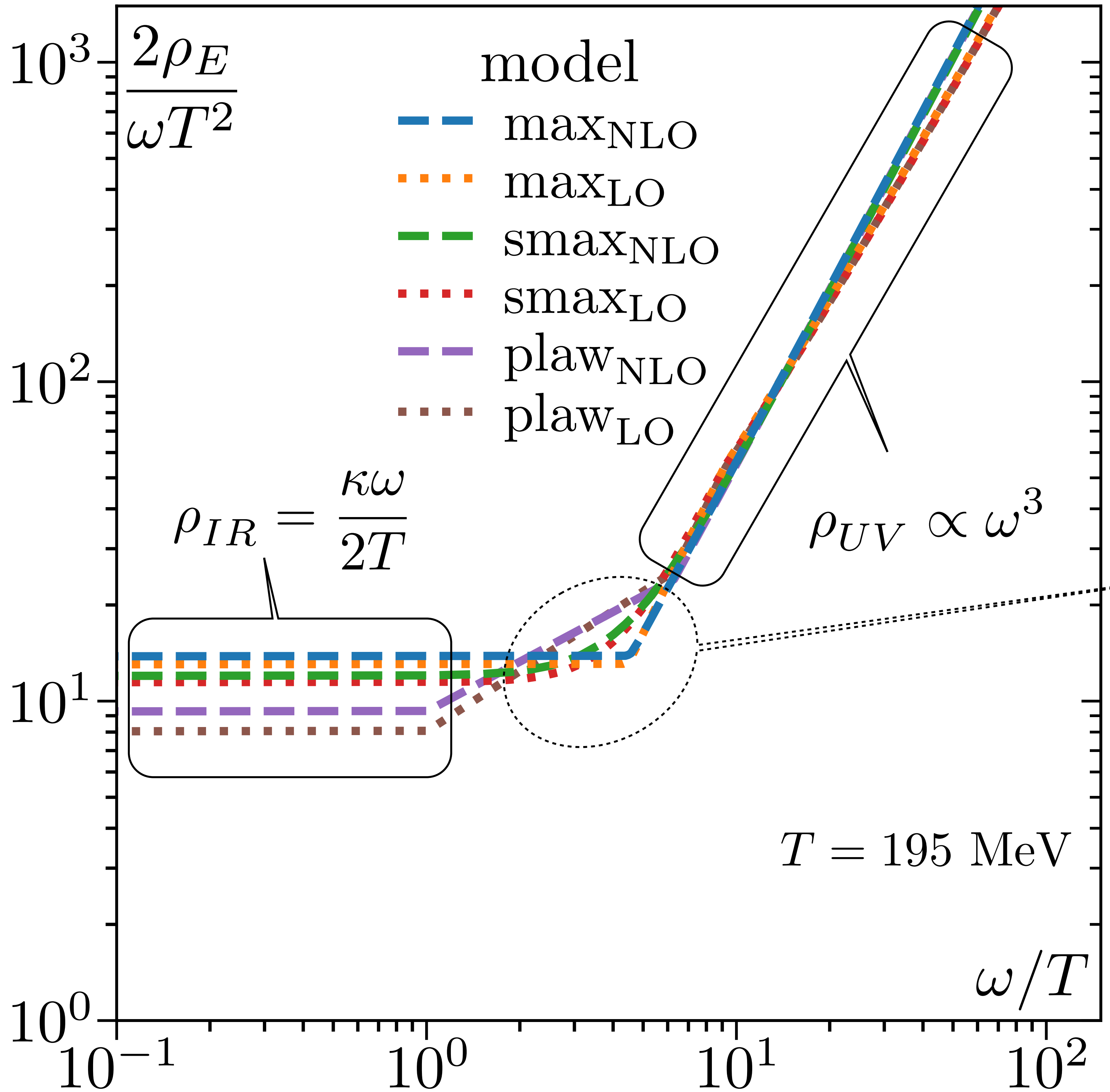
Extension to QCD: EE correlators

First QCD calculation of κ_E !   [LA, OK, RL, SM, PP, **HTS**, SS, PRL 130 (2023) 23, 231902]



- $N_f = 2 + 1$, HISQ
- 195 MeV $\leq T \leq$ 352 MeV
- Valence pion mass 320 MeV

Extension to QCD: spectra analysis

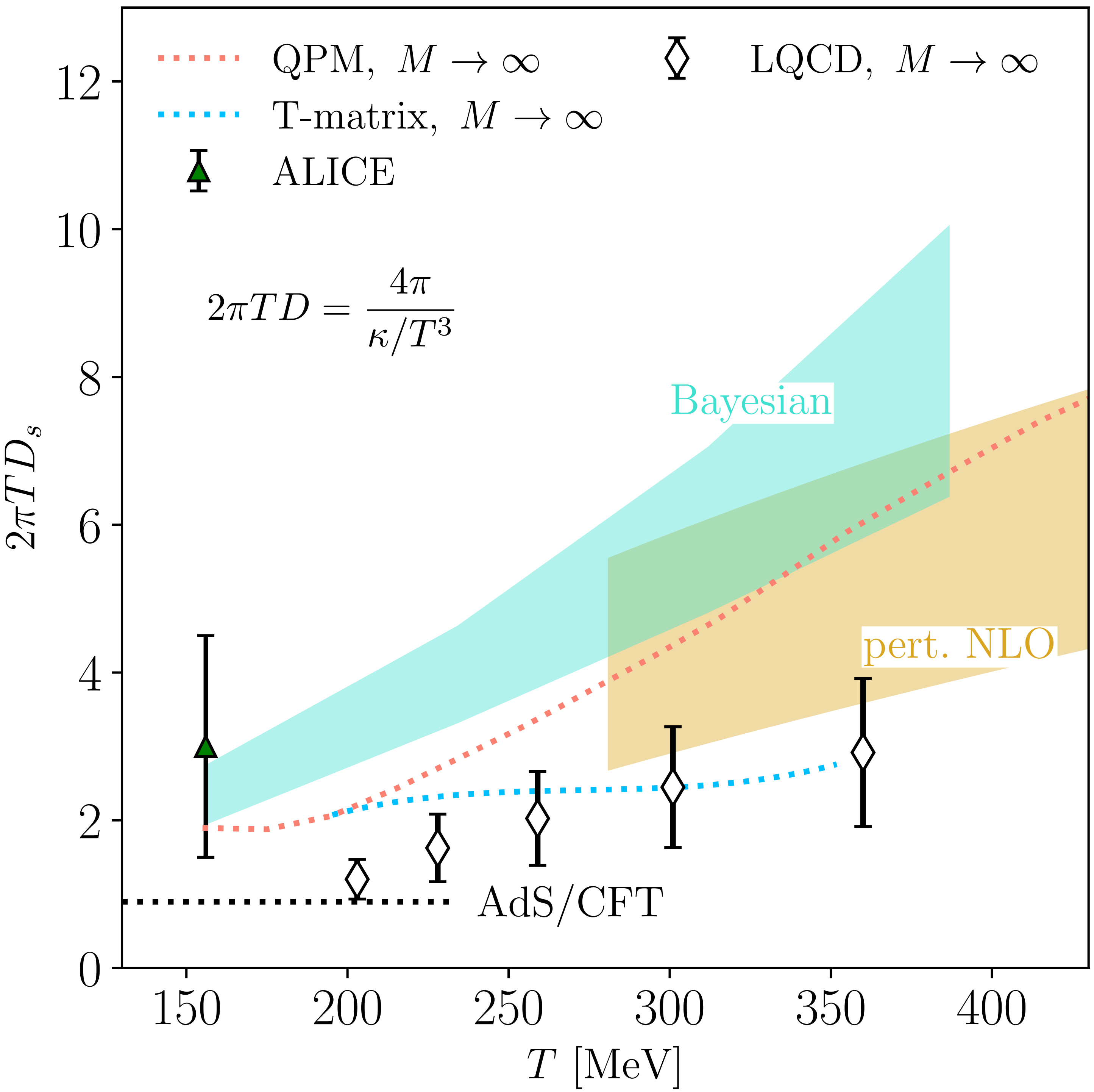


$$G(\tau, T) = \int \frac{d\omega}{\pi} K(\omega, \tau, T)\rho(\omega, T)$$

$$\begin{cases} \rho_{\max} \equiv \max(\phi_{\rm IR}, \phi_{\rm UV}) \\ \rho_{\rm smax} \equiv \sqrt{\phi_{\rm IR}^2 + \phi_{\rm UV}^2} \\ \rho_{\rm plaw} \equiv \begin{cases} \phi_{\rm IR} & \omega \le \omega_{\rm IR}, \\ a\omega^b \quad \text{for} & \omega_{\rm IR} < \omega < \omega_{\rm UV}, \\ \phi_{\rm UV} & \omega \ge \omega_{\rm UV}, \end{cases} \end{cases}$$

[LA, OK, RL, SM, PP, **HTS**, SS, PRL 130 (2023) 23, 231902]

Infinite heavy quark diffusion coefficient



- Agree with AdS/CFT at $\sim T_c$
- Close to the phenomenological extraction using the ALICE data
- Agree with T-matrix estimate at moderate and high temperature
- Agree with NLO perturbative estimate at high temperature
- Lower than Bayesian&QPM estimate
- Mild temperature dependence

[LA, OK, RL, SM, PP, **HTS**, SS, PRL 130 (2023) 23, 231902]

Finite mass correction

Physical charm & bottom quark not infinitely heavy!

$$M_c : \sim 1.3 \text{ GeV}$$
$$M_b : \sim 4.5 \text{ GeV}$$
$$\kappa_E : \ M_Q \to \infty$$

D. Guazzini, et al., JHEP 10 (2007) 081

$$\langle \mathcal{F}(t')\mathcal{F}(t)\rangle = q^2\left\{\langle E_i(t')E_j(t)\rangle + \frac{1}{3}\langle \mathbf{v}^2\rangle\langle \delta_{ij}B_k(t')B_k(t) - B_j(t')B_i(t)\rangle\right\}$$

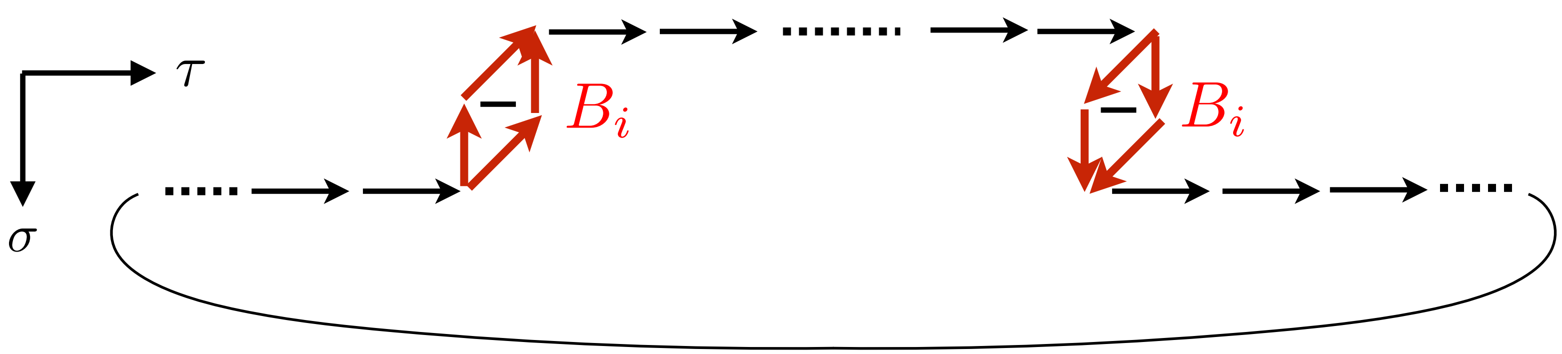
Infinite heavy limit

Finite mass correction

$$\kappa = \kappa_E + \frac{2}{3}\langle \mathbf{v^2}\rangle \kappa_{\mathbf{B}}$$

$$2/3 \cdot \langle \mathbf{v}^2\rangle_{\text{charm}} : \ 18\% \sim 30\%$$
$$2/3 \cdot \langle \mathbf{v}^2\rangle_{\text{bottom}} : \ 7\% \sim 13\%$$

$$D_s = \frac{2T^2}{\kappa} \Longrightarrow D_s = \frac{2T^2}{\kappa} \cdot \frac{\langle p^2\rangle}{3MT}$$



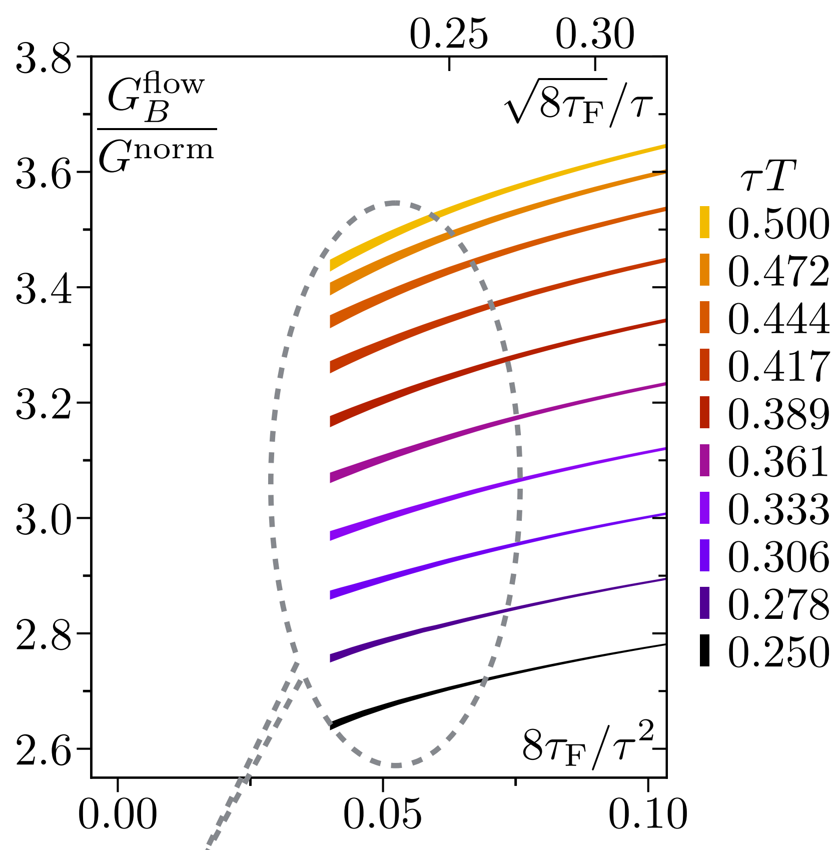
Color-magnetic field correlation function

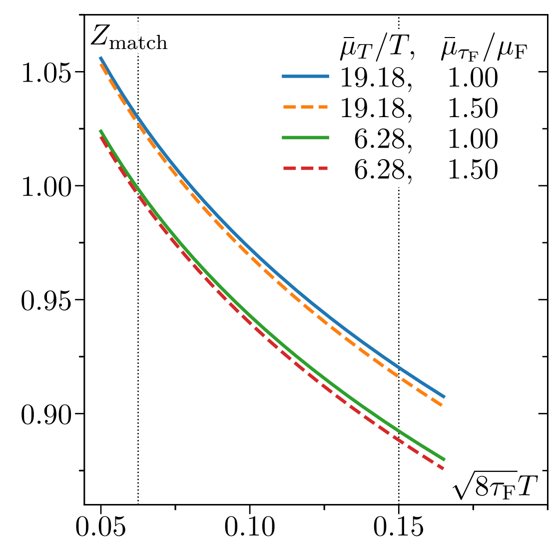
A. Bouttefeux, M. Laine, JHEP 12 (2020) 150

Intractabilities in B-field correlators

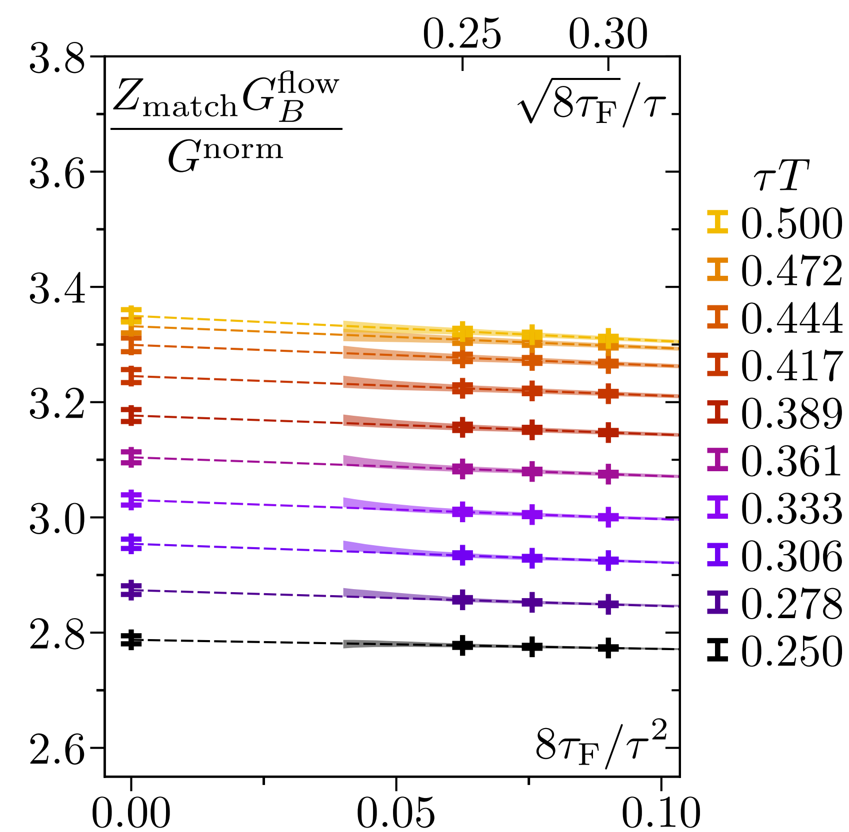
- Anomalous dimension in B-field
- Log divergence in flow time

A. Bouttefeux and M. Laine, JHEP 12 (2020) 150
M. Laine, JHEP 06(2021)139





$$G_B^{\rm phys.} = Z_{\rm match} G_B^{\rm flow}(\tau_F)$$



breaks down of linear $\tau_F \to 0$ extrapolation

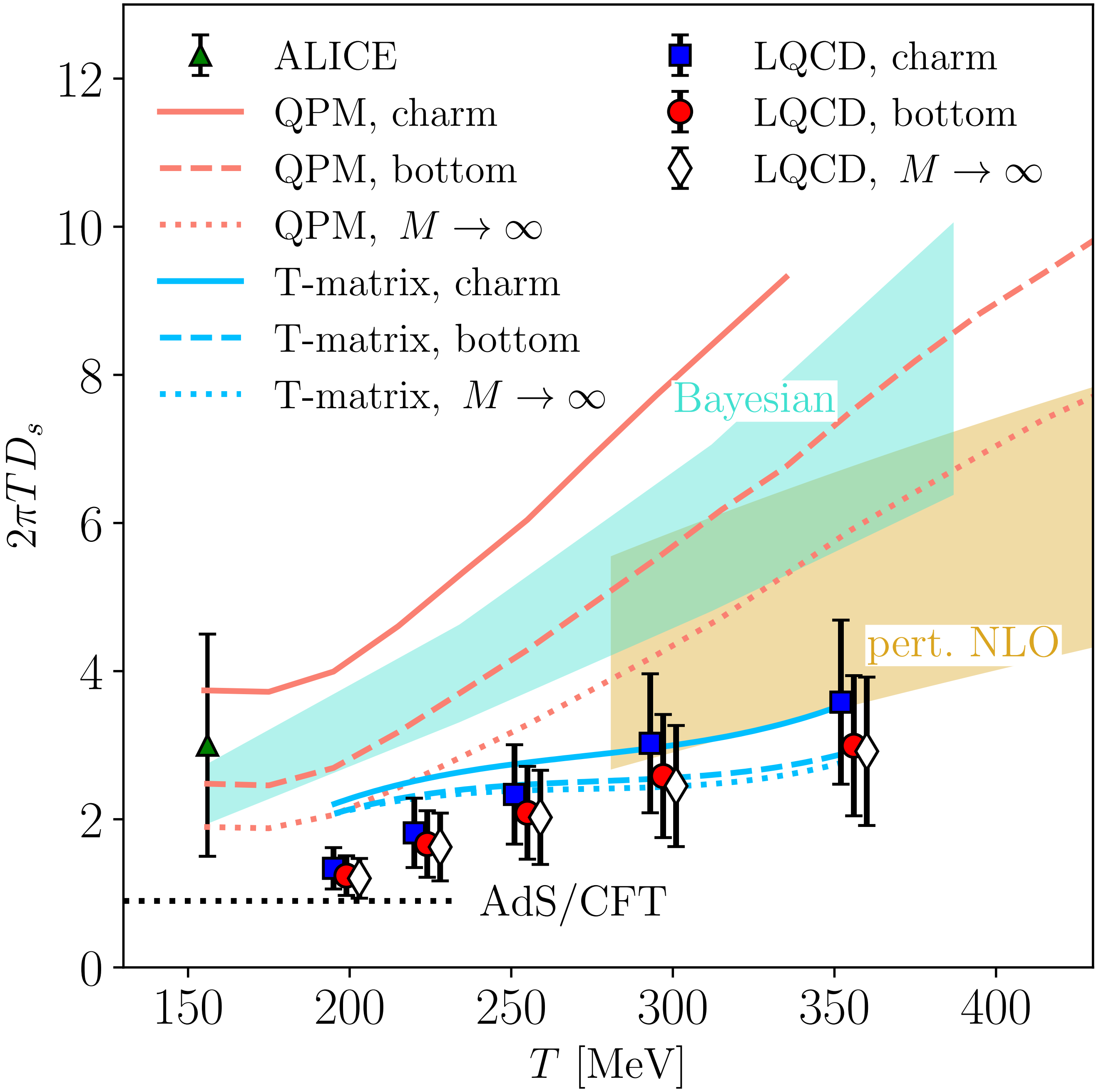
[LA, DC, OK, GDM, **HTS**, PRD 109 (2024) 11, 114505]

Renormalization issue solved by a matching factor:

$$\ln Z_{\rm match} = \int_{\bar{\mu}_T^2}^{\bar{\mu}_{\tau_F}^2} \gamma_0 g^2_{\overline{\rm MS}}(\bar{\mu}) \frac{d\bar{\mu}^2}{\bar{\mu}^2} + \gamma_0 g^2_{\overline{\rm MS}}(\bar{\mu}_T)\left[\ln\frac{\bar{\mu}_T^2}{(4\pi T)^2} - 2 + 2\gamma_E\right] - \gamma_0 g^2_{\overline{\rm MS}}(\bar{\mu}_{\tau_F})\left[\ln\frac{\bar{\mu}_{\tau_F}^2}{4\,\mu_F^2} + \gamma_E\right]$$

Charm and Bottom quark diffusion

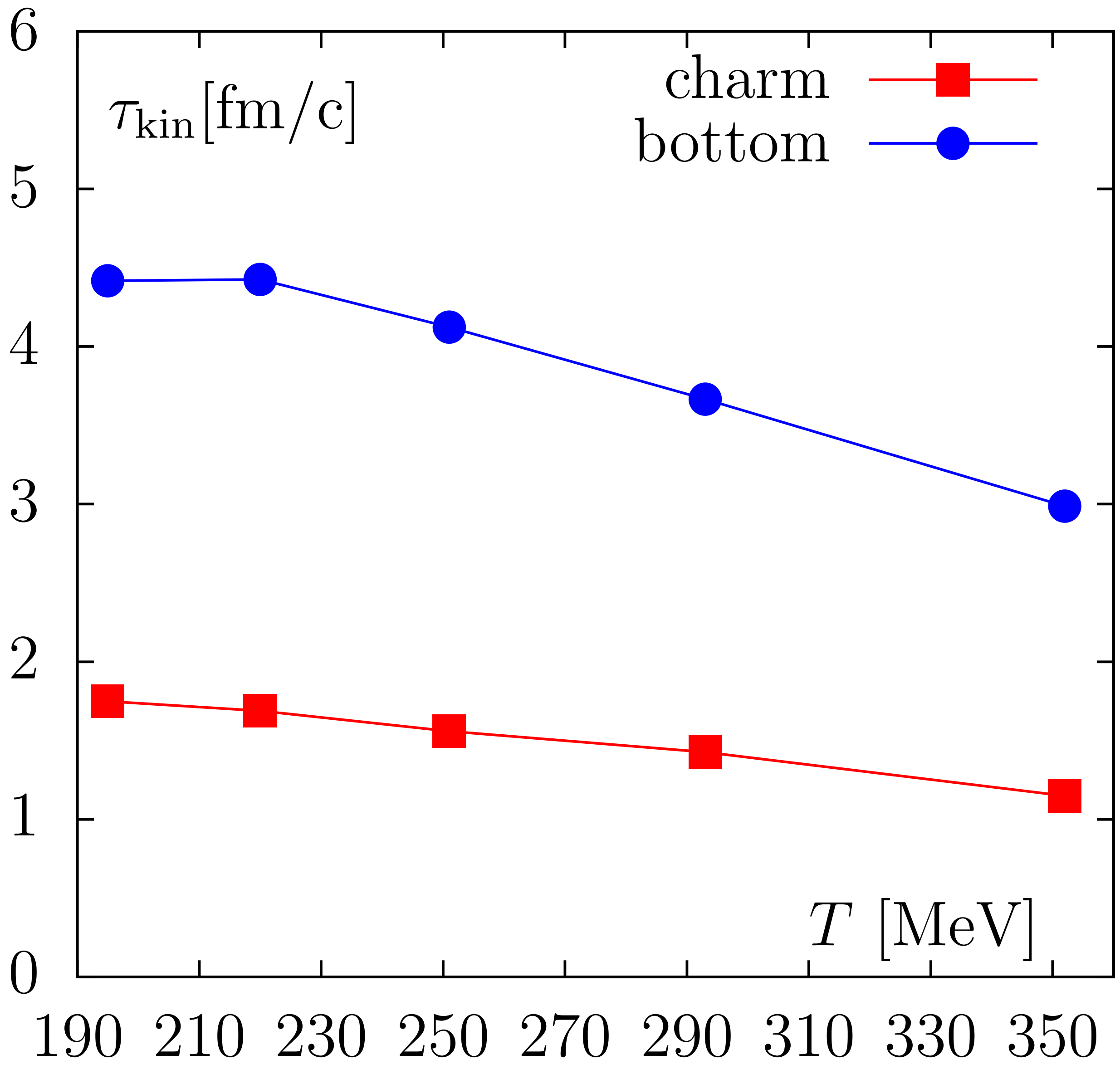
First QCD determination of charm&bottom quark diffusion!



- HQ mass dependence of HQ diffusion: mild
- Universal change pattern with quark mass
- Weak quark mass dependence in LQCD & T-matrix
- Weaker than quasi-particle model (QPM) calculations

[LA, DC, OK, RL, GDM, SM, PP, **HTS**, SS, PRL 132 (2024) 5, 051902]

Equilibration time of charm&bottom quark

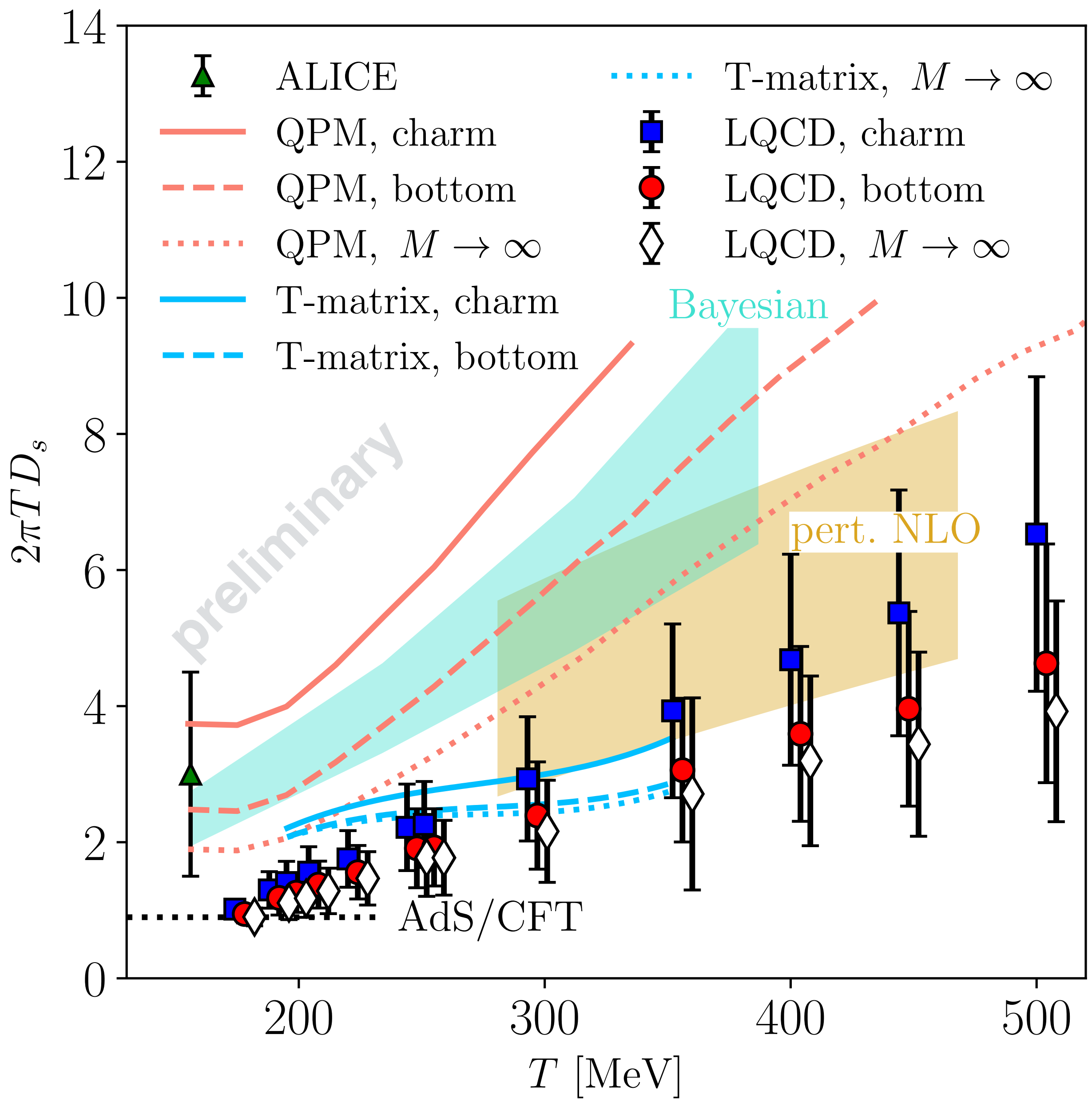


$$\tau_{\rm kin} = \frac{1}{\eta_D} = \frac{1}{\kappa/T^3}\left(\frac{T_c}{T}\right)^2\left(\frac{M}{1.5\ \mathrm{GeV}}\right)\frac{3\ \mathrm{GeV}}{T_c^2}$$

- Lattice provides short equilibration time for charm&bottom quark
- Rapid equilibrium <—> QGP is near perfect fluid
- Lattice determination of $\tau_{\rm kin}^{\rm charm}$ favors the experimental estimate (~1 fm/c for all)

[LA, DC, OK, RL, GDM, SM, PP, **HTS**, SS, PRL 132 (2024) 5, 051902]

HQ diffusion at the physical point



[JDG, SM, PP, **HTS**, JHW, work in progress]

$195 \text{ MeV} \leq T \leq 352 \text{ MeV}$ $\quad m_\pi \sim 320\text{MeV}$

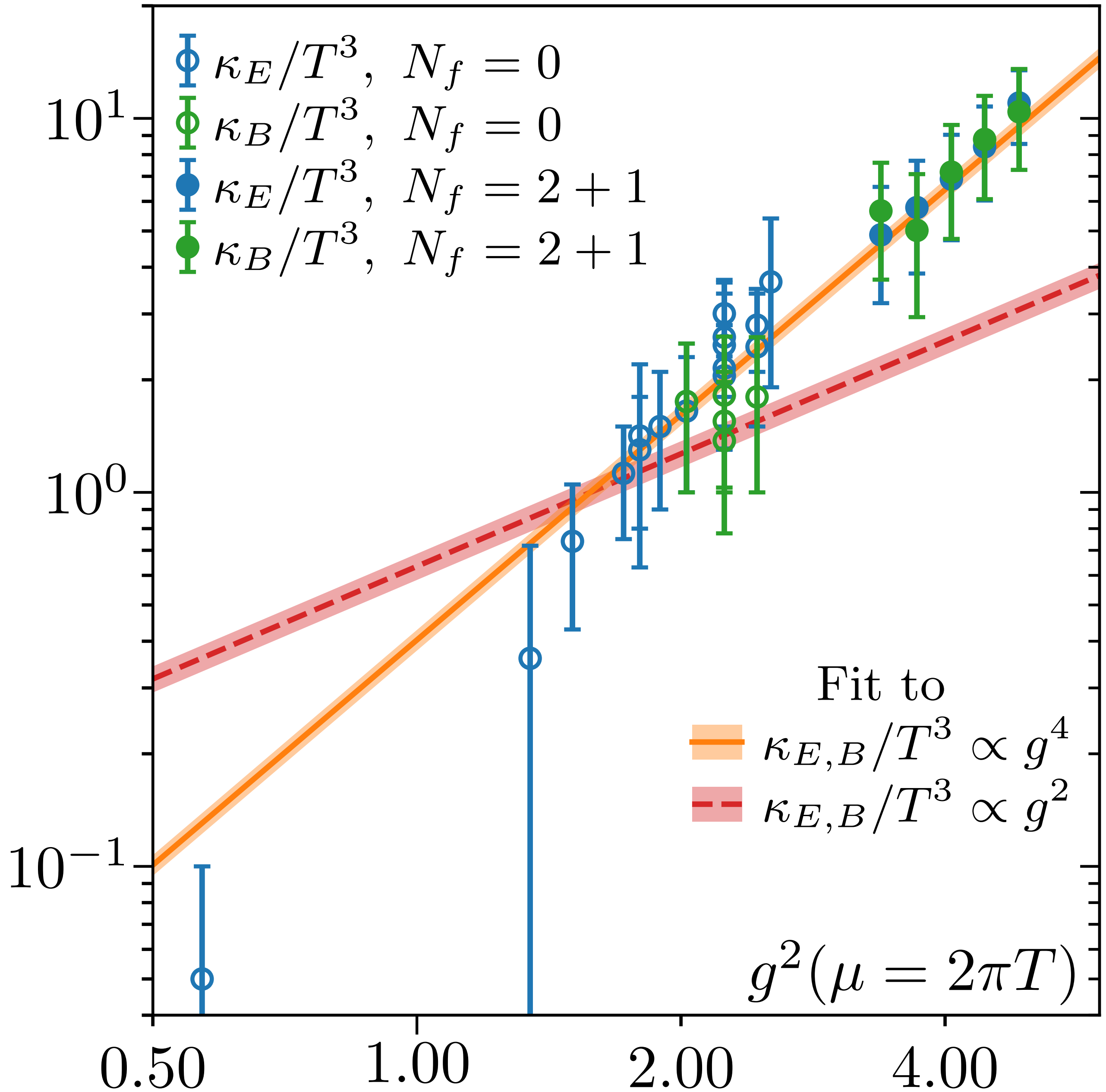
↓ ↓

$174 \text{ MeV} \leq T \leq 500 \text{ MeV}$ $\quad m_\pi \sim 160\text{MeV}$

Increasing stat. for $T = 153 \text{ MeV}, 164 \text{ MeV}$

- Physical pion mass
- Wide temperature range
- Consistent observation as previous studies
- Almost invisible light quark mass dependence

κ_E V.S. κ_B



[LA, DC, OK, GDM, **HTS**, PRD 109 (2024) 11, 114505]

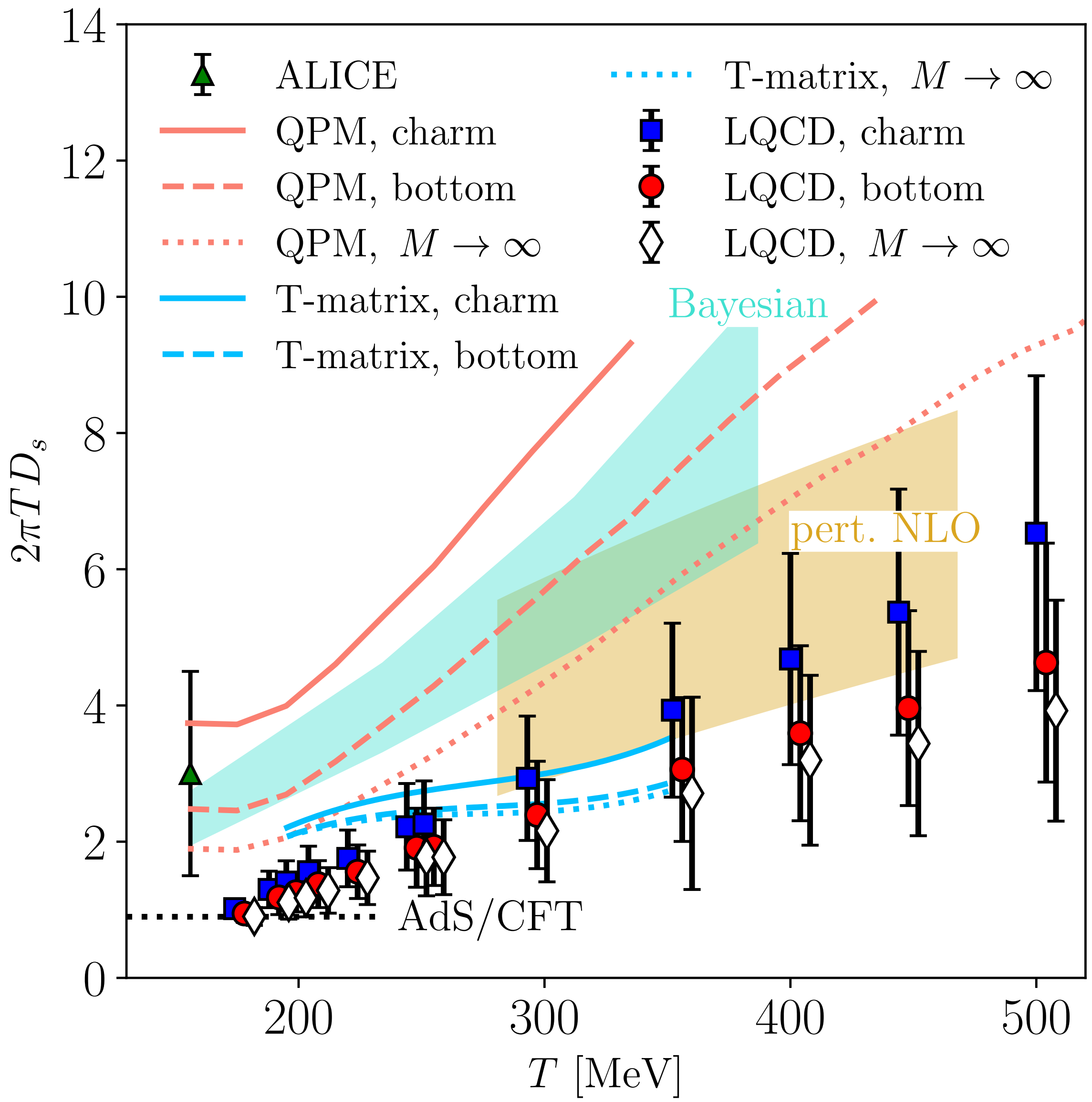
Quenched results from:

A. Francis, et al., PRD92, 116003
B. L. Altenkort, et al., PRD103,014511
D. Banerjee, et al., Nucl.Phys.A.2023.122721
D. Banerjee, et al., JHEP 08 (2022) 128
N. Brambilla, et al., PRD107, 054508

==> S. Caron-Huot and G. D. Moore, PRL. 100, 052301 (2008)

- Similar magnitude for κ_E and κ_B in full QCD & quenched
- Smooth connection between quenched and full QCD in temperature
- Lattice results confirms the form suggested by pert. computations

Summary



- Heavy quark diffusion coefficient is an important parameter in understanding the HICs
- Nonperturbative determination of HQ diffusion is a non-trivial task
- Remarkable progress has been achieved thanks to the gradient flow method
- Fast equilibration of charm&bottom quark has been found
- HQ diffusion calculation is being extended to the physical point
- Lattice calculation of HQ diffusion is switching to a precision study phase

Backup: identify the heavy quark diffusion

Phenomenological diffusion picture of classical particle

Equilibrium -> Relaxation -> Equilibrium

$$\langle A(\mathbf{x})\rangle_{\text{eq}} = 0 \qquad \partial_t \langle A(\mathbf{x},t)\rangle = D\nabla^2 \langle A(\mathbf{x},t)\rangle$$

Solution:

$$\langle A(\mathbf{k},\omega)\rangle = \frac{i}{\omega + iD\mathbf{k}^2}\langle A(\mathbf{k},t=0)\rangle$$

Linear response theory

Perturbation to Hamiltonian:

$$H(t) = H_0 - \int \mathrm{d}\mathbf{x}\; A(\mathbf{x})h(x)e^{\epsilon t}\Theta(-t)$$

Solution:

$$\frac{\partial}{\partial t}\Big(\delta\langle A(\mathbf{k},t=0)\rangle\Big) = -\frac{G_R(\mathbf{k},t)}{\chi_q(\mathbf{k})}\delta\langle A(\mathbf{k},0)\rangle$$

Kubo formula: $$G_R(\mathbf{k},\omega) = \frac{iD\mathbf{k}^2}{\omega + iD\mathbf{k}^2}\chi_q(\mathbf{k}) \sim \rho(\vec{k},\omega)$$

$$A \to J^\mu = \bar{\psi}\gamma^\mu\psi$$

$$D = \frac{1}{3\chi_q}\lim_{\omega\to 0}\sum_{i=1}^{3}\frac{\rho^{ii}(\omega)}{\omega}$$

Backup: full QCD setup

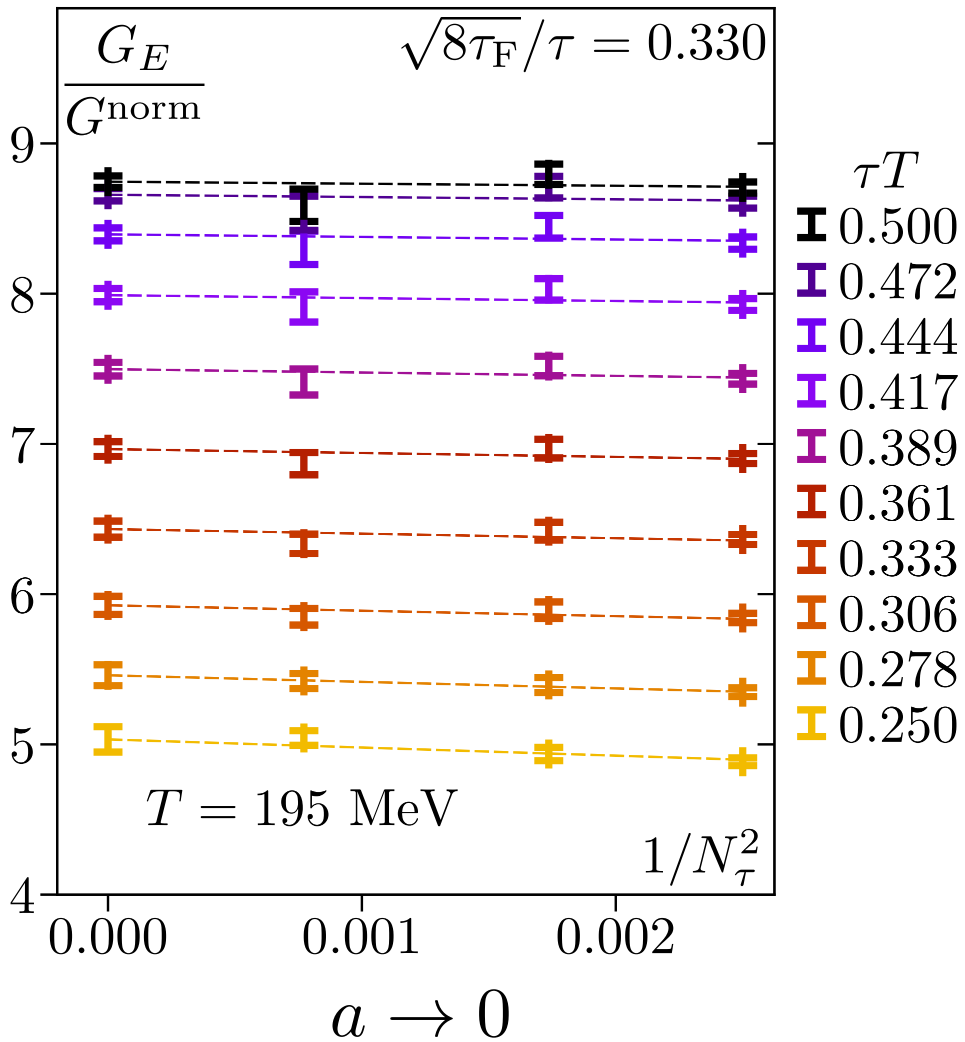
$$N_f = 2+1,\ \text{HISQ},\ m_\pi = 320\ \text{MeV}$$

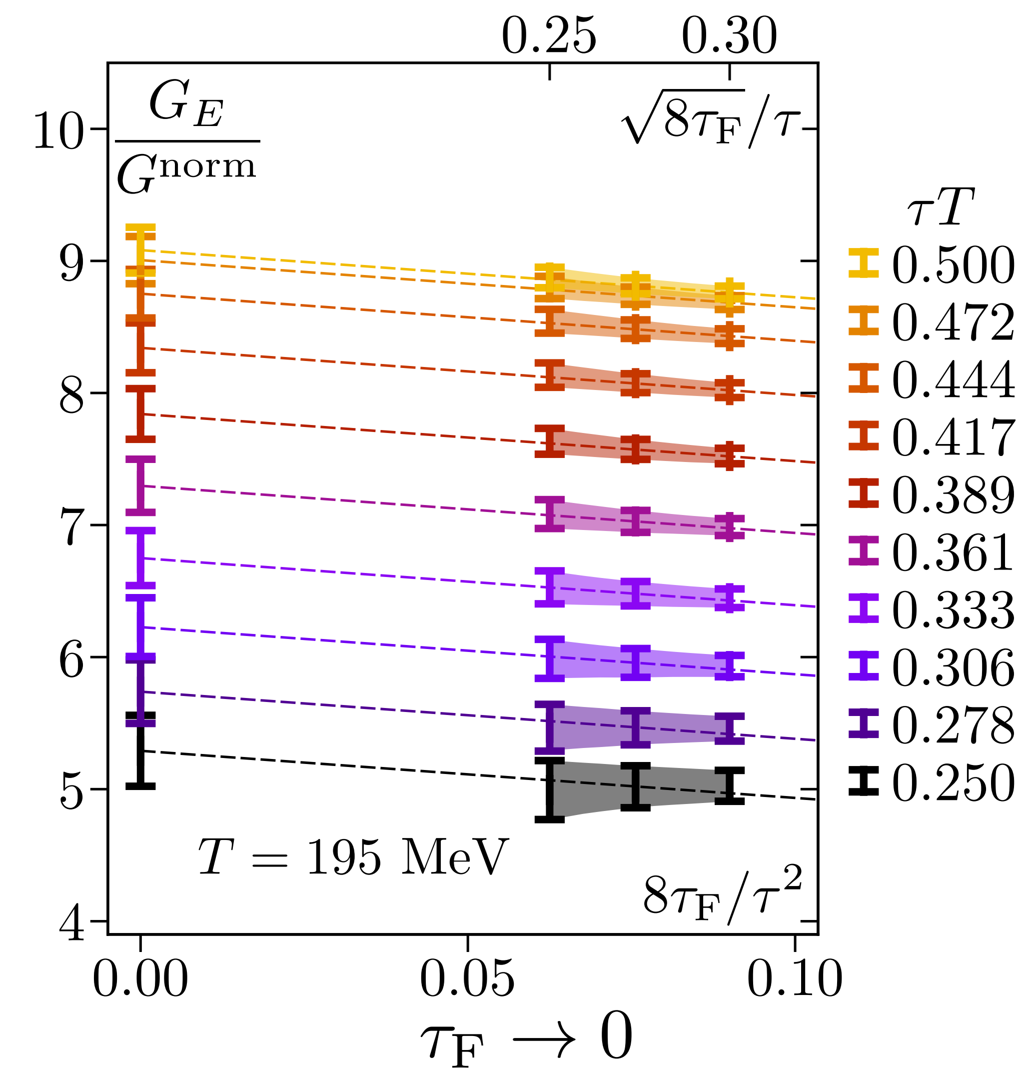
T [MeV]	β	am_s	am_l	N_σ	N_τ	# conf.
195	7.570	0.01973	0.003946	64	20	5899
	7.777	0.01601	0.003202	64	24	3435
	8.249	0.01011	0.002022	96	36	2256
220	7.704	0.01723	0.003446	64	20	7923
	7.913	0.01400	0.002800	64	24	2715
	8.249	0.01011	0.002022	96	32	912
251	7.857	0.01479	0.002958	64	20	6786
	8.068	0.01204	0.002408	64	24	5325
	8.249	0.01011	0.002022	96	28	1680
293	8.036	0.01241	0.002482	64	20	6534
	8.147	0.01115	0.002230	64	22	9101
	8.249	0.01011	0.002022	96	24	688
352	8.249	0.01011	0.002022	96	20	2488

- Wide temperature range
- Different lattice spacings
- Large lattices towards thermodynamic limit

Backup: double extrapolation

First QCD calculation of kappa (u+d+s quarks in the sea)





- Extrapolation Ansatz describes lattice data well

[LA, OK, RL, SM, PP, **HTS**, SS, PRL 130 (2023) 23, 231902]

Backup: anomalous dimension of B-field

- Anomalous dimension in MSbar-scheme

$$Z_B = 1 + \frac{g^2 C_A}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2\ln\left(\frac{\bar{\mu}e^{\gamma_E}}{4\pi T}\right) - 2\right] + \mathcal{O}(g^4)$$

- Gradient flow-scheme —> MSbar-scheme —> physical values
- Scale dependence must go for "WeWant" and $\langle BB \rangle_{\tau_F}$

$$Z^2 = \left(1 - 2\frac{g^2 C_A}{16\pi^2}\ln(\mu^2\tau_F)\right)\left(1 + 2K\frac{g^2 C_A}{16\pi^2}\right) \equiv Z_f^2 Z_K^2$$

$$\mathrm{WeWant} = Z_B^2 \langle BB \rangle_{\mathrm{MS}}$$

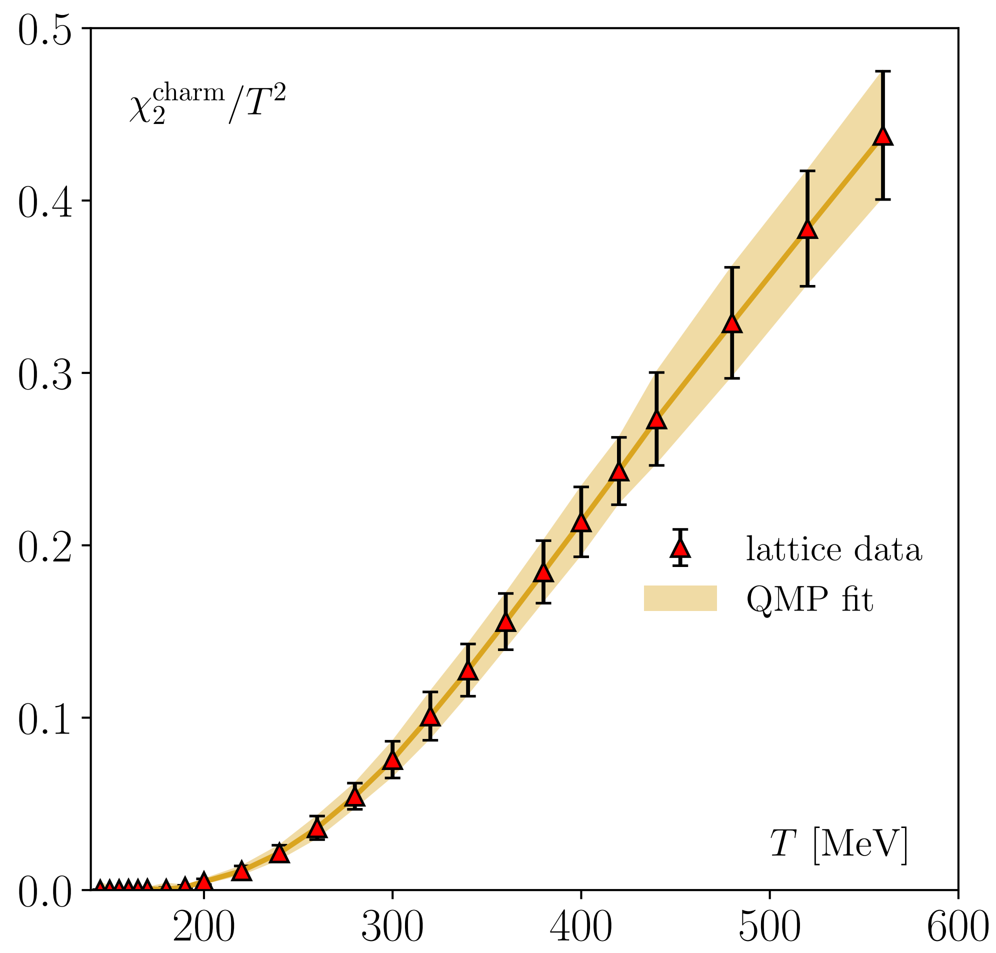
$$\langle BB \rangle_{\tau_F} \equiv Z^{-2} \langle BB \rangle_{\mathrm{MS}}$$

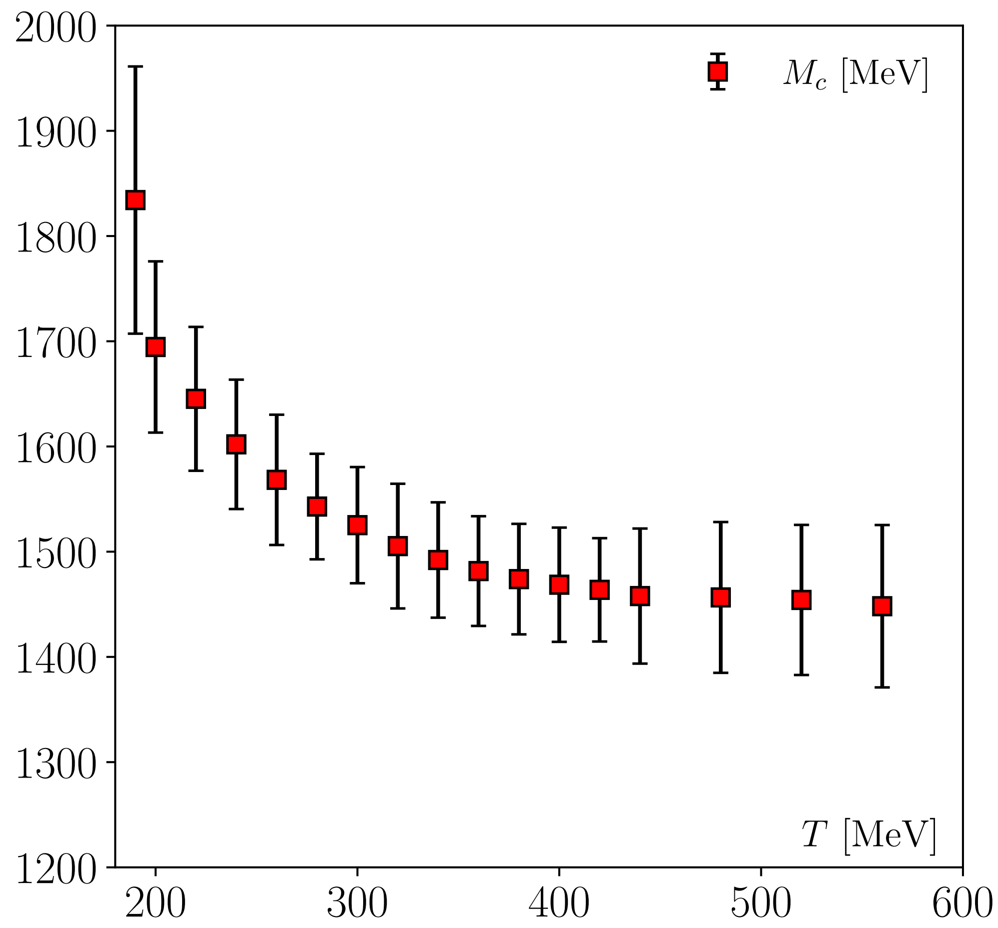
$$\mathrm{WeWant} = Z_B^2 Z^2 \langle BB \rangle_{\tau_F}$$

- Determination of the matching factor

$$\ln Z_{\mathrm{match}} = \int_{\bar{\mu}_T^2}^{\bar{\mu}_{\tau_F}^2} \gamma_0 g^2_{\overline{\mathrm{MS}}}(\bar{\mu})\frac{d\bar{\mu}^2}{\bar{\mu}^2} + \gamma_0 g^2_{\overline{\mathrm{MS}}}(\bar{\mu}_T)\left[\ln\frac{\bar{\mu}_T^2}{(4\pi T)^2} - 2 + 2\gamma_{\mathrm{E}}\right] - \gamma_0 g^2_{\overline{\mathrm{MS}}}(\bar{\mu}_{\tau_F})\left[\ln\frac{\bar{\mu}^2_{\tau_F}}{4\,\mu_F^2} + \gamma_{\mathrm{E}}\right]$$

Backup: T-dependent charm quark mass





[PRL 132 (2024) 5, 051902]

$$\frac{\chi_2^{\text{charm}}}{T^2} = \frac{4N_c}{(2\pi T)^3}\int d^3p\ e^{-E_p/T}$$

$$E_p^2(T) = m^2(T) + p^2$$

$\longrightarrow m(T)$

$$\langle v^2\rangle = \left(\int d^3p\ \frac{p^2}{E_p^2} e^{-E_p/T}\right) \Big/ \left(\int d^3p\ e^{-E_p/T}\right)$$

$$\langle p^2\rangle = \left(\int d^3p\ p^2 e^{-E_p/T}\right) \Big/ \left(\int d^3p\ e^{-E_p/T}\right)$$

$$\kappa = \kappa_E + \frac{2}{3}\langle \mathbf{v^2}\rangle \kappa_{\mathbf{B}}$$

$$D_s = \frac{2T^2}{\kappa}\cdot\frac{\langle p^2\rangle}{3MT}$$

Backup: scattering from various models