Heavy tailed distributions in high energy physics

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Outline

- Stable distributions and their representations
- Application in HBT measurements
- Application in multiplicity measurements
- Summary

Stable distributions

- Let X to be a random variable and X_1, X_2 its independent copies
- If

$$aX_1 + bX_2 = cX + d$$

- holds for $a, b, c, d > \mathbb{R}^+$ then X is stable in the *broad sense*
- If $d = 0$ then in the *strict sense*
- X is symmetric stable if it is symmetrically distributed around 0.
- Examples: Gaussian, Cauchy, Levy

Examples

- Gaussian: $X \sim N(\mu, \sigma^2)$

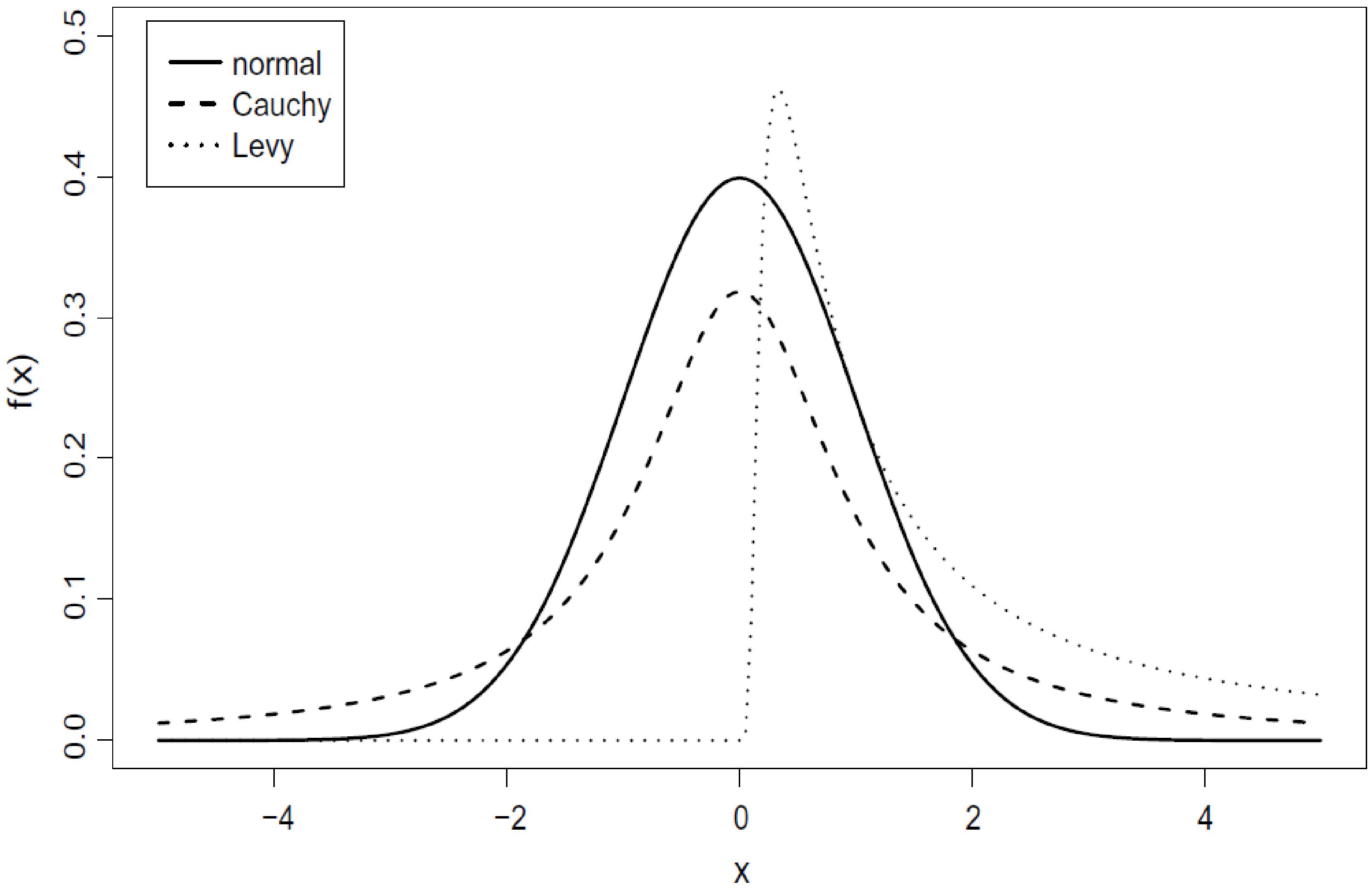
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- Cauchy: $X \sim Cauchy(\gamma, \delta)$

$$f(x) = \frac{\frac{\gamma}{\pi}}{\gamma^2 + (x-\delta)^2}$$

- Levy: $X \sim Levy(\gamma, \delta)$

$$f(x) = \sqrt{\frac{\gamma}{2\pi}} \frac{1}{(x-\delta)^{3/2}} \exp\left(-\frac{\gamma}{2(x-\delta)}\right)$$



Parametrizations – 1

- No unique but general formula is known for stable distributions
- The most concrete way is through the characteristic function

$$\phi(u) = \begin{cases} \exp\left(-\gamma^{\alpha}|u|^{\alpha}\left[1 - i\beta\tan\left(\frac{\pi\alpha}{2}\right)\left(sign(u)\right)\right] + i\delta u\right) & if\ \alpha \neq 1 \\ \exp\left(-\gamma|u|\left[1 + i\beta\left(sign(u)\right)\log(u)\right] + i\delta u\right) & if\ \alpha = 1 \end{cases}$$

- where $0 < \alpha \leq 2, \ \ -1 \leq \beta \leq 1, \ \ \gamma \geq 0, \quad \delta \in \mathbb{R}$
- Usual high energy physics application $\beta = 0, \delta = 0$, i. e.

$$\phi(u) = \exp(-\gamma^{\alpha}|u|^{\alpha}) \ \Rightarrow \ \ \mathrm{S}(\alpha, 0, \gamma, 0; x) = \int e^{iux}\phi(u)$$

Parametrizations – 2

- Utilization of Fox's H-function (generalized hyperbolic functions)

$$H_{p,q}^{m,n}\left(z\middle|\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_p,B_p)\end{matrix}\right)=\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{z^{-s}\left(\left(\prod_{j=1}^{m}\Gamma(b_j+B_js)\right)\left(\prod_{j=1}^{n}\Gamma(1-a_j-A_js)\right)\right)}{\left(\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\right)\left(\prod_{j=n+1}^{p}\Gamma(a_j+A_js)\right)}$$

- Symmetric ($\beta = 0$) and centralized ($\delta = 0$) stable distributions

$$S(\alpha,0,\gamma,0;x)=\frac{1}{\alpha\gamma}H_{2,2}^{1,1}\left(-\frac{x}{\gamma}\middle|\begin{matrix}\left(1-\frac{1}{\alpha},\frac{1}{\alpha}\right),\left(\frac{1}{2},\frac{1}{2}\right)\\(0,1),\left(\frac{1}{2},\frac{1}{2}\right)\end{matrix}\right)=\cdots$$

$$=\frac{1}{\alpha\gamma\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\left(-\frac{x}{\gamma}\right)^{\nu}\Gamma\left(\frac{1}{\alpha}(1+\nu)\right)\cos\left(\frac{\pi\nu}{2}\right)=\cdots=\int_0^{\infty}e^{-\gamma^{\alpha}w^{\alpha}}\cos(xw)\,dw$$

Levy parametrization of the C_2

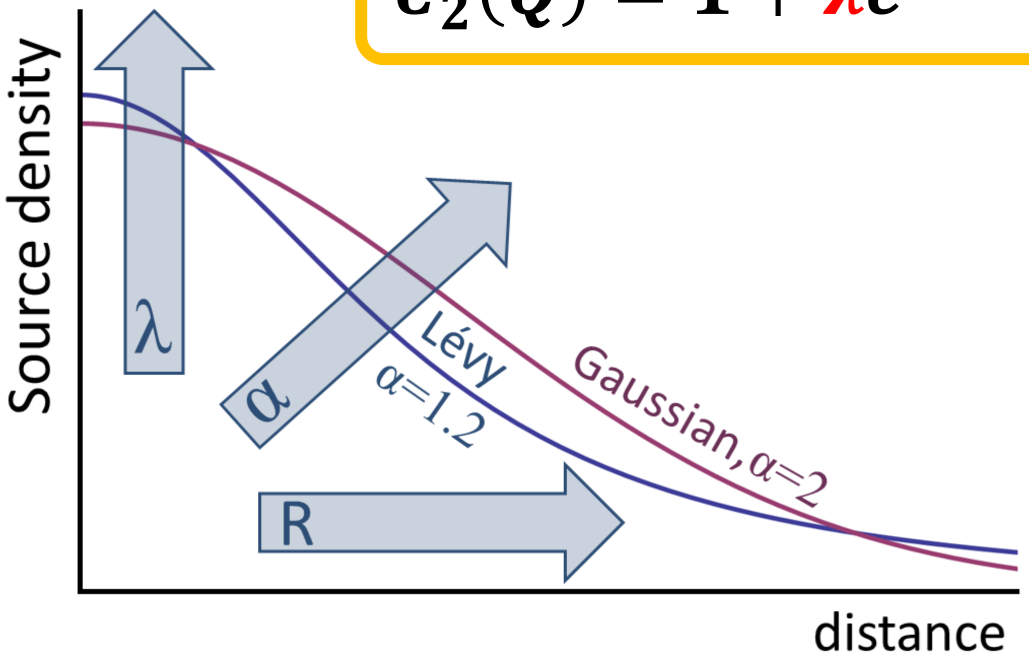
- Generalized Gaussian – Levy distribution

$$\mathcal{L}(\alpha, R, r) = \frac{1}{(2\pi)^3} \int d^3q \, e^{iqr} e^{-\frac{1}{2}|qR|^\alpha}$$

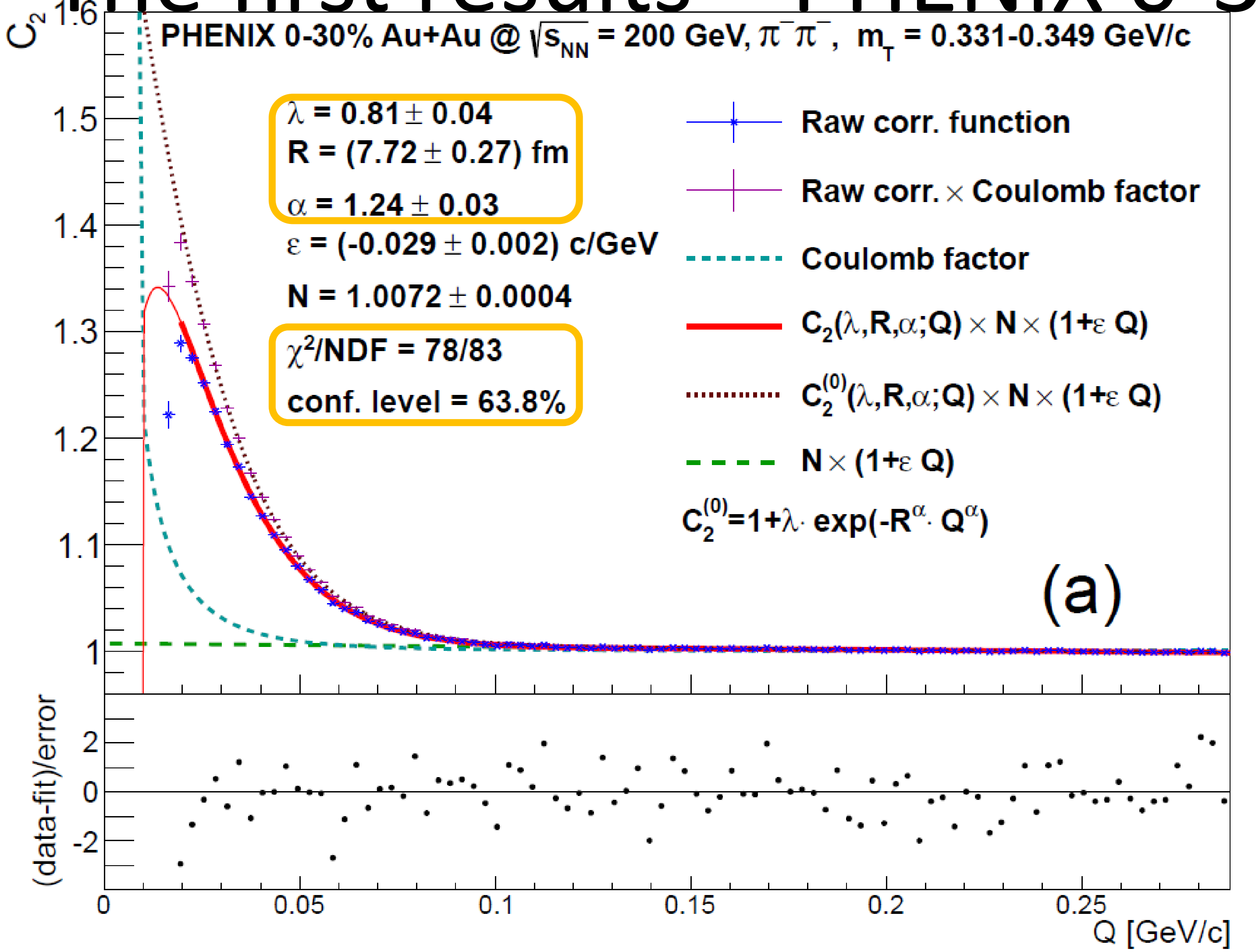
- $\alpha = 2$: Gaussian, $\alpha = 1$: Cauchy, $0 < \alpha \leq 2$: Levy
- Assume the source to be Levy!

$$\boldsymbol{C_2(Q) = 1 + \lambda e^{(RQ)^\alpha}}$$

- $\lambda(K)$: core-halo parameter
- $R(\mathrm{K})$: Levy-scale parameter
- $\alpha(K)$: Levy index of stability

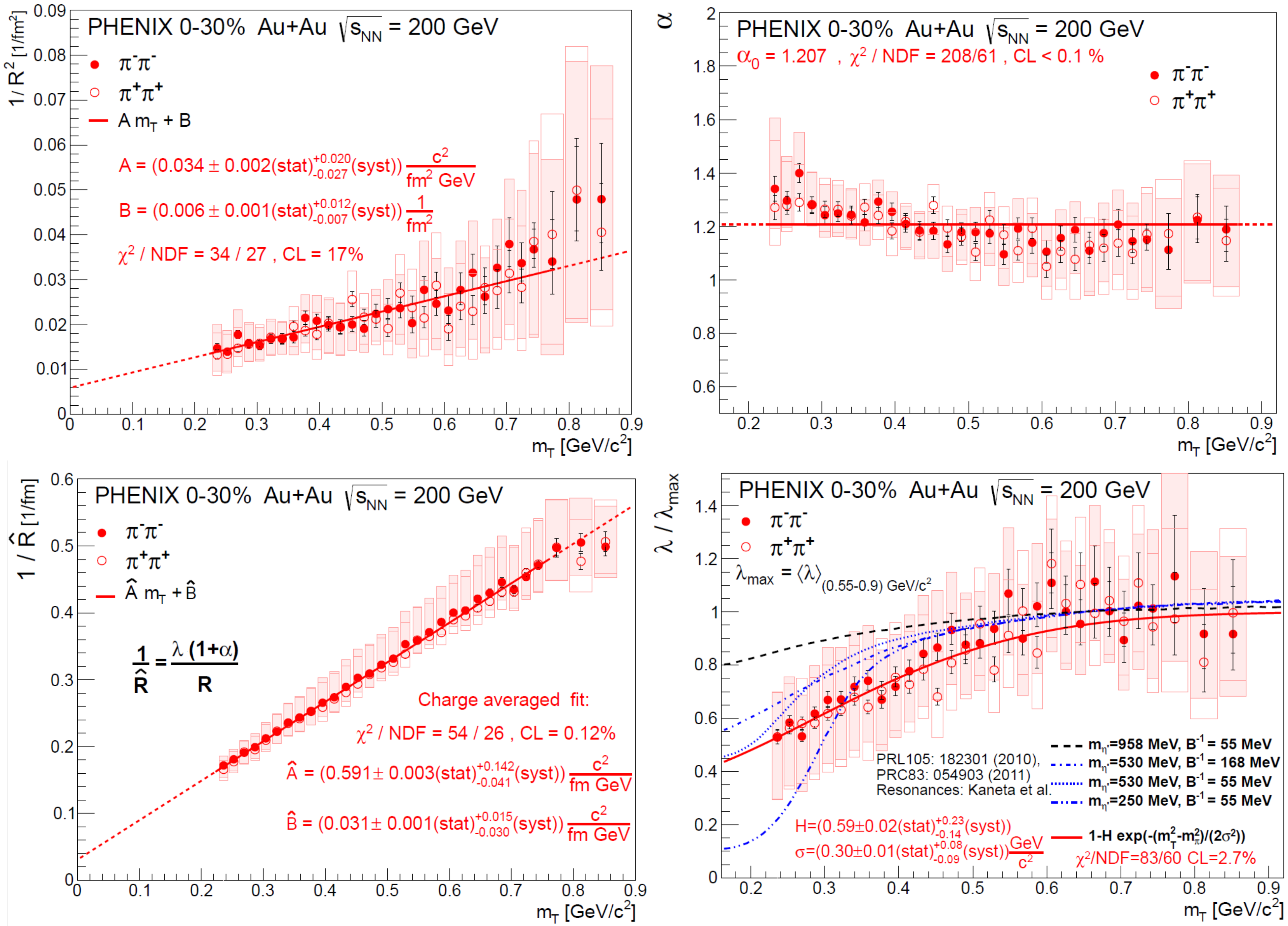


The first results – PHENIX 0-30% Au+Au



- Measured correlation function in 31 m_T bin with 0-30% cent.
- Coulomb correction incorporated into the fit function
- $\alpha \neq 2$ nor $\alpha \neq 1$
- The fits are acceptable in terms of confidence level and χ^2/NDF
- Gaussian parametrization cannot describe the data

The first results – PHENIX 0-30%, Au+Au

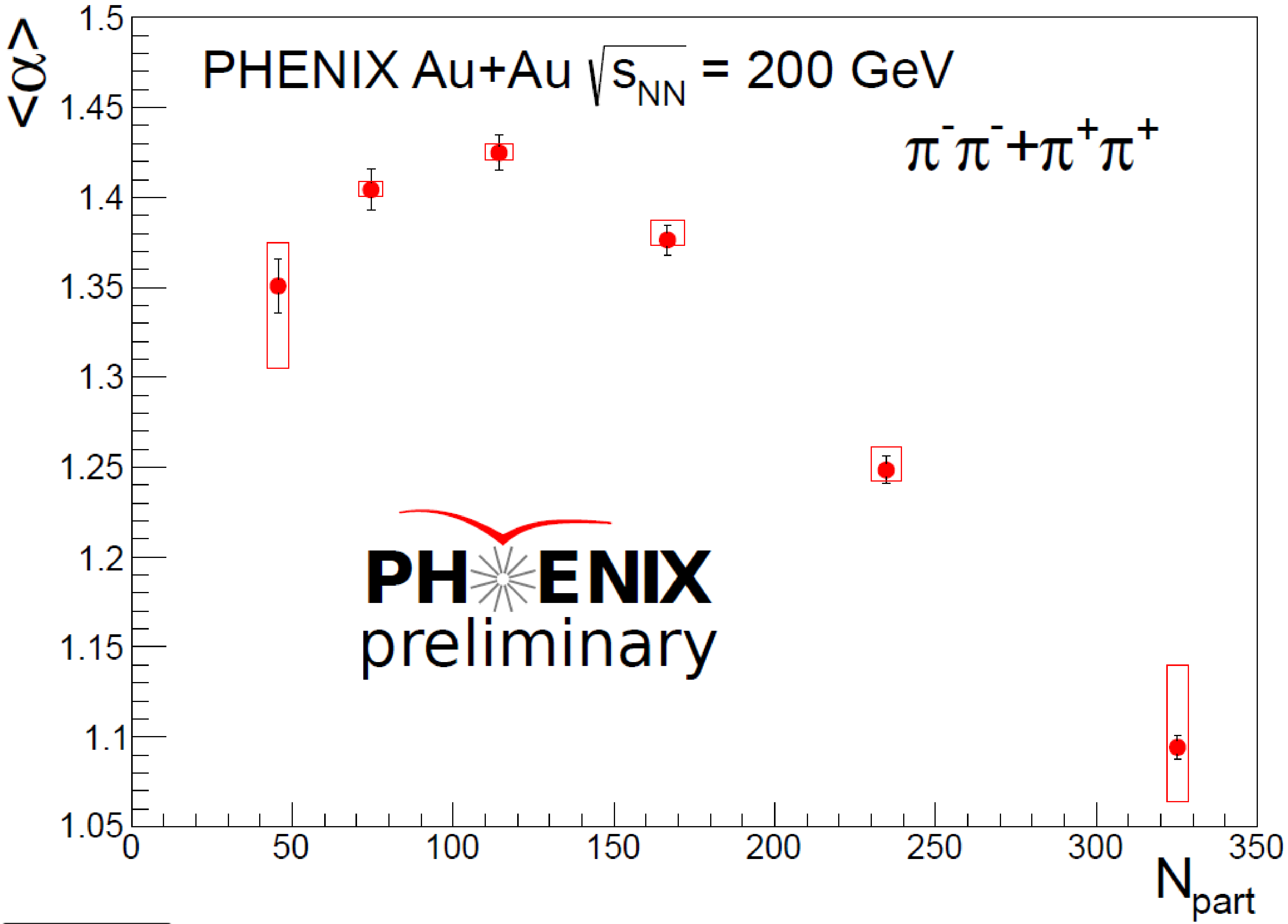


- R exhibits hydro scaling
- $1 < \alpha < 2$, $\langle\alpha\rangle \approx 1.2$
- $\lambda(m_T)$ suppressed which compatible with modified η' mass in the medium (compared with a resonance model)

- New scaling parameter
  - Interpretation?
- Interpretation of α ?

- Let's see the N_{part} and $\sqrt{s_{NN}}$ dependence

N_{part} dependence – PHENIX Au+Au



- R exhibits hydro scaling
- $1 < \langle\alpha\rangle < 2$
- $\langle\alpha\rangle$ depends on N_{part}
- $\lambda(m_T)$ suppressed
- The suppression doesn't depend on centrality
- Models can be ruled out

- Preliminary results!
- Improved, final results very soon

Application in multiplicity measurement

- Charged particle multiplicity
  - One of the simplest observable (dealing with positive integers)
- Monte Carlo models having hard time to reproduce them
- Particle creation and the structure of the QCD → KNO scaling

$$P_n = \frac{1}{\langle n \rangle} \Psi\left(\frac{n}{\langle n \rangle}\right)$$

- Functional form of Ψ can be constrained from P_n measurement
- If $\Psi(\mathrm{z}) = \frac{\lambda^k}{\Gamma(k)} z^{k-1} e^{-\lambda z}$ then $P_n = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)} \left(\frac{k}{k+\langle n \rangle}\right)^k \left(\frac{\langle n \rangle}{k+\langle n \rangle}\right)^n$ NBD
- A generalization from S. Hegyi

Generalized NBD

- Let's calculate P_n as a Poisson transform

$$P_n = \int_0^\infty \Psi(z) \frac{(\langle n \rangle z)^n}{n!} e^{-\langle n \rangle z} dz$$

- and let $\Psi(z)$ be the generalized gamma distribution

$$\Psi(z) = \frac{\mu}{\Gamma(k)} \lambda^{\mu k} z^{\mu k - 1} \exp(-[\lambda z]^{\mu})$$

- $\mu = 1$ restore the gamma distribution
- It can be shown that the above integral can be expressed as Fox's H function

Generalized NBD – Fox’s H function

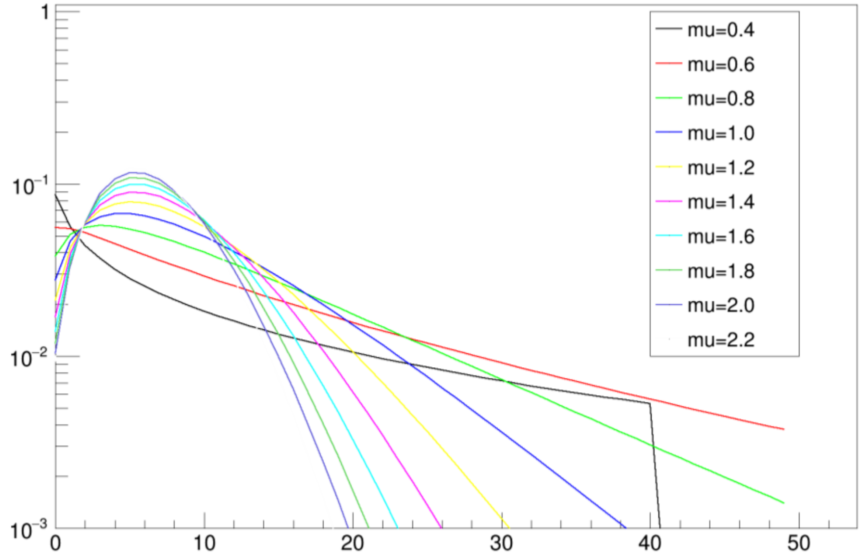
$$\Psi(z) = \frac{\lambda}{\Gamma(k)} H^{1,0}_{0,1}\left(\lambda z \,\middle|\, \begin{matrix} - \\ \left(k - \frac{1}{\mu}, \frac{1}{\mu}\right) \end{matrix}\right)$$

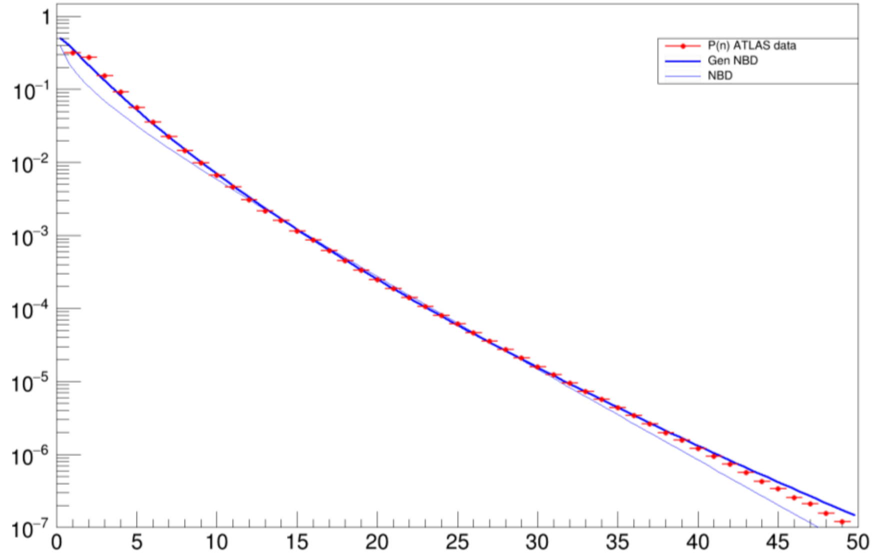
$$P_n = \frac{1}{n!\Gamma(k)} H^{1,1}_{1,1}\left(\frac{\lambda}{\langle n \rangle} \,\middle|\, \begin{matrix} (1-n, 1) \\ \left(k, \frac{1}{\mu}\right) \end{matrix}\right) \quad if \;\; 0 < \mu < 1$$

$$P_n = \frac{1}{n!\Gamma(k)} H^{1,1}_{1,1}\left(\frac{\lambda}{\langle n \rangle} \,\middle|\, \begin{matrix} \left(1-k, \frac{1}{\mu}\right) \\ (n, 1) \end{matrix}\right) \quad if \;\; \mu > 1$$

$$P_n = \frac{1}{n!\,\Gamma(k)} \left(\frac{\langle n \rangle}{\lambda}\right)^n \int_0^\infty t^{k+\frac{n}{\mu}-1} e^{-t-\left(\frac{\langle n \rangle}{\lambda}\right) t^{\frac{1}{\mu}}}$$

Details and more applications:
[1] , [2] , [3] , [4] , [5]





Summary

- More precise measurements → heavy tailed distributions
- Mathematically more involved but worth it
- Can give insight to real physics or quantify deviation from expectations
- Learn it now while everybody stucks with Gauss ☺

Thank you for your attention!