Lévy Stochastic Processes and Limit Theorems

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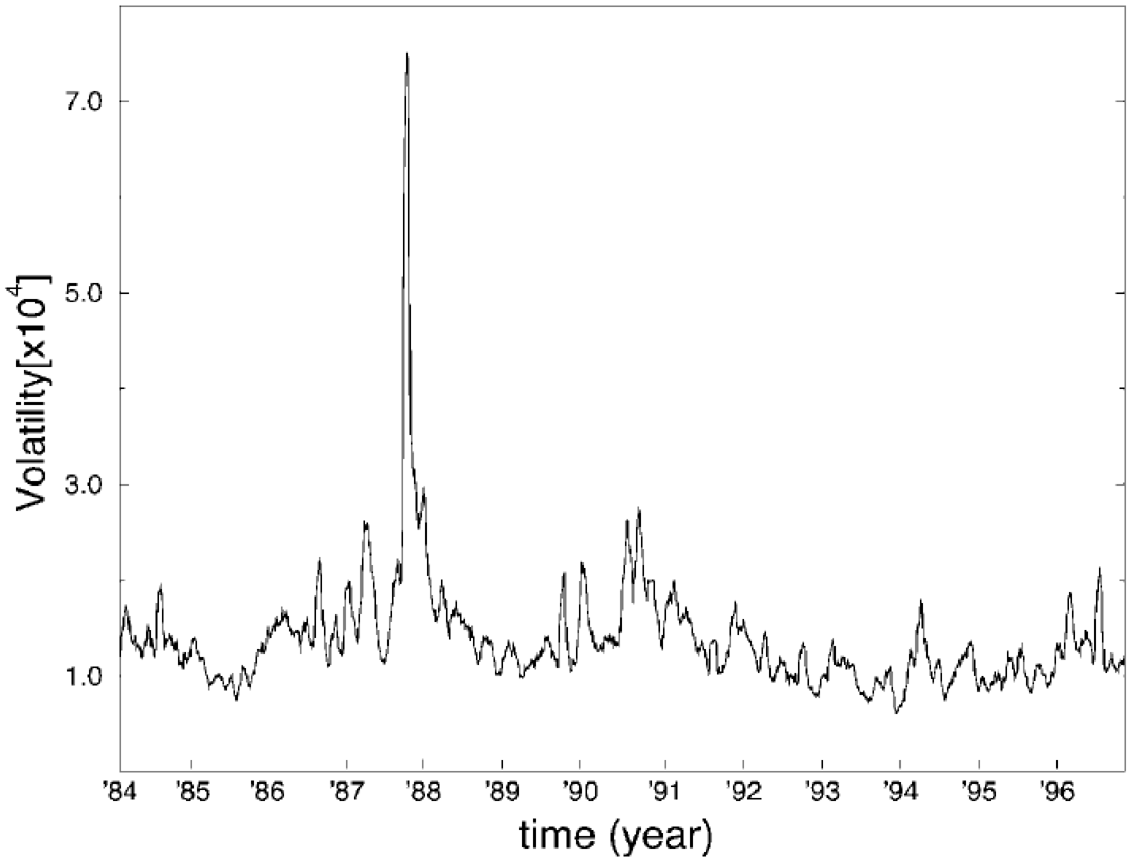

Stable Distributions

- **Consider the sum of *n* independent identically distributed (i.i.d.) random variables x_i:**

1.1 Definition of stable

https://edspace.american.edu/jpnolan/wp-content/uploads/sites/1720/2020/09/Chap1.pdf

An important property of normal or Gaussian random variables is that the sum of two of them is itself a normal random variable. One consequence of this is that if X is normal, then for X_1 and X_2 independent copies of X and any positive constants a and b,

$$aX_1 + bX_2 \stackrel{d}{=} cX + d, \tag{1.1}$$

for some positive c and some $d \in \mathbb{R}$. (The symbol $\stackrel{d}{=}$ means equality in distribution, i.e. both expressions have the same probability law.) In words, equation (1.1) says that the shape of X is preserved (up to scale and shift) under addition. This book is about the class of distributions with this property.

Definition 1.1 A random variable X is *stable* or *stable in the broad sense* if for X_1 and X_2 independent copies of X and any positive constants a and b, (1.1) holds for some positive c and some $d \in \mathbb{R}$. The random variable is *strictly stable* or *stable in the narrow sense* if (1.1) holds with $d = 0$ for all choices of a and b. A random variable is *symmetric stable* if it is stable and symmetrically distributed around 0, e.g. $X \stackrel{d}{=} -X$.

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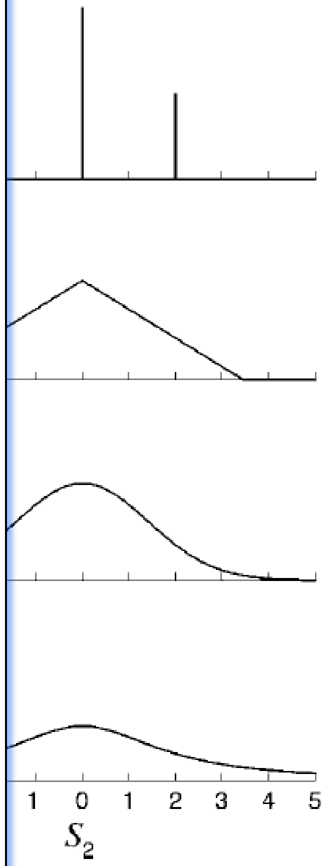


Stable Distributions (Lorentzian case)

- **Formal proof (The Lorentzian and Gaussian distributions are stable.)**

- **For Lorentzian random variables, the probability density function is**

$$P(x) = \frac{\gamma}{\pi} \frac{1}{\gamma^2 + x^2}$$

- **The Fourier transform of the pdf:** $\varphi(q) \equiv \int_{-\infty}^{+\infty} P(x) e^{iqx} dx$ **, thus** $\varphi(q) = e^{-\gamma|q|}.$

- **The convolution theorem states that the Fourier transform of a convolution of two functions is the product of the Fourier transforms of the two functions**

$$\mathscr{F}\left[f(x) \bigotimes g(x)\right] = \mathscr{F}[f(x)]\mathscr{F}[g(x)] = F(q)G(q).$$

Stable Distributions (Lorentzian case)

- **For i.i.d. random variables:** $S_2 = x_1 + x_2.$
- **The pdf $P_2(S_2)$ of the sum of two i.i.d. random variables is given by the convolution of the two pdfs of each random variable:** $P_2(S_2) = P(x_1) \bigotimes P(x_2),$
- **Implying the convolution theorem:** $\varphi_2(q) = [\varphi(q)]^2.$
- **In the general case:**

$$P_n(S_n) = P(x_1) \bigotimes P(x_2) \bigotimes \cdots \bigotimes P(x_n),$$

- **Hence**

$$\varphi_n(q) = [\varphi(q)]^n.$$

Stable Distributions (Lorentzian case)

- **The importance of using the characteristic function comes into play here.**
- **For Lorentzian distribution:** $\varphi_2(q) = e^{-2|q|\gamma}.$
- **By performing the inverse Fourier transform:** $P(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(q) e^{-iqx} dq,$
- **We obtain the pdf:** $P_2(S_2) = \frac{2\gamma}{\pi}\frac{1}{4\gamma^2 + x^2}.$ $P(x) = \frac{\gamma}{\pi}\frac{1}{\gamma^2 + x^2}$
- **Hence a Lorentzian distribution is a stable distribution.**

Stable Distributions (Gaussian case)

- **For Gaussian random variables, the pdf is**

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}.$$

- **The characteristic function is:** $\varphi(q) = e^{-(\sigma^2/2)q^2} = e^{-\gamma q^2}$, **where** $\gamma \equiv \sigma^2/2.$

- **Hence** $\varphi_2(q) = e^{-2\gamma q^2}.$

Stable Distributions (Gaussian case)

- **By performing the inverse Fourier transform, we obtain**

$$P_2(S_2) = \frac{1}{\sqrt{8\pi\gamma}} e^{-x^2/8\gamma}.$$

- **Thus the Gaussian distribution is also a stable distribution. Writing**

$$P_2(S_2) = \frac{1}{\sqrt{2\pi}(\sqrt{2}\sigma)} e^{-x^2/2(\sqrt{2}\sigma)^2},$$ **we find** $$\sigma_2 = \sqrt{2}\sigma.$$

- **We have verified that at least two stable stochastic processes exist: Lorentzian and Gaussian. The characteristic functions of both processes have the same functional form** $\varphi(q) = e^{-\gamma|q|^\alpha}$, **where α=1 for the Lorentzian, and α=2 for the Gaussian form.**

General Class of Stable Distributions

Lévy [92] and Khintchine [80] solved the general problem of determining the entire class of stable distributions. They found that the most general form of a characteristic function of a stable process is

$$\ln \varphi(q) = \begin{cases} i\mu q - \gamma |q|^{\alpha} \left[1 - i\beta \frac{q}{|q|} \tan\left(\frac{\pi}{2}\alpha\right)\right] & [\alpha \neq 1] \\ i\mu q - \gamma |q| \left[1 + i\beta \frac{q}{|q|} \frac{2}{\pi} \ln |q|\right] & [\alpha = 1] \end{cases}, \qquad (4.20)$$

where $0 < \alpha \leq 2$, γ is a positive scale factor, μ is any real number, and β is an asymmetry parameter ranging from -1 to 1.

<u>Special cases:</u>

- $\alpha = 1/2$, $\beta = 1$ (Lévy–Smirnov)
- $\alpha = 1$, $\beta = 0$ (Lorentzian)
- $\alpha = 2$ (Gaussian)

References:

[92] P. Lévy: *Calcul des probabilités,* (Gauthier-Villars, Paris, 1925)

[80] A. Ya. Khintchine and P. Lévy, *Sur Les Loi Stables,* C. T. Acad. Sci. Paris **202** (1936) 374-376

Power-law behavior

- **Henceforth we consider here only the symmetric stable distribution (β = 0)**
- **with a zero mean (μ = 0). The symmetric stable distribution of index α and scale factor γ is,**

$$P_{\mathrm{L}}(x) \equiv \frac{1}{\pi} \int_0^\infty e^{-\gamma |q|^\alpha} \cos(qx) dq.$$

- **we find the asymptotic approximation of a stable distribution of index α valid for large values of |*x*|,**

$$P_{\mathrm{L}}(|x|) \sim \frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi |x|^{1+\alpha}} \sim |x|^{-(1+\alpha)}.$$

The asymptotic behavior for large values of x is a power-law behavior, a property with deep consequences for the moments of the distribution. Specifically, $E\{|x|^n\}$ diverges for $n \geq \alpha$ when $\alpha < 2$. In particular, all Lévy stable processes with $\alpha < 2$ have *infinite* variance. Thus non-Gaussian stable stochastic processes do not have a characteristic scale – the variance is infinite!

The St. Petersburg paradox

The St. Petersburg game:

A casino offers a game of chance for a single player in which a fair coin is tossed at each stage. The initial stake begins at 2 dollars and is doubled every time tails appears. The first time heads appears, the game ends and the player wins whatever is the current stake. How much would we pay to play this game?

Let's calculate the expected value:

$$E = \frac{1}{2} \times 2 + \frac{1}{4} \times 4 + \frac{1}{8} \times 8 + \frac{1}{16} \times 16 + \ldots$$

$$= 1 + 1 + 1 + 1 + \ldots$$

$$= \infty$$

The bank would ask for its loss on this game - which is an infinitely large sum. But there is no player who would pay more than $30 for it. The two parties cannot come to an agreement. Why?

They are trying to determine a characteristic scale for a problem that has no characteristic scale.

https://en.wikipedia.org/wiki/St._Petersburg_paradox

Price Change Statistics I.

Stable non-Gaussian distributions are of interest because they obey limit theorems. However, we should not expect to observe price change distributions that are stable. The reason is related to the hypotheses underlying the limit theorem for stable distributions:

(i) pairwise-independent,

(ii) identically distributed.

Hypothesis (ii) is not generally verified by empirical observation because, e.g., the standard deviation of price changes is strongly time-dependent. This is known in finance as time-dependent volatility.

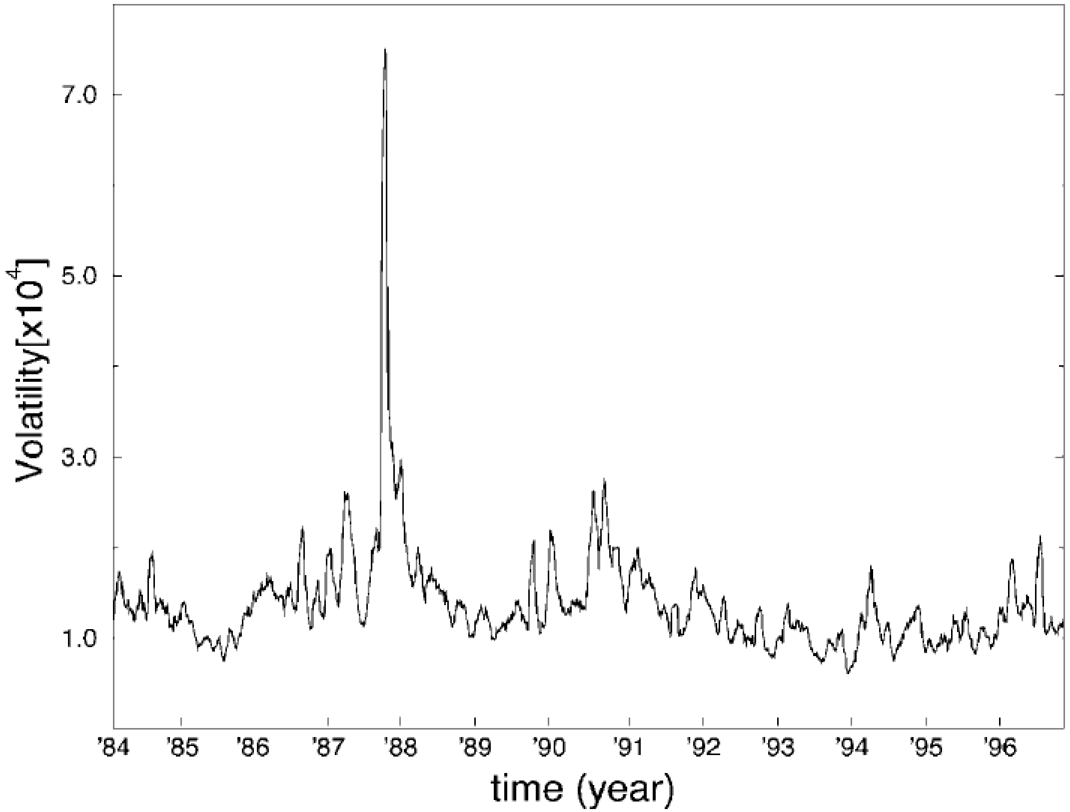


Fig. 4.2. Monthly volatility of the S&P 500 index measured for the 13-year period January 1984 to December 1996. Courtesy of P. Gopikrishnan.

Price Change Statistics II.

There is a more appropriate limit distribution theorem, in which the xi v.v.'s are independent, but not necessarily of the same distribution (it was first presented by Bawly and Khintchine).

In the sum S_n there is no single stochastic variable x_i that dominates the sum. The Khintchine theorem states that it is necessary and sufficient that $F_n(S)$, the limit distribution function, be infinitely divisible.

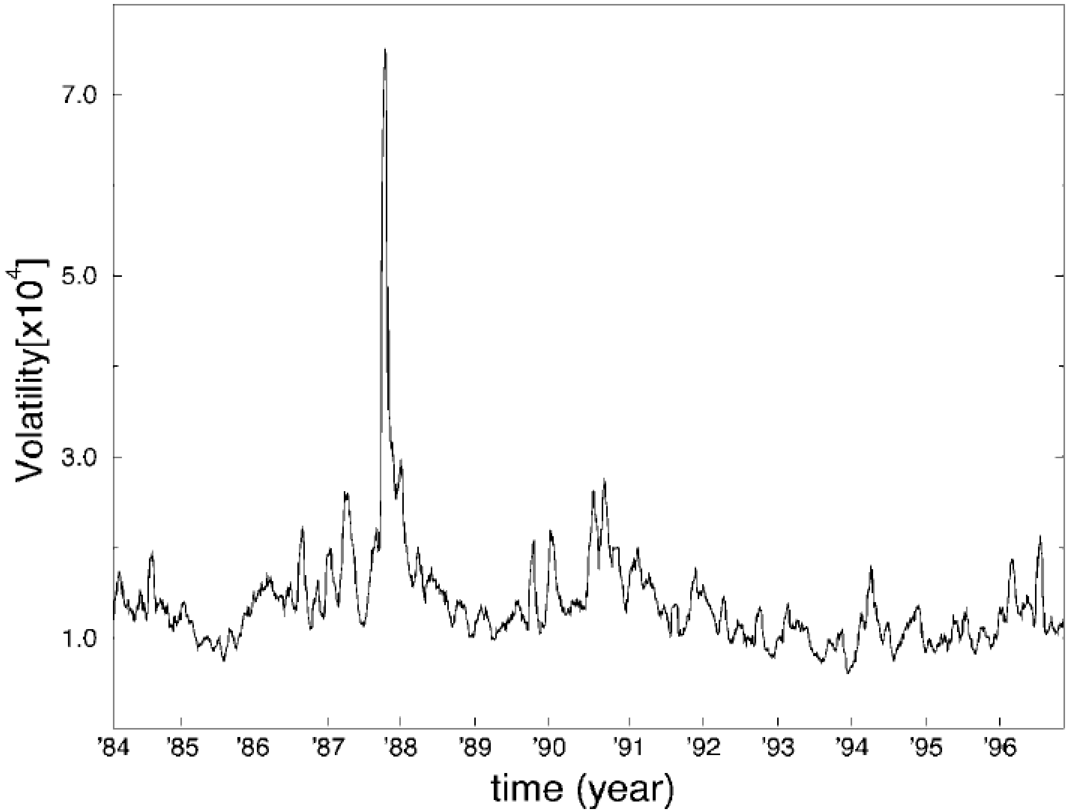


Fig. 4.2. Monthly volatility of the S&P 500 index measured for the 13-year period January 1984 to December 1996. Courtesy of P. Gopikrishnan.

Infinitely divisible random processes

A random process y is infinitely divisible if, for every natural number k, it can be represented as the sum of k i.i.d. random variables $\{x_i\}$. The distribution function $F(y)$ is infinitely divisible if and only if the characteristic function $\varphi(q)$ is, for every natural number k, the kth power of some characteristic function $\varphi_k(q)$. In formal terms

$$\varphi(q) = [\varphi_k(q)]^k, \qquad (4.33)$$

with the requirements (i) $\varphi_k(0) = 1$ and (ii) $\varphi_k(q)$ is continuous.

Example (Stable processes)

$$\varphi(q) = \exp\left[i\mu q - \frac{\sigma^2}{2}q^2\right] \Longrightarrow \varphi_k(q) = \exp\left[\frac{i\mu q}{k} - \frac{\sigma^2}{2k}q^2\right]$$

$$\varphi(q) = \exp[i\mu q - \gamma|q|^\alpha], \Longrightarrow \varphi_k(q) = \exp\left[\frac{i\mu q}{k} - \frac{\gamma}{k}|q|^\alpha\right]$$

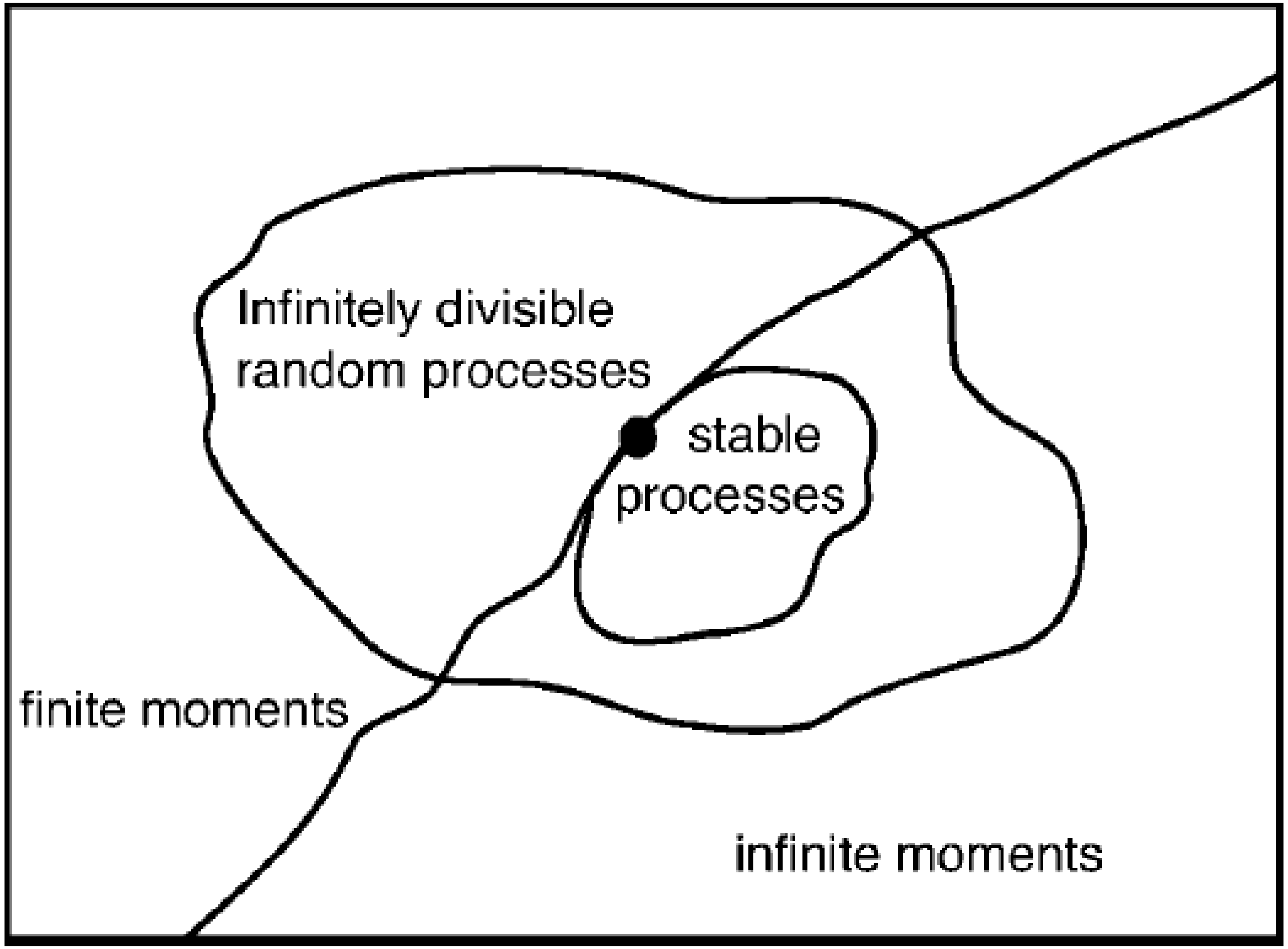
Example (Poisson processes)

$$P(m;\lambda) = e^{-\lambda}(\lambda^m/m!), \text{ with } m = 0, 1, \ldots, n,$$

$$\varphi(q) = \exp[\lambda(e^{iq} - 1)], \Longrightarrow \varphi_k(q) = \exp\left[\frac{\lambda}{k}(e^{iq} - 1)\right].$$

Classes of Random Processes

- The class of infinitely divisible random processes is a large class that includes the class of stable random processes.
- They may have finite or infinite variance.
- Stable non-Gaussian random
- processes have infinite variance.
- The Gaussian process is the only
- stable process with finite variance.



Thank you for your attention!

- **Definition of stable distributions**
- **Formal proof for Lorentzian and Gaussian cases**
- **Price Change Statistics**
- **Class of Random processes**