# Tau Polarization Discussion

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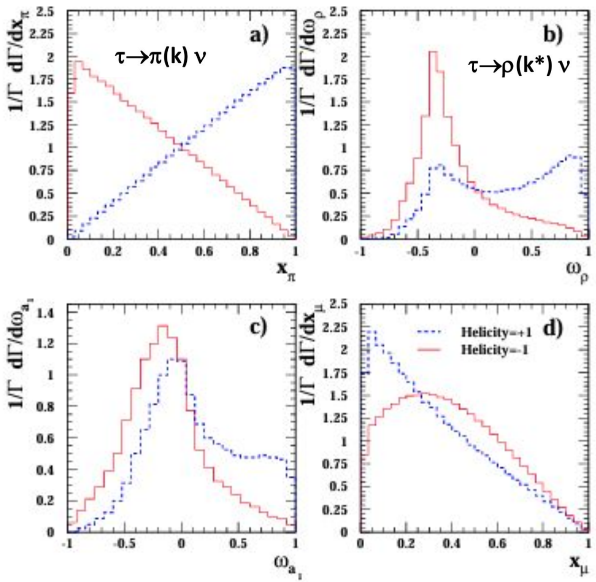
# Goal: Tau polarisation study using  full simulation




Z pole LEP+SLD paper: Physics Reports 427:257-454,2006

Figure 4.2: Monte Carlo simulated distributions of polarisation sensitive kinematic variables defined in the text for (a) $\tau \to \pi\nu$, (b) $\tau \to \rho\nu$, (c)$\tau \to a_1\nu$ and (d) $\tau \to \mu\nu\overline{\nu}$ decays for positive and negative helicity $\tau$ leptons excluding the effects of selection and detector response.

$$\mathcal{P}_\tau(\cos\theta_{\tau^-}) = -\frac{\mathcal{A}_\tau(1+\cos^2\theta_{\tau^-}) + 2\mathcal{A}_e\cos\theta_{\tau^-}}{(1+\cos^2\theta_{\tau^-}) + \frac{8}{3}A^{\tau}_{FB}\cos\theta_{\tau^-}}$$

Plan for this study:

➔ Check performance in full sim samples: impact of reconstruction in identification of decay channels and associated systematics

➔ Once understood, propagate to a physics analysis

➔ effect on templates!  ➔ sensitivity

Using **CLD** fullsim samples

**For details: see the introductory talk (27/May)**

# ZTauTau Pythia8 CLD Sample

**Performance studies based on ZTauTau events (CLD sample)**

Note: we are using the same card as the one used for Delphes. Polarization still to be checked. For now we are focusing on tau reconstruction.

Now we have all tau decays: generalization of the method

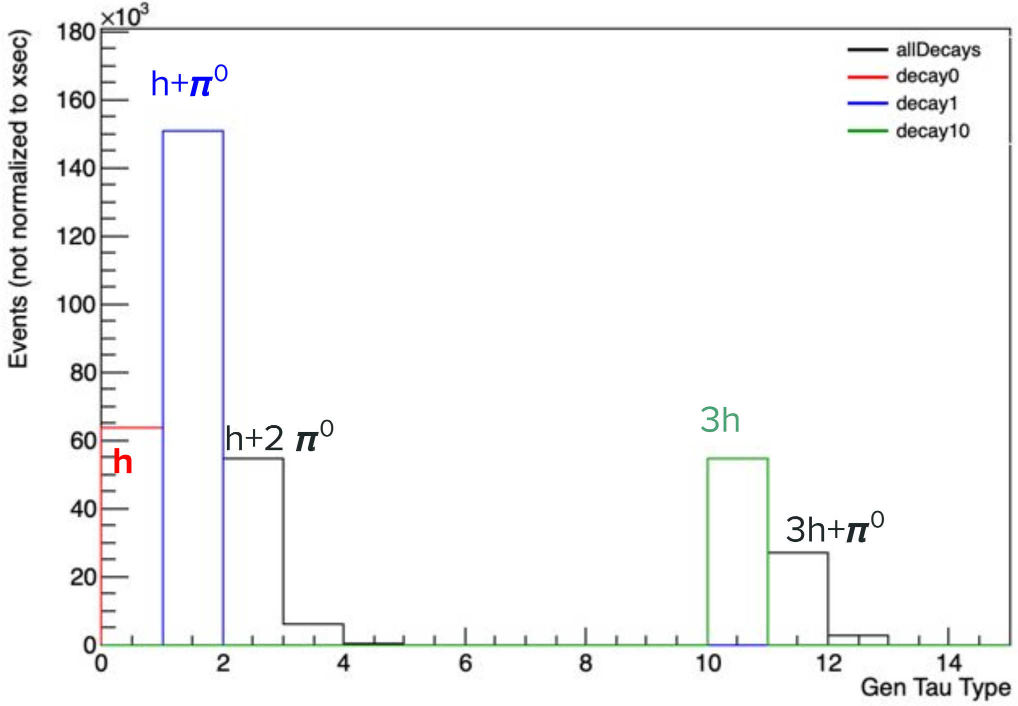
Heuristic approach (Pandora-based) expanded ➔ reconstruct tau based on hadrons and pi0s (gen) photons (reco)

- Considering h, h+Pi0, and 3h cases
- Cone of dR=1 around a pion seed ➔ this cut is configurable. Started with a very loose cut as a result of feedback from the first meeting (will be tightened & tuned)
- For now no cut on pi+- / pi0 ‘components’ ➔ check momentum of candidates to tune

First checks:

- Check performance wrt to Generator Level Tau (Efficiency / Resolution)
- Focus on specific decays
- Migration

# Labelling of Decay Modes at Generator Level




One prong 3 prongs

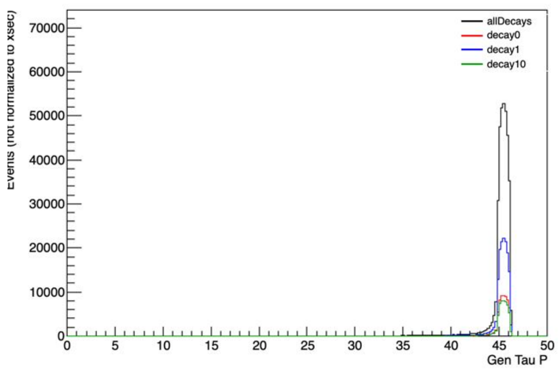
Labelling of GEN 'Types / DecayModes':

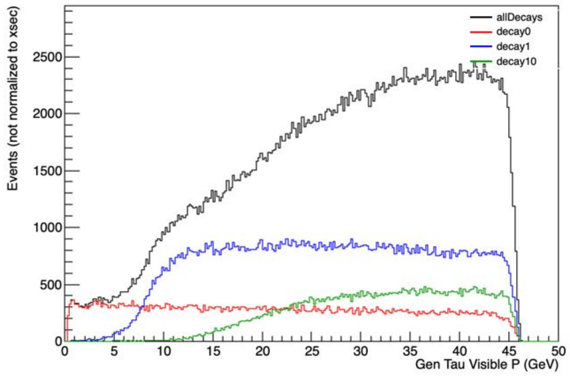
- Decays With 1 Charged Hadron: DM = NPi0s
- Decays With 3 Charged Hadron: DM = 10 + NPi0s

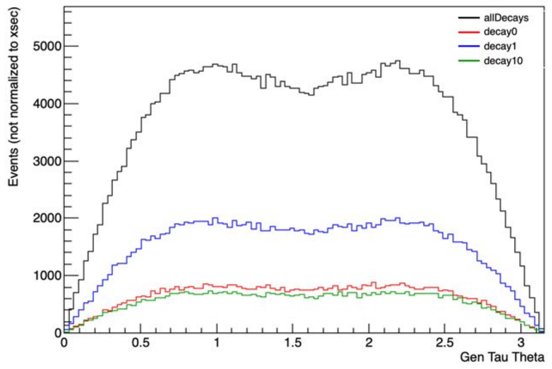
Tau->Muon and Tau->Electron decays removed from the plots (in the pipeline to study for Ae)

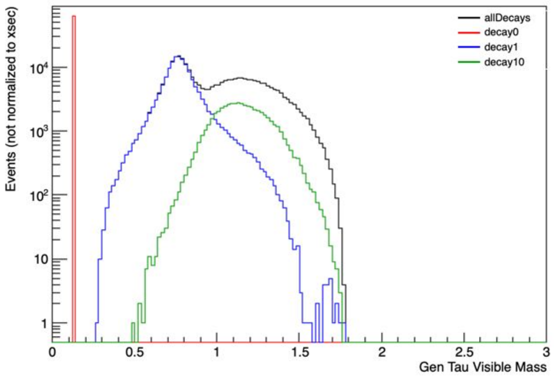
Decays with kaons ignored for now

# Characterization of the Taus in this sample



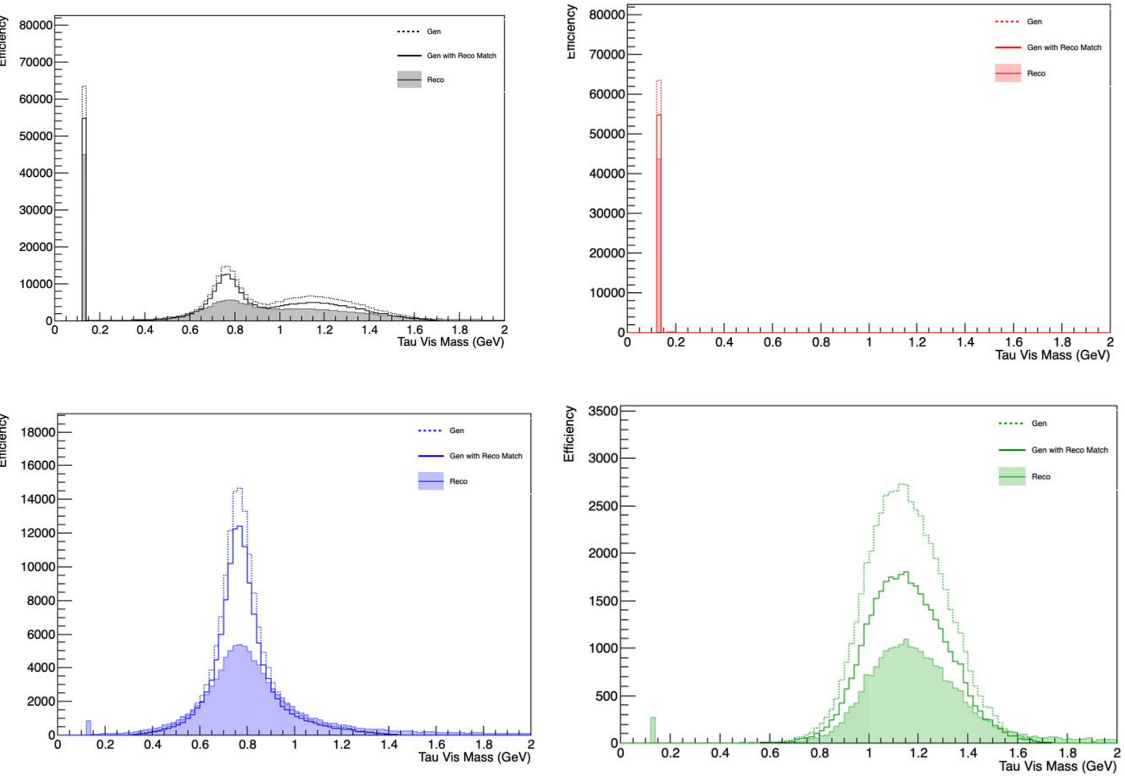



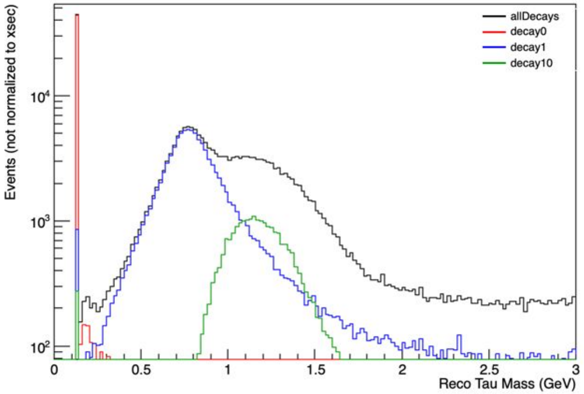

In order to have a easier comparison to offline: building ‘Visible Tau’ at gen level from the decay products of the tau, removing the neutrinos

# Tau Visible Mass Per Decay Mode

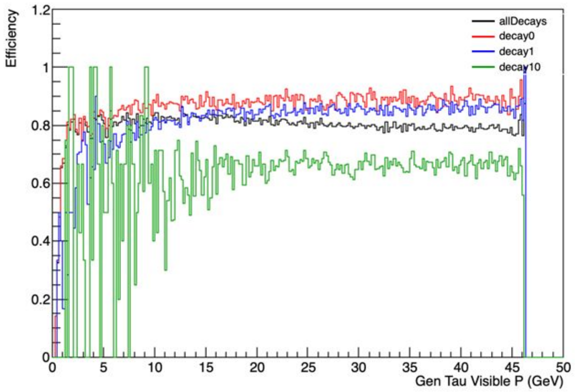
## Gen Level

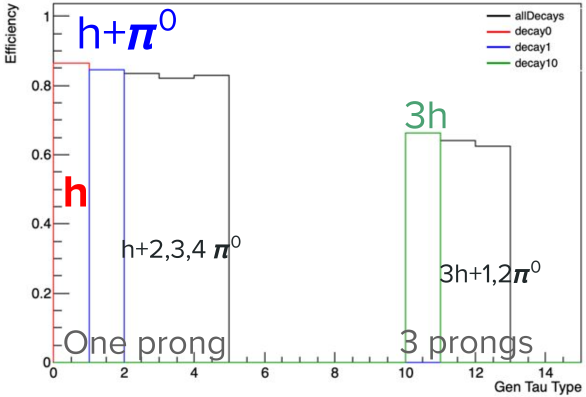


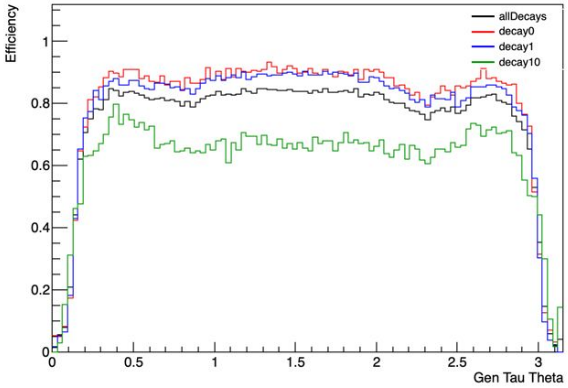

## Reco (from PFO candidates)

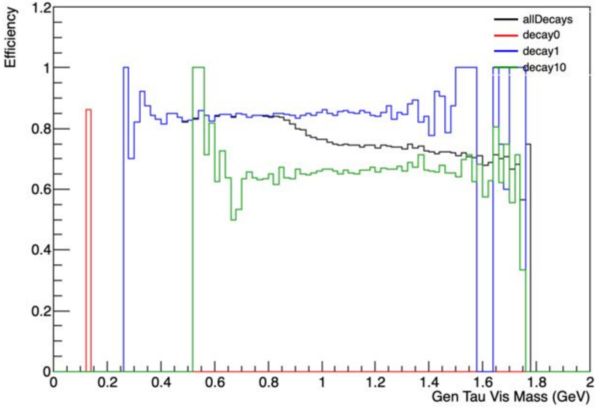


# Tau Reconstruction Efficiencies



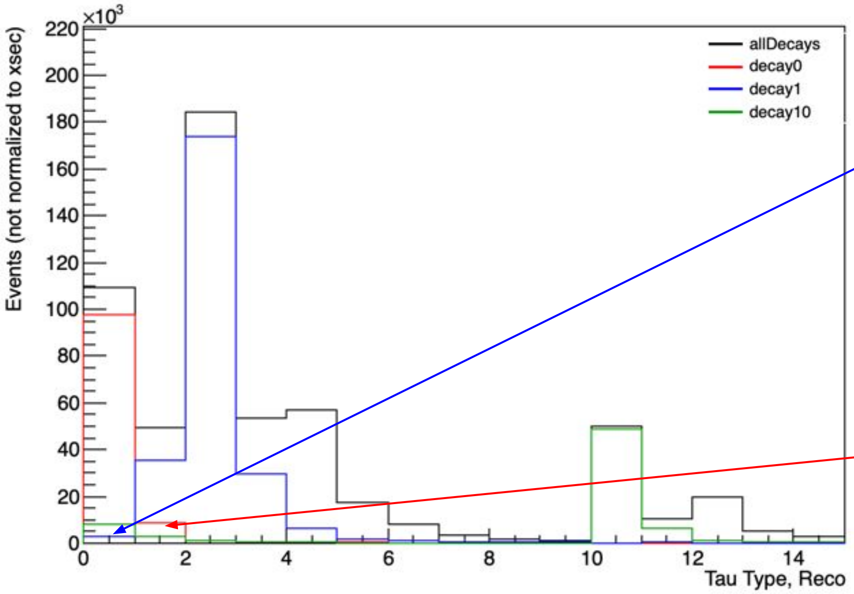




Efficiencies ~80% for tau reconstruction (here not checking the migration between categories, just total reco efficiency)

Note lower efficiency for the 3h case

# Migration?




Very small fraction of Tau->Rho (labelled as decay1 at gen level, blue) that are reconstructed as DM0 at recolevel (only 1 pion, no photons)

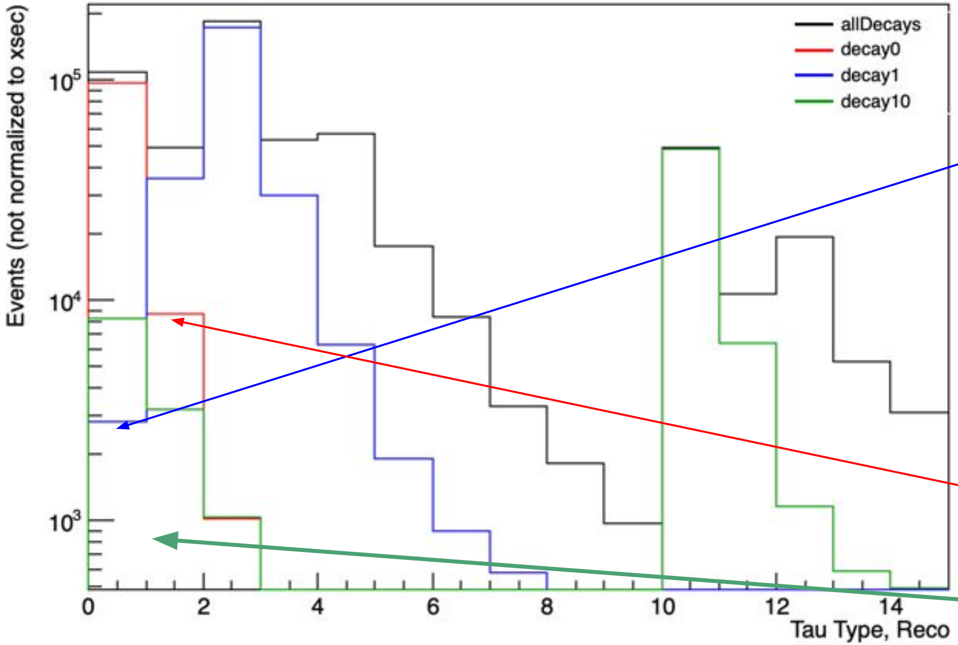
Slightly larger number of events that are true Tau->Pion events (decay0, red) labelled as Rhos at reco level (Reco DM 1 or 2), incorporating a photon

Cases with 3 pions need some more investigation

**Labelling of Reco 'Types / DecayModes':**

- **Decays With 1 Charged Hadron: Reco DM = NPhotons**
- **Decays With 3 Charged Hadron: Reco DM = 10 + NPhotons**

# Migration?

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**Cases with 3 pions need some more investigation**

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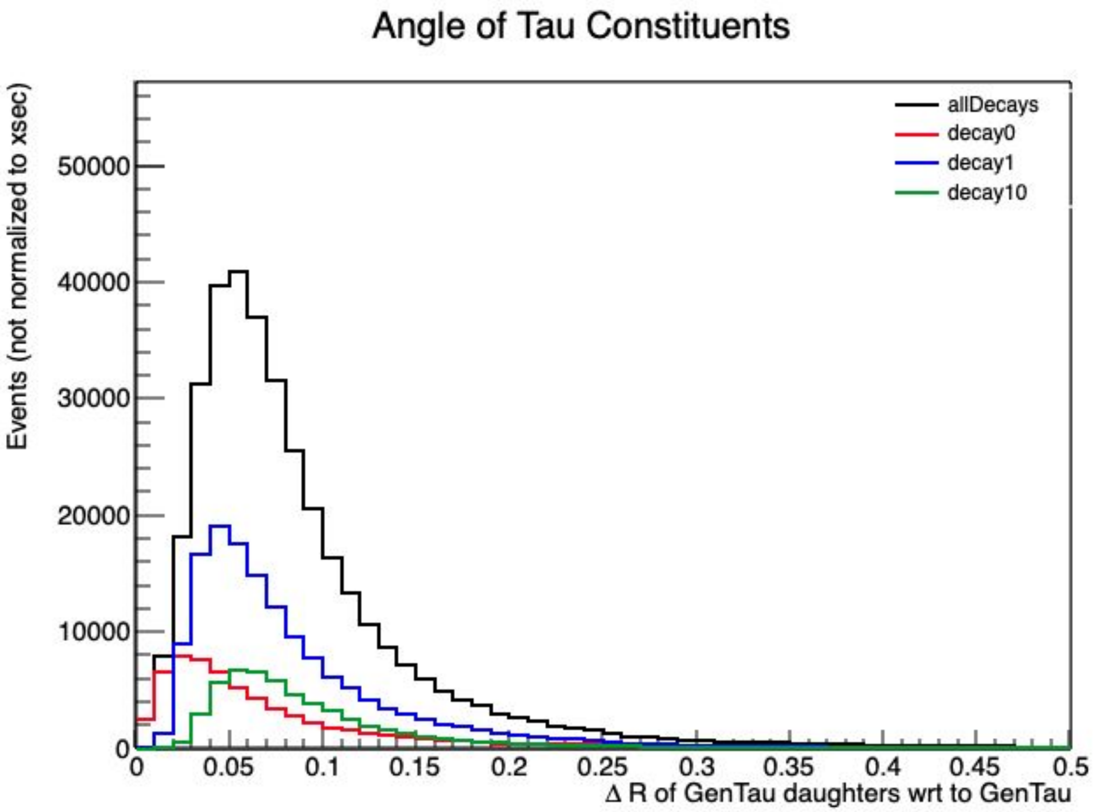
# Understanding the efficiency & migration

Investigate Pion&Photon in PF:

- **DRAngle** of Tau Candidates ("Size" of the tau) → tuning
- **Min P of the Photon & Pion Candidates** ? → tuning
- **Why do we have missing Pions? Pandora / track Efficiency?**
    - First check: check PFO candidates
        - Spoiler: sizeable misidentified Pion→Neutron rate
    - Check SiTrack Efficiency
- Check Fiducial Volume (|cos theta|<0.95 ?)
- Additional check to be done: right now using the 'default' PFO collection. Loose/TightSelected PFO collections exist, to be checked.

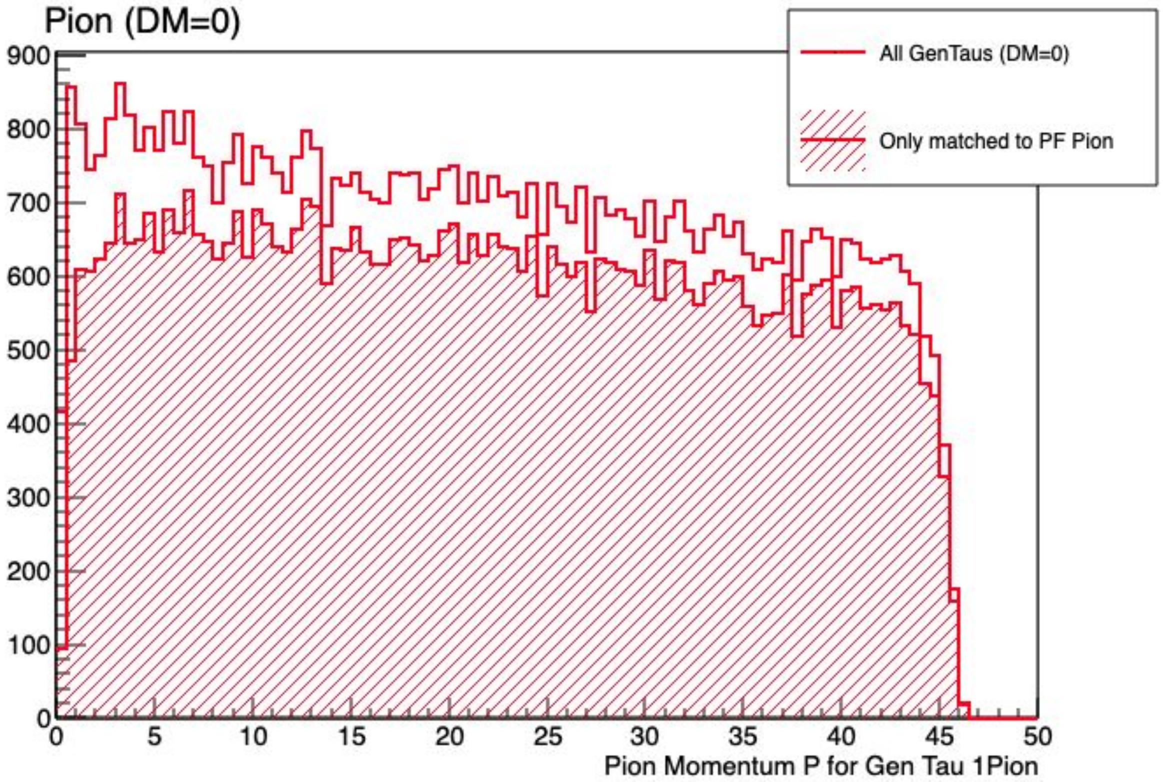
# Angle of Tau constintuents




The DR (1.0) used in the previous slides is too wide ➔ Reconfigure to 0.4 or below

The ‘Size’ of the Tau is typically <~0.1 (depends on DM)

# Momentum of the Constituents: DM 0 (Pion)

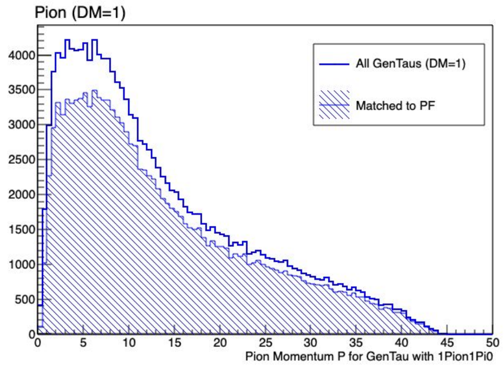


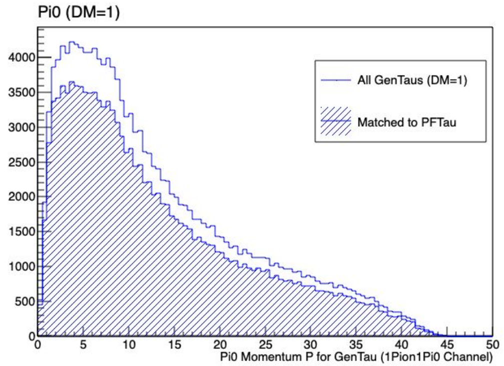

Rather flat

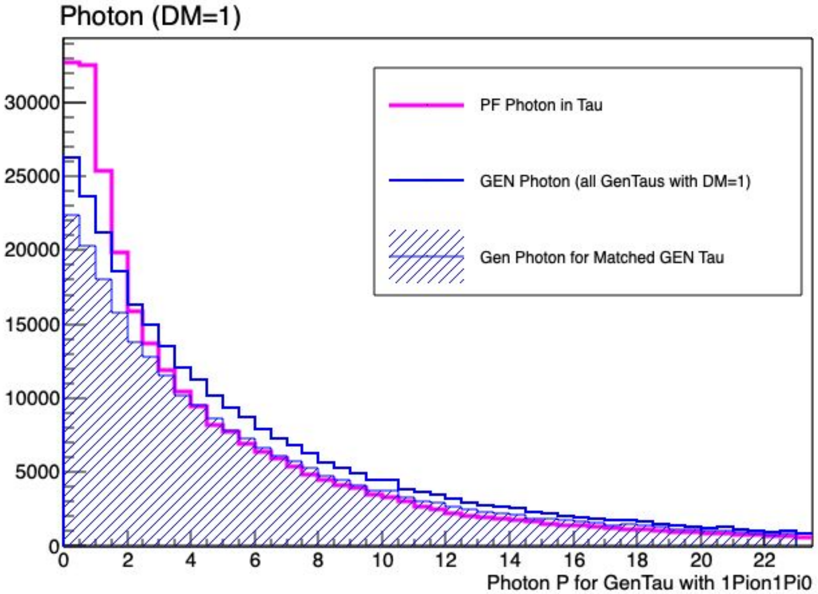
➡ The efficiency loss does not depend on P, we lose high P Pions

# Momentum of the Constituents: DM 1 (Pion+Photon)

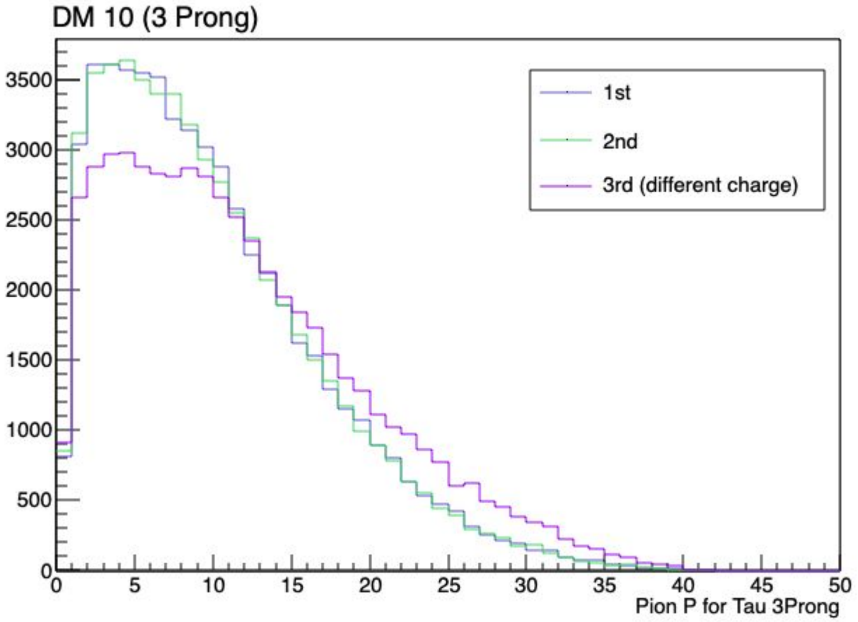
Pion: same message as previous slide
Photon: excess of low P photons at reco level ➔ This can be cleaned with a cut.

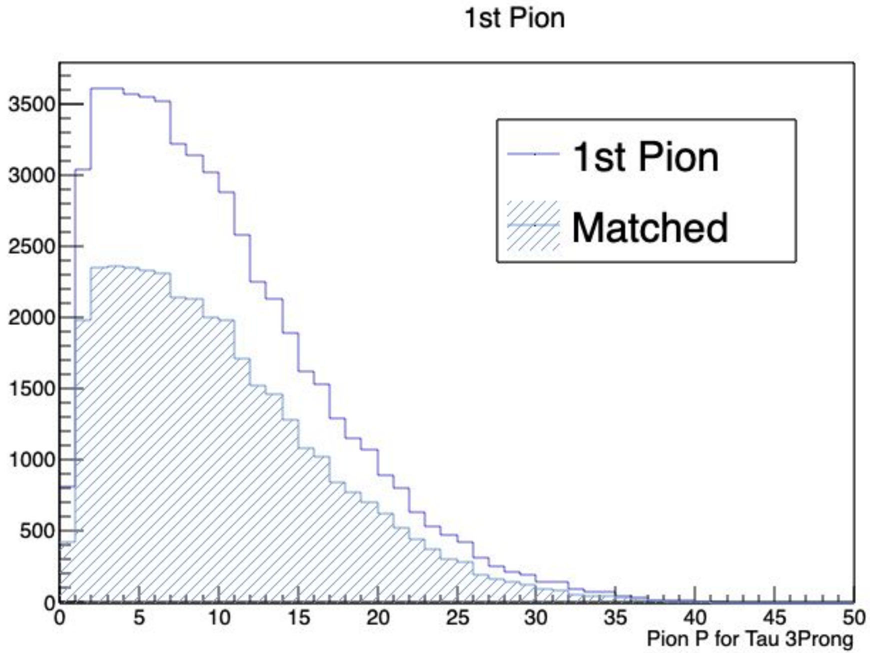


# Momentum of the Constituents: DM 1 (Pion+Photon)



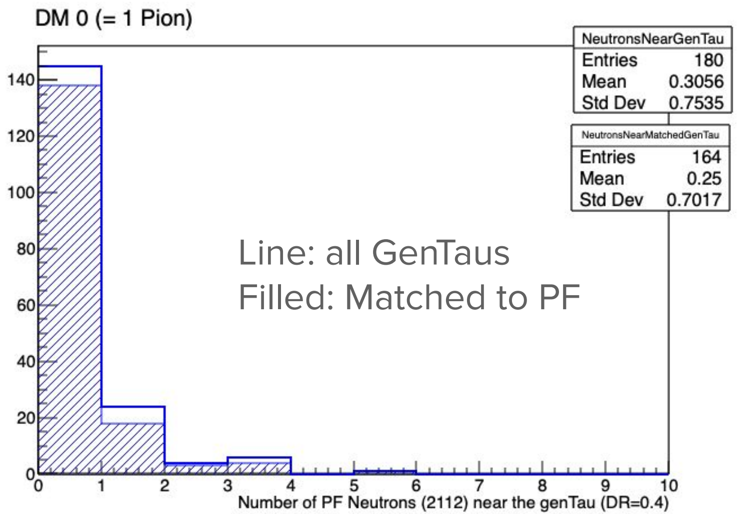



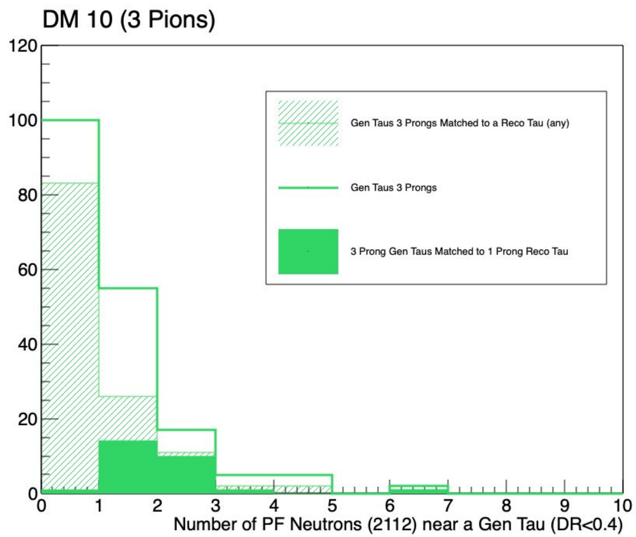

Since there is a efficiency loss per pion, and there are 3 pions here ➔ larger effect
Why do we ‘miss’ these pions?

# PF Neutrons within the Tau DR radius ?

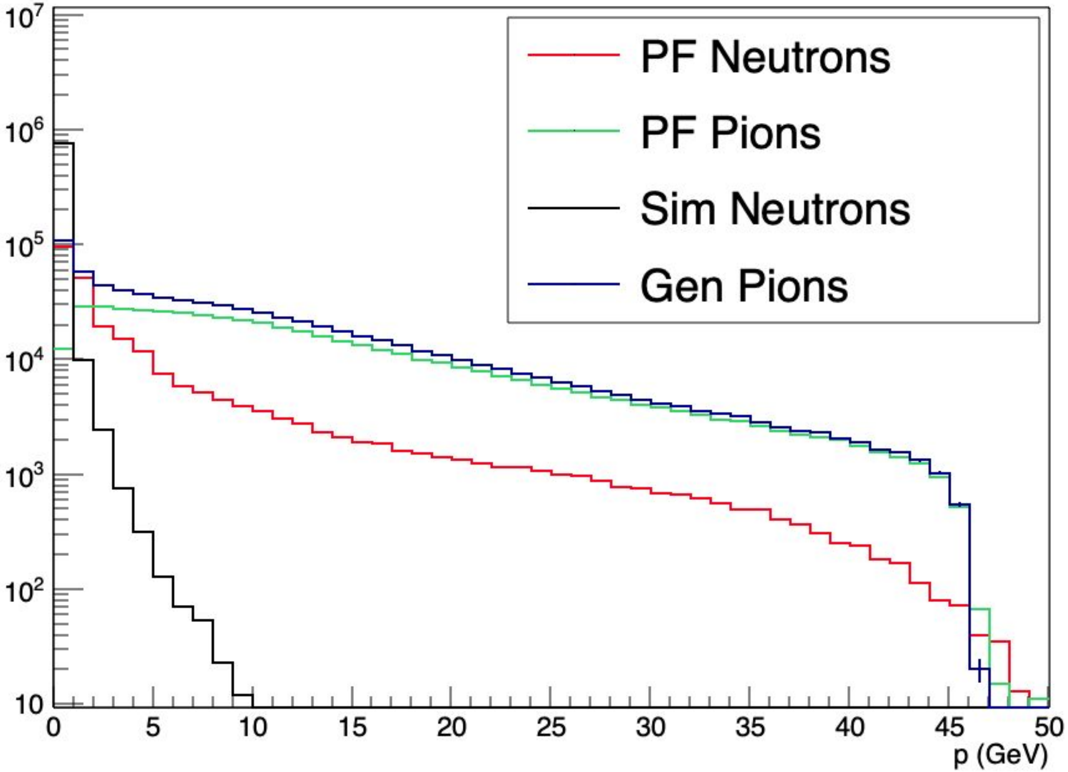
Sizeable amount of high P neutrons can be found accompanying the charged Pions (inside the tau cone) ➔ **pion gets misclasified as neutron?**

Note particularly visible in the ‘3Pion misidentified as 1Pion’ class (filled green plot)



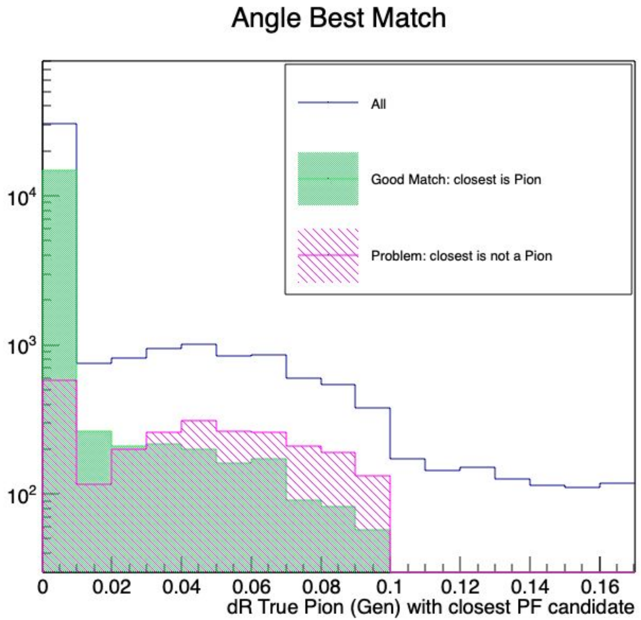



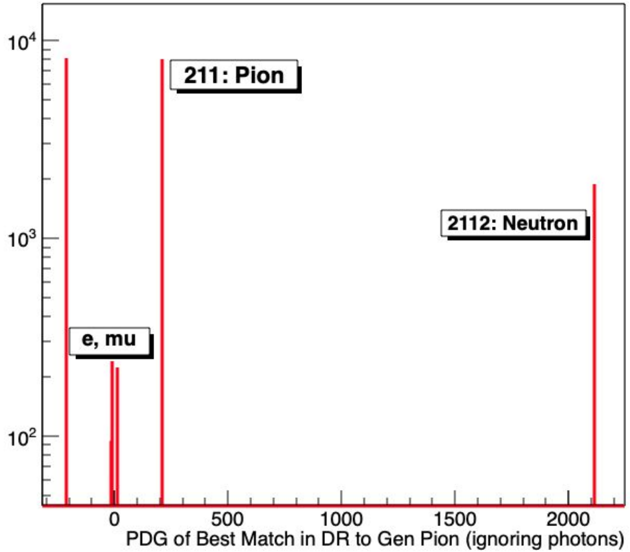
# Neutrons? Gen/Sim vs PF?



# Investigating the PFO collection

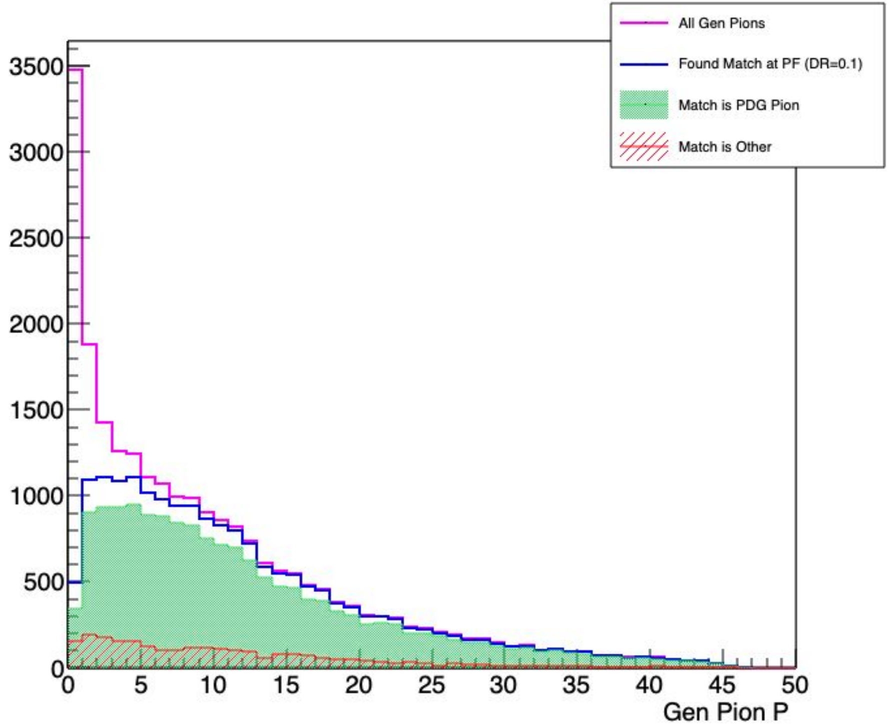
- 'PF Efficiency' ➔ For each Generated Pion, loop over the PF candidates to find the closest particle in DR ➔ check their assigned PF PDG Id  (note: check both neutral and charged PFOs, but exclude photons)
- Within DR=0.1, PF identifies many Neutrons ➔ Pandora  is misidentifying  Pions as Neutrons
- To be investigated further looking at the track collection that pandora uses (SiTrackRefitted) and the matching in pandora (in progress!)

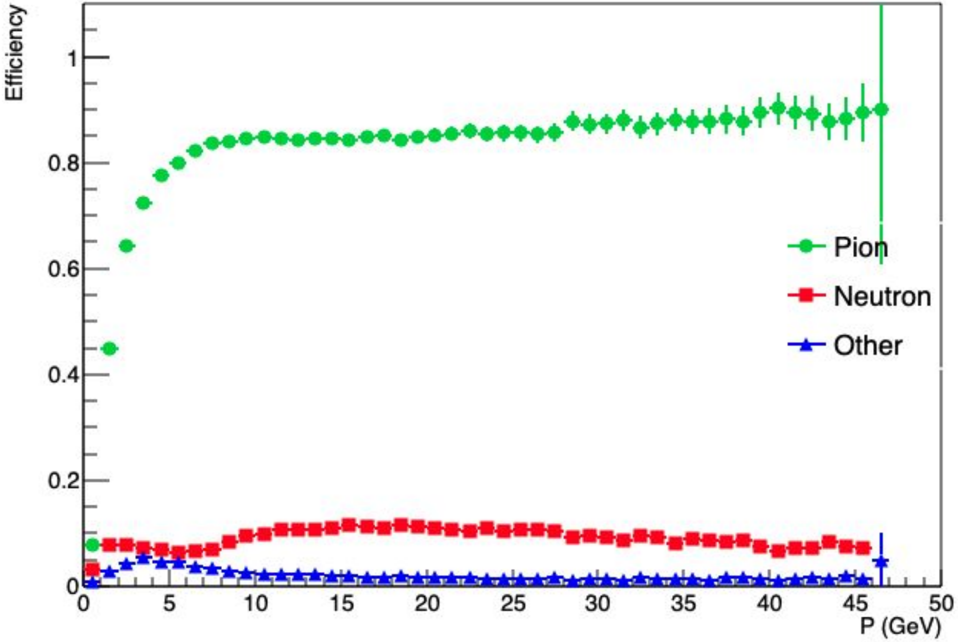


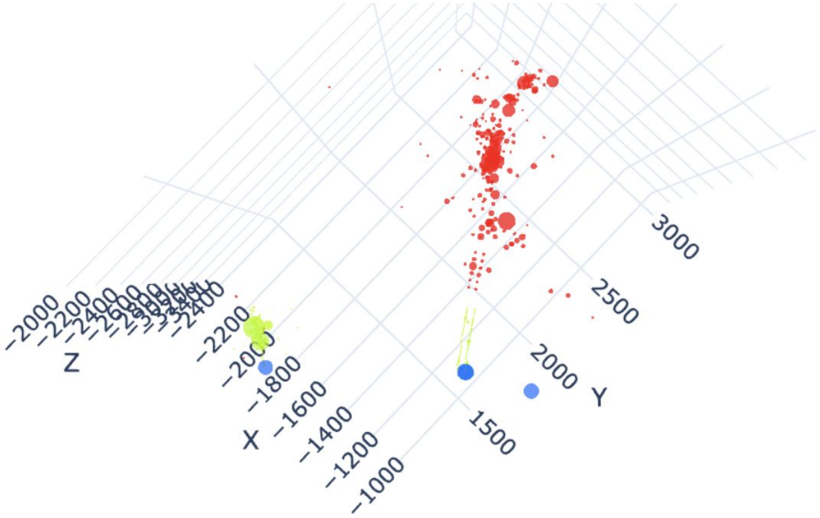
# PFOs: Behaviour vs P

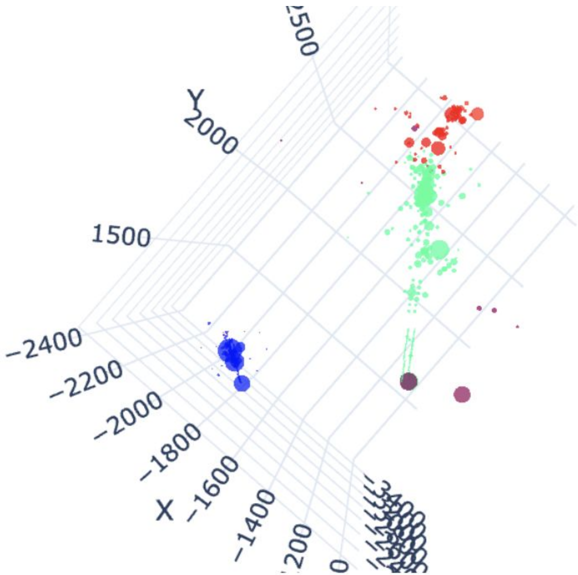
After the turnon → Flat Plateau with
~85% efficiency (Gen Pion → PF Pion)
~10% misID (Gen Pion → Neutron/E/Mu within a 0.1 cone)
Note the P turn-on will affect 3Pion more than 1Pion



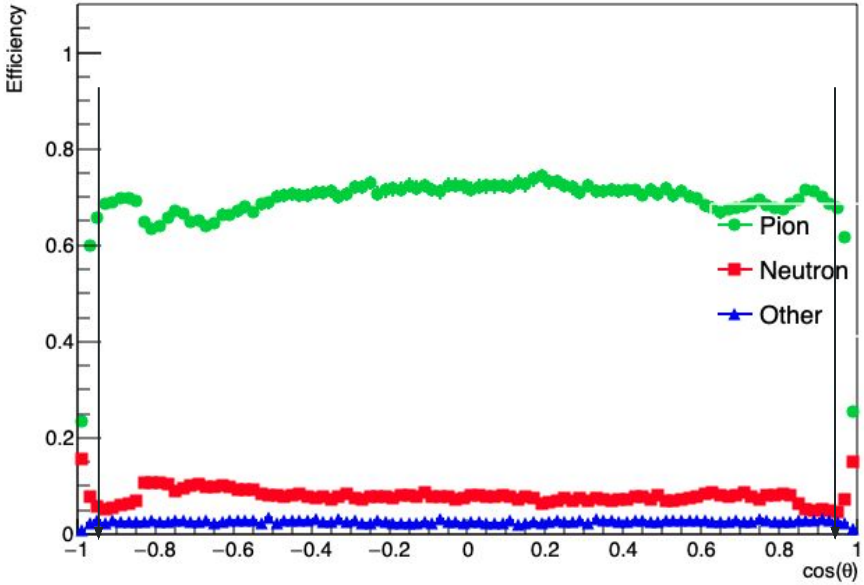



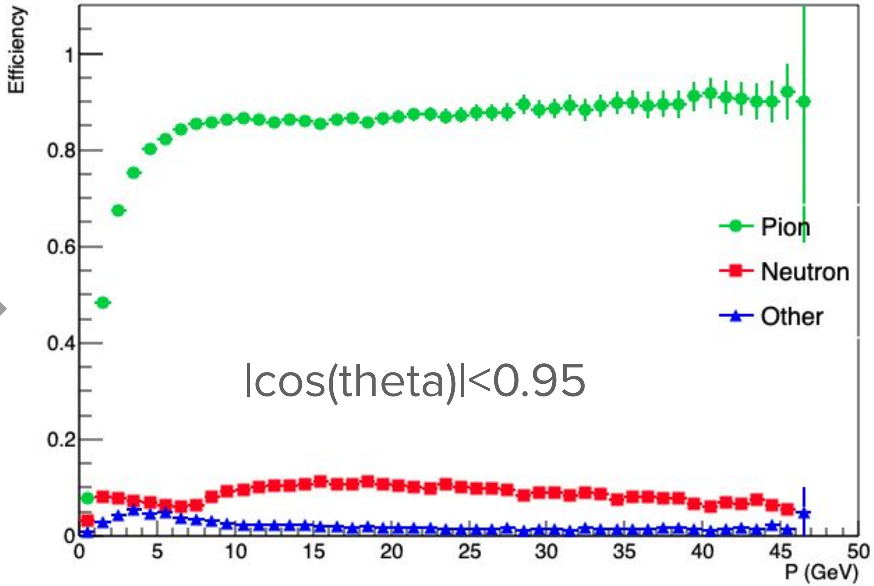
# Example: event reconstruction of cases with missing PF Pions



# Effect of fiducial volume?



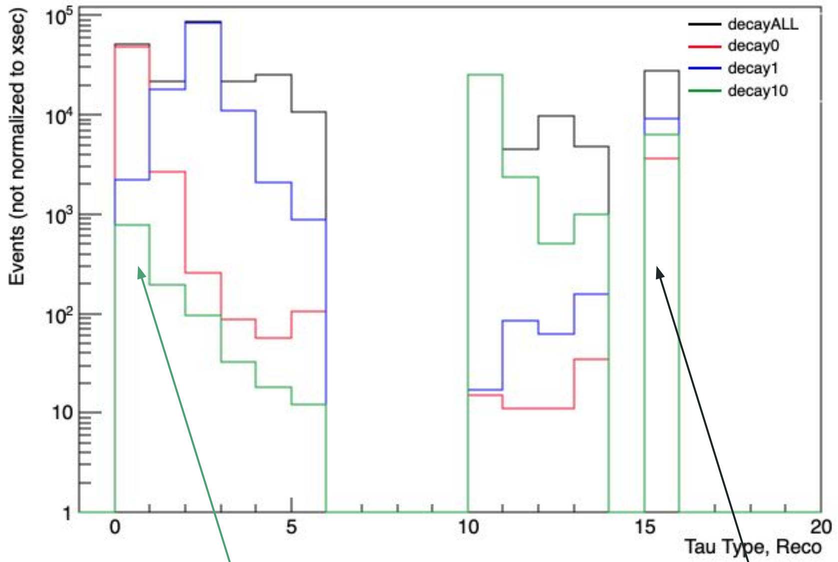

Remove cos(theta)>0.95 from fiducial volume ➔ almost no change

# How to mitigate this issue for 3Pion➔1Pion?

The proper solution is to understand & fix the track efficiency

In the meanwhile, check for neutrons while building the tau candidate ➔ cases with 1Pion+2 “Neutrons” can go into their own class ➔ this cleans better the 1 Prong category

Careful: this affects the other decays, needs to be tuned.




Contamination of 3Pion in the 1Pion sample down

Cases with neutrons in the tau cone (DR=0.4). P(Neutron)>2

# ID: Migration matrix

dR=0.4
MisId Pions (Neutrons): P>1 GeV
No P cut on photons/pions

| Taus by ID<br>Gen vs Reco | | Reco Taus | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | $\pi$ | $\pi+\gamma$<br>$\pi+2\gamma$ | $\pi+3\gamma$,<br>$\pi+4\gamma$ | $3\pi$ | $3\pi+\gamma$<br>$3\pi+2\gamma$ | $\pi$+<br>MisId | Total |
| Truth | $\pi$ | 76.2% | 4.6% | 0.2% | <0.1% | <0.1% | 5.6% | 86.7% |
| | $\pi+\pi^0$ | 1.5% | 67.9% | 8.6% | <0.1% | <0.1% | 6.0% | 84.0% |
| | $\pi+2\pi^0$ | 0.5% | 5% | 59.8% | <<0.1% | <<0.1% | 7.7% | 83.6% |
| | $3\pi$ | 1.4% | 0.5% | 0.1% | 46.3% | 4.4% | 11.5% | 66.9% |

# Effect of Photon P

dR=0.4

MisId Pions (Neutrons): P>1 GeV

**Min P Cut on Photons and Pions: 0.5 GeV**

| | | Reco Taus | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | $\pi$ | $\pi+\gamma$, $\pi+2\gamma$ | $\pi+3\gamma$, $\pi+4\gamma$ | $3\pi$ | $3\pi+\gamma$, $3\pi+2\gamma$ | $\pi$+ MisId | Total |
| Truth | $\pi$ | 77.8% | 3.1% | 0.2% | <0.1% | <0.1% | 5.6% | 86.9% |
| | $\pi+\pi^0$ | 2.4% | 69.6% | 6.4% | <0.1% | <0.1% | 6.0% | 84.9% |

Tested the effect on the migration of adding cuts on the P of the Photon&Pion → Interplay between $\pi+\pi^0$ and $\pi$ channels.

Should be tuned more systematically, this is just a check. Secondary for now, the pion efficiency is the main problem for now

# Summary

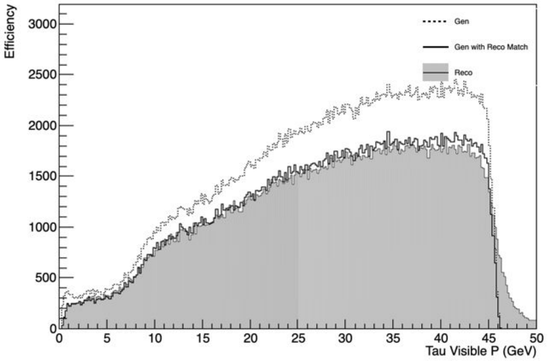
- First checks done with a ZTauTau Sample
  - Focusing on the phase space of taus in Z decays
  - **Comparison of gen to reco:**
    - Overall efficiency ~80% . Worse for 3h decays.
    - Good mass/pt resolution, good agreement of reco variables to visible tau at gen level
    - Migration between categories ➔ Reco&ID efficiency is lower
      - ~70-80% for 1 Pion, ~45% for 3Pion ➔ too low
    - Efficiency loss driven by missing Charged Pions in PFO sample ➔ appearance of high P neutrons ➔ Misidentification of Pions as Neutrons. Missing track?
    - First looks at tuning.
      - For now, DR=0.4 (larger than typical size of tau).
      - Studying the impact of selection cuts on the photon/pion on the migration between rho/pion categories
  - **Main to do: investigate&improve charged pion identification efficiency**

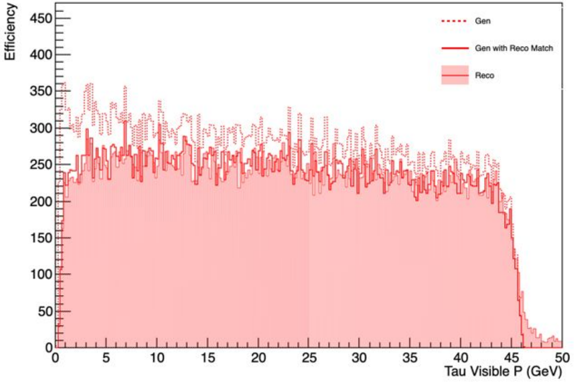
# Further steps

- In parallel to the reconstruction study, physics analysis to be done:
  - We need alternative polarization samples / templates to move from reconstruction to a polarization study
  - For this:
    - Checking Pythia8 implementation of polarization ( hep-ph:1211.6730 )
    - Reweighting of current SM sample to build the templates (modelling thanks to J. Alcaraz) ➔ should be possible to reuse the current Py8 sample
- Improvements in ML coming:
  - Move to the multi-class classification on the Z->tautau samples

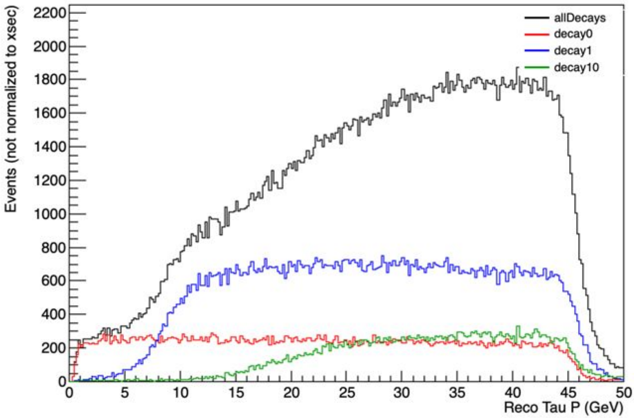
# backup

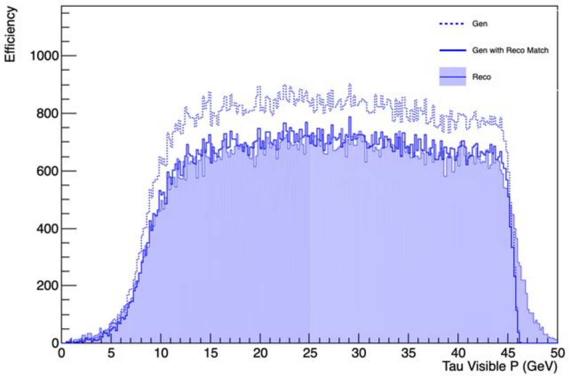
# Tau Visible Momentum (P) Per Decay Mode

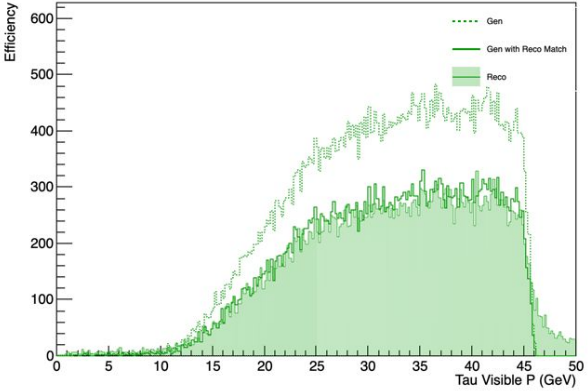



Reco (from PFO candidates)



# ID: Migration matrix

dR=0.4

**MisId Pions (Neutrons): P>3 GeV**

No P cut on photons/pions

| Taus by ID<br>Gen vs Reco | | Reco Taus | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | π | π+γ<br>π+2γ | π+3γ,<br>π+4γ | 3π | 3π+γ<br>3π+2γ | π+<br>MisId | Total |
| Truth | π | 78.8% | 5.0% | 0.4% | <0.1% | <0.1% | **2.0%** | 86.9% |
| | $\pi+\pi^0$ | 1.6% | 70.9% | 9.2% | <0.1% | <0.1% | 1.9% | 84.8% |
| | $\pi+2\pi^0$ | 0.5% | 5.8% | 63.5% | <<0.1% | <<0.1% | 2.3% | 83.6% |
| | 3π | **1.6%** | 0.7% | 0.2% | 46.3% | 4.4% | 10.9% | 66.9% |

# Effect of Photon P

dR=0.4
MisId Pions (Neutrons): P>1 GeV
**Min P Cut on Photons and Pions: 1 GeV**

| | | Reco Taus | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | π | π+$\gamma$<br>π+2$\gamma$ | π+3$\gamma$,<br>π+4$\gamma$ | 3π | 3π+$\gamma$<br>3π+2$\gamma$ | π+<br>MisId | Total |
| Truth | π | 78.7% | 2.3% | 0.1% | <0.1% | <0.1% | 5.6% | 87.0% |
| | π+$\pi^0$ | 4.4% | 69.4% | 5.0% | <0.1% | <0.1% | 6.0% | 85.2% |

Tested the effect on the migration of adding cuts on the P of the Photon&Pion ➔ Interplay between π+$\pi^0$ and π channels.

Should be tuned more systematically, this is just a check. Secondary for now, the pion efficiency is the main problem for now

# Effect of Photon P

dR=0.4

MisId Pions (Neutrons): P>1 GeV

**Min P Cut on Photons and Pions: 2 GeV**

| | | Reco Taus | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | π | π+γ<br>π+2γ | π+3γ,<br>π+4γ | 3π | 3π+γ<br>3π+2γ | π+<br>MisId | Total |
| Truth | π | 81.8% | 1.9% | 0.1% | <0.1% | <0.1% | 3.1% | 87.1% |
| | $\pi+\pi^0$ | 10.4% | 68.9% | 3.1% | <0.1% | <0.1% | 2.9% | 85.5% |

Just to check the effect

# LEP Fiducial Cuts

Table 2.1
Ideal acceptances, selection efficiencies[a] and background contribution at the peak of the resonance (1994 data)

| | ALEPH | DELPHI | L3 | OPAL |
|---|---|---|---|---|
| $q\bar{q}$ *Final state* | | | | |
| Acceptance | $s'/s > 0.01$ | $s'/s > 0.01$ | $s'/s > 0.01$ | $s'/s > 0.01$ |
| Efficiency (%) | 99.1 | 94.8 | 99.3 | 99.5 |
| Background (%) | 0.7 | 0.5 | 0.3 | 0.3 |
| $e^+e^-$ *Final state* | | | | |
| Acceptance | $-0.9 < \cos\theta < 0.7$ | $\|\cos\theta\| < 0.72$ | $\|\cos\theta\| < 0.72$ | $\|\cos\theta\| < 0.7$ |
| | $s' > 4m_\tau^2$ | $\eta < 10°$ | $\eta < 25°$ | $\eta < 10°$ |
| Efficiency (%) | 97.4 | 97.0 | 98.0 | 99.0 |
| Background (%) | 1.0 | 1.1 | 1.1 | 0.3 |
| $\mu^+\mu^-$ *Final state* | | | | |
| Acceptance | $\|\cos\theta\| < 0.9$ | $\|\cos\theta\| < 0.94$ | $\|\cos\theta\| < 0.8$ | $\|\cos\theta\| < 0.95$ |
| | $s' > 4m_\tau^2$ | $\eta < 20°$ | $\eta < 90°$ | $m_{f\bar{f}}^2/s > 0.01$ |
| Efficiency (%) | 98.2 | 95.0 | 92.8 | 97.9 |
| Background (%) | 0.2 | 1.2 | 1.5 | 1.0 |
| $\tau^+\tau^-$ *Final state* | | | | |
| Acceptance | $\|\cos\theta\| < 0.9$ | $0.035 < \|\cos\theta\| < 0.94$ | $\|\cos\theta\| < 0.92$ | $\|\cos\theta\| < 0.9$ |
| | $s' > 4m_\tau^2$ | $s' > 4m_\tau^2$ | $\eta < 10°$ | $m_{f\bar{f}}^2/s > 0.01$ |
| Efficiency (%) | 92.1 | 72.0 | 70.9 | 86.2 |
| Background (%) | 1.7 | 3.1 | 2.3 | 2.7 |

[a]The lepton selection efficiencies given by the experiments were in some cases quoted with respect to full acceptance in $\cos\theta$; for the purpose of comparison, they were corrected to the fiducial cuts in $\cos\theta$ actually used in the analyses, assuming a shape of the differential cross-section according to $(1 + \cos^2\theta)$.

The Monte Carlo generators are used to apply corrections at the edges of the experimental acceptance, and for small extrapolations of the measured cross-sections and forward–backward asymmetries from the true experimental cuts to sets of simple cuts that can be handled at the fitting stage. In the case of $q\bar{q}$ final states, this ideal acceptance is defined by the single requirement $s' > 0.01\,s$, where $\sqrt{s'}$ is the effective centre-of-mass energy after initial-state photon radiation. The idealised acceptances chosen for each lepton decay channel vary among the experiments and are specified in Table 2.1. The results quoted for the $e^+e^-$ final state either include contributions originating from $t$-channel diagrams, or the $t$ and $s$-$t$ interference effects are explicitly subtracted, allowing the same treatment of $e^+e^-$ and $\mu^+\mu^-$ or $\tau^+\tau^-$ final states in the fits for the Z parameters.

# Overview

**First target: Aτ . Focus on π vs ρ channels, photon&π0 identification / rejection**

Second target: muon/electron vs tau discrimination (Ae)

Starting point: full simulation of single particles (pi+-, pi0, rho, muons, electrons) in CLD

Two approaches: ML (based on hits, Graph) and heuristic (based on PFO candidates)

| Decay mode | Resonance | $\mathcal{B}$ (%) |
|---|---|---|
| Leptonic decays | | 35.2 |
| $\tau^- \to e^- \bar{\nu}_e \nu_\tau$ | | 17.8 |
| $\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau$ | | 17.4 |
| Hadronic decays | | 64.8 |
| $\tau^- \to h^- \nu_\tau$ | | 11.5 |
| $\tau^- \to h^- \pi^0 \nu_\tau$ | $\rho(770)$ | 25.9 |
| $\tau^- \to h^- \pi^0 \pi^0 \nu_\tau$ | $a_1(1260)$ | 9.5 |
| $\tau^- \to h^- h^+ h^- \nu_\tau$ | $a_1(1260)$ | 9.8 |
| $\tau^- \to h^- h^+ h^- \pi^0 \nu_\tau$ | | 4.8 |
| Other | | 3.3 |

➔ From LEP we know we can focus on the pion and rho decays

# Notes from LEP

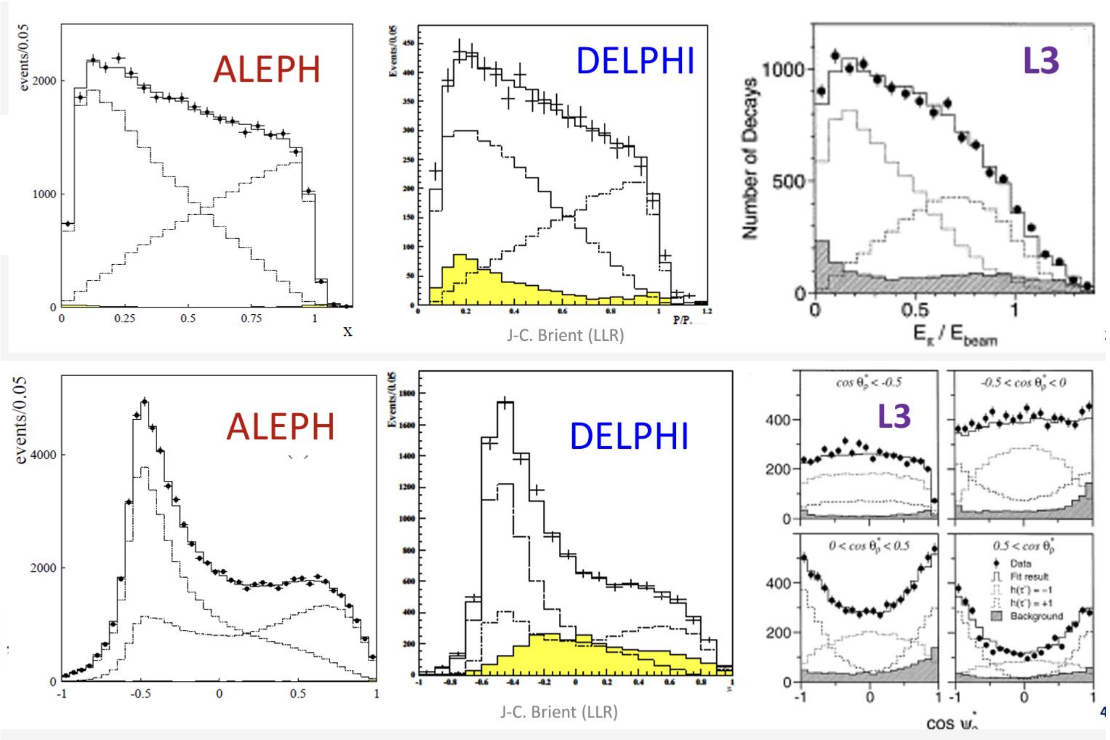
See J-C. Brient's presentation in Krakow Physics Workshop for a detailed discussion

**$\tau \to \pi(k)\ \nu$**

$$x_\pi = E_\pi / E_\tau$$

**$\tau \to \rho(k^*)\ \nu$**

$$\omega_\rho = \frac{W_+(\theta^*, \psi) - W_-(\theta^*, \psi)}{W_+(\theta^*, \psi) + W_-(\theta^*, \psi)}$$




Effect of reconstruction on the templates?

Discrimination pion vs rho

# Notes from LEP

**ALEPH**

Systematics errors estimation as a function of the source of the uncertainties

| | $A_\tau$ | | | | | | |
|---|---|---|---|---|---|---|---|
| Source | $h$ | $\rho$ | $3h$ | $h\,2\pi^0$ | $e$ | $\mu$ | Incl. $h$ |
| selection | - | 0.01 | - | - | 0.14 | 0.02 | 0.08 |
| tracking | 0.06 | - | 0.22 | - | - | 0.10 | - |
| ECAL scale | 0.15 | 0.11 | 0.21 | 1.10 | 0.47 | - | - |
| PID | 0.15 | 0.06 | 0.04 | 0.01 | 0.07 | 0.07 | 0.18 |
| misid. | 0.05 | - | - | - | 0.08 | 0.03 | 0.05 |
| photon | 0.22 | 0.24 | 0.37 | 0.22 | - | - | - |
| non-$\tau$ back. | 0.19 | 0.08 | 0.05 | 0.18 | 0.54 | 0.67 | 0.15 |
| $\tau$ BR | 0.09 | 0.04 | 0.10 | 0.26 | 0.03 | 0.03 | 0.78 |
| modelling | - | - | 0.70 | 0.70 | - | - | 0.09 |
| MC stat | 0.30 | 0.26 | 0.49 | 0.63 | 0.61 | 0.63 | 0.26 |
| TOTAL | 0.49 | 0.38 | 1.00 | 1.52 | 0.96 | 0.93 | 0.87 |

| | $A_e$ | | | | | | |
|---|---|---|---|---|---|---|---|
| Source | $h$ | $\rho$ | $3h$ | $h\,2\pi^0$ | $e$ | $\mu$ | Incl. $h$ |
| tracking | 0.04 | - | - | - | - | 0.05 | - |
| non-$\tau$ back. | 0.11 | 0.09 | 0.04 | 0.22 | 0.91 | 0.24 | 0.17 |
| modelling | - | - | 0.40 | 0.40 | - | - | - |
| TOTAL | 0.12 | 0.09 | 0.40 | 0.47 | 0.91 | 0.25 | 0.17 |

Krakow Physics Workshop

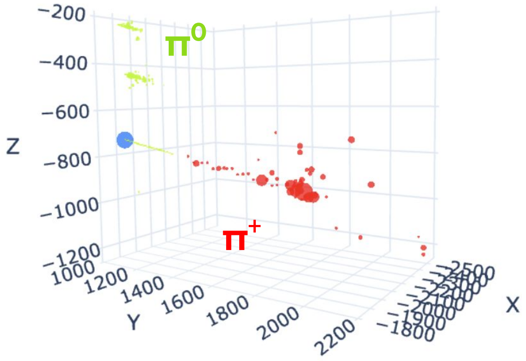
| experiments | $\mathcal{A}_\tau$ (x100) |
|---|---|
| ALEPH | 14.51 ± 0.52 ± 0.29 |
| DELPHI | 13.59 ± 0.79 ± 0.55 |
| L3 | 14.76 ± 0.88 ± 0.62 |
| OPAL | 14.56 ± 0.76 ± 0.57 |
| Combined | **14. 39 ± 0.35 ± 0.26** |

At FCC Dominated by systematics

# Full simulation of CLD samples

Data available @ /eos/experiment/fcc/ee/datasets/mlpf/condor/train/single_particles_flat/*_1/

- Gun hepmc
  - 0-50 GeV (uniformly distributed in energy)
  - 1 particle per event (π or ρ)
  - Random angle in the detector
- CLD simulation and reconstruction [1]
- Inputs to the ML approach:
  - All ECAL and HCAL hits and the track state at calorimeter (4)
    - Hits inputs are (x,y,z) coordinates in the detector, energy
    - Track inputs (x,y,z), p
- Input to heuristic analysis: particle flow reconstruction (with pandora)
- **NEW: Now using ZTauTau sample**

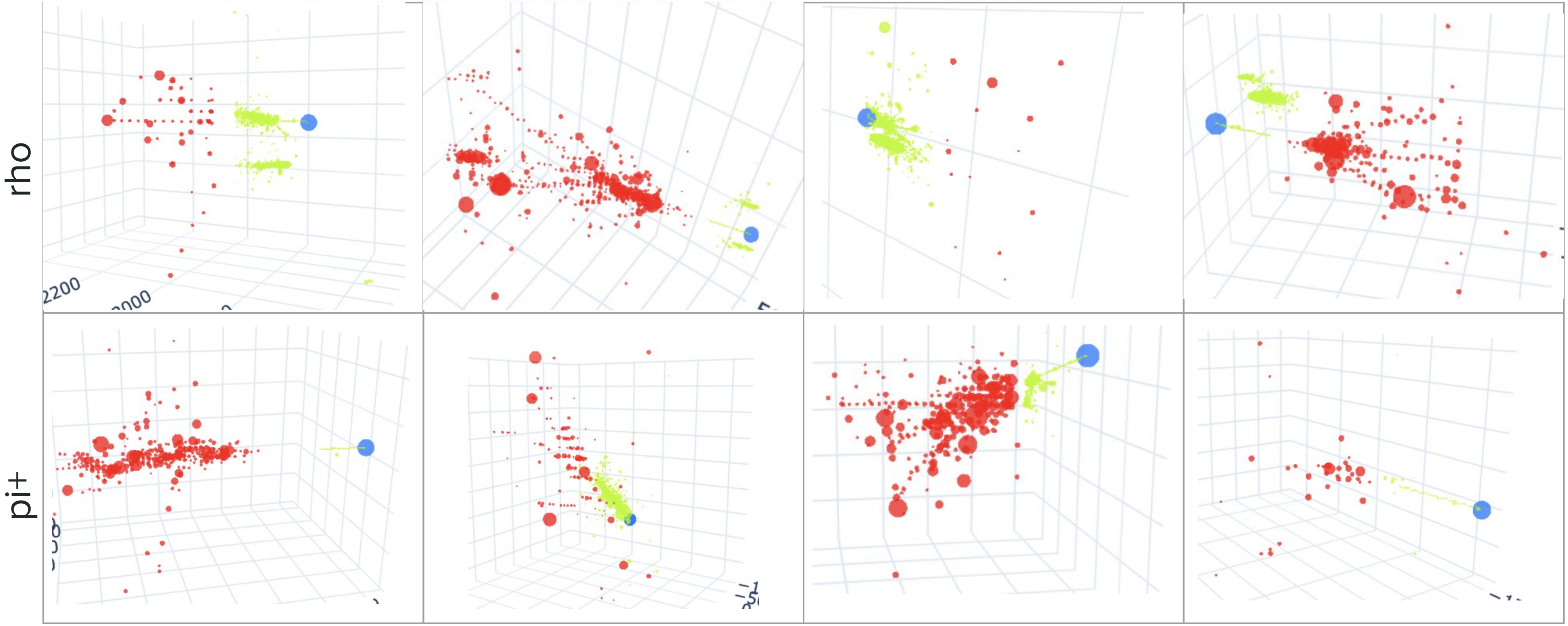



Example rho event. Track (blue), ECAL hits (green) HCAL hit (red). The size of the hits represents the energy deposit.

[1] https://github.com/key4hep/CLDConfig/tree/main/CLDConfig

# Example events

Track (blue), ECAL hits (green) HCAL hit (red). The size of the hits represents the energy deposit.



# Heuristic approach: Decay reconstruction ( Pi/Rho Guns)

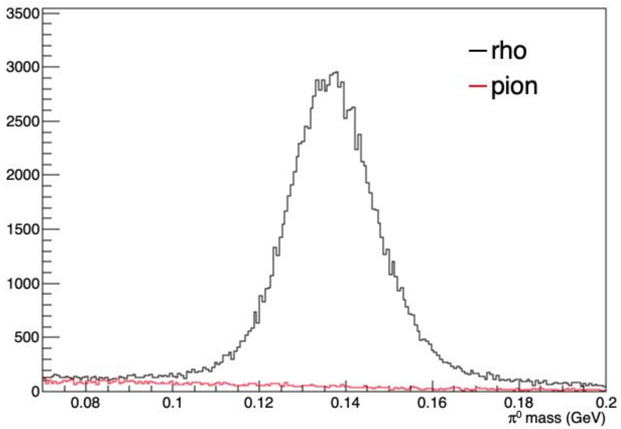
Target specific decays of the tau: identify the decay by looking at the particle flow candidates in the event (similar to the simple reconstruction presented last year, designed using Delphes )

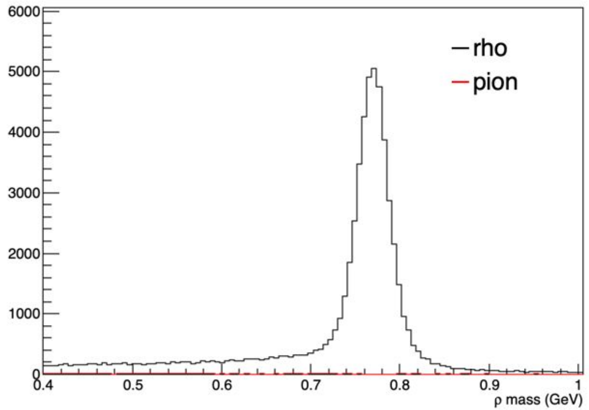
**Start directly from charged and neutral PFOs:** charged pions and photons as elements for building the tau decay:

- Reconstruct Pi0 (from pairs of photons) and Rho (clustering photons around the pion - allowing for only one photon to recover events in which either the two photons are very collimated or we have lost one photon)
- For now no pt/energy requirements on PFO candidates, and maximum distance photons and photon/pion dR=0.1

**Exclusive samples of rho, pion+-**

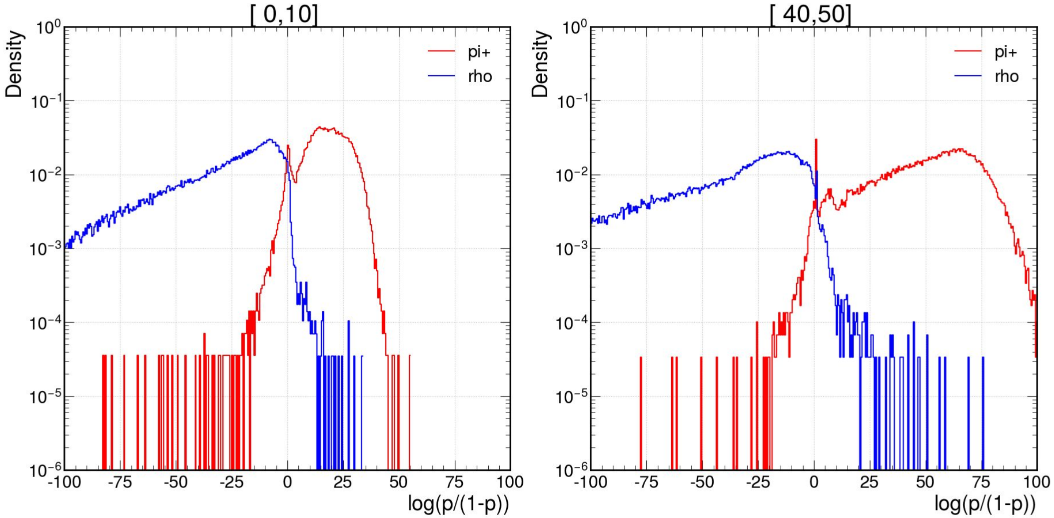
Careful: arbitrary normalization (~100k events in each samples)



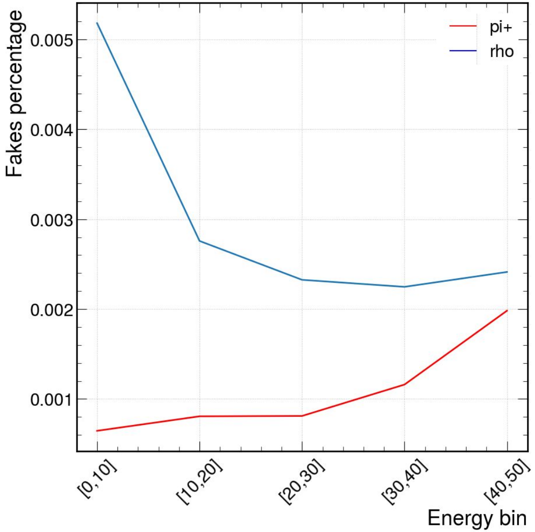



# ML approach and results

- The GNN takes hits as inputs and outputs a score for rho and pi+




P= probability to be classified as pi+



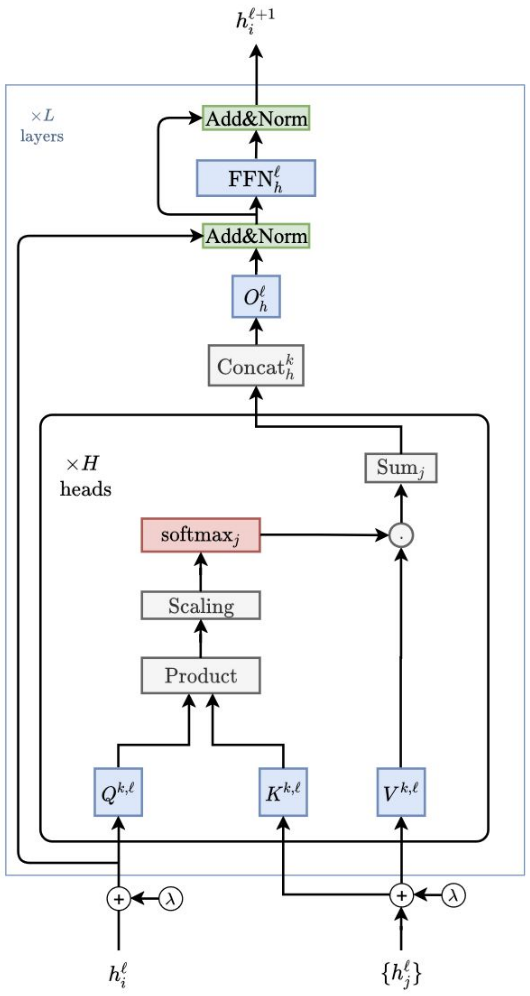

Mistag rate at 99% efficiency for pi+ (red), rho (blue)
Fakes percentage < 1% !

Reminder: Inputs to the ML approach: All ECAL and HCAL hits and the track state at calorimeter , Hits inputs are (x,y,z) coordinates in the detector, energy , Track inputs (x,y,z), p

# Architecture

Graph Neural Network transformer

- Each hit attends to all other hits
- Graph is consider fully connected




Graph Transformer layer

1] https://arxiv.org/pdf/2012.09699