Estimates of loss function concentration in noisy parametrized quantum circuits

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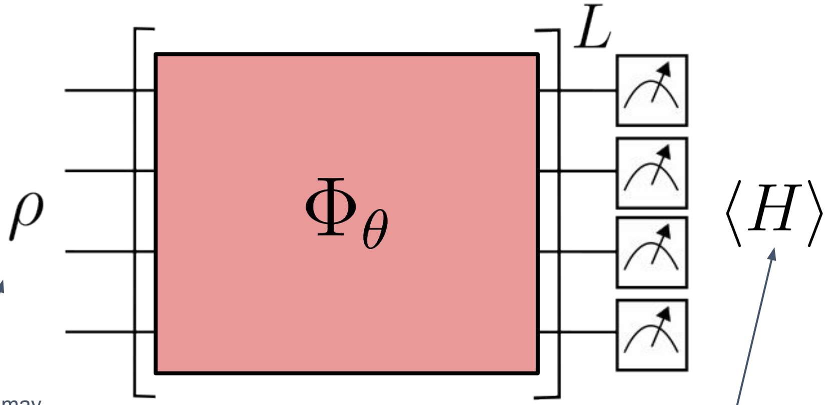
CERN | QUANTUM TECHNOLOGY INITIATIVE

UNIVERSITÀ DEGLI STUDI DI TRIESTE

Variational Quantum Algorithms in a nutshell

Variational Quantum Algorithms are based on a **hybrid** quantum classical optimization scheme.



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An observable, generally the Hamiltonian for applications to condensed matter physics.

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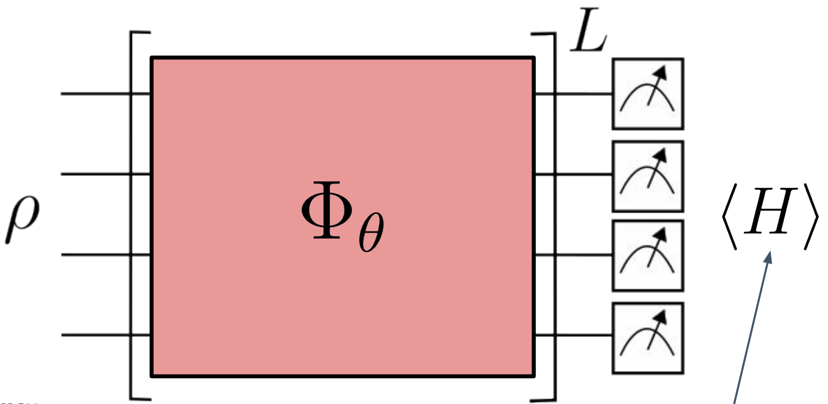

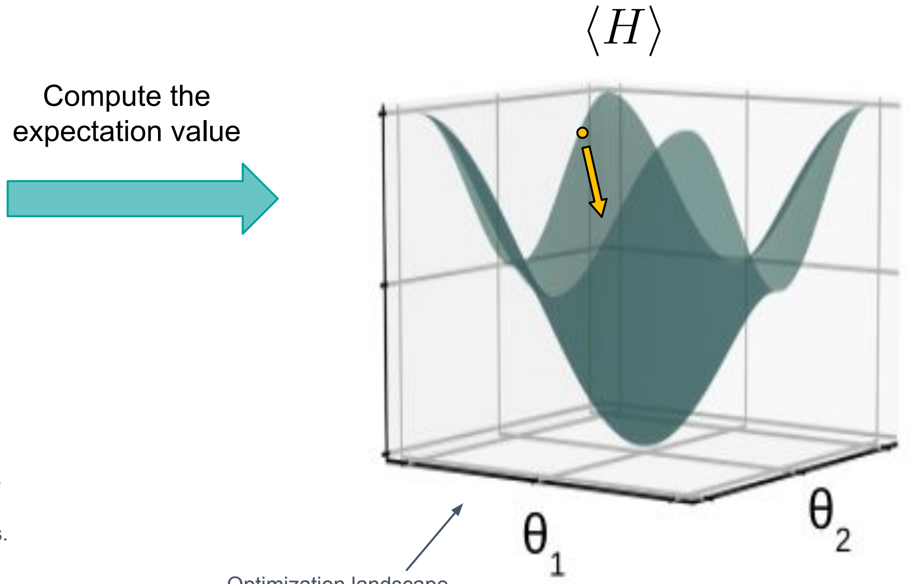

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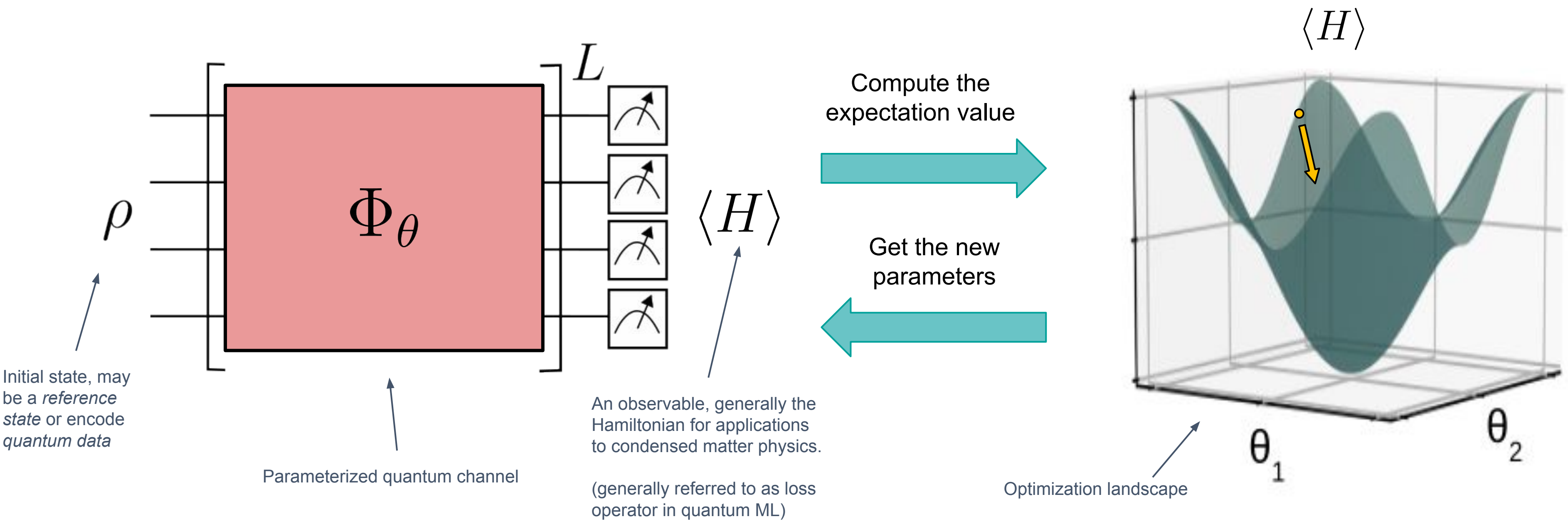

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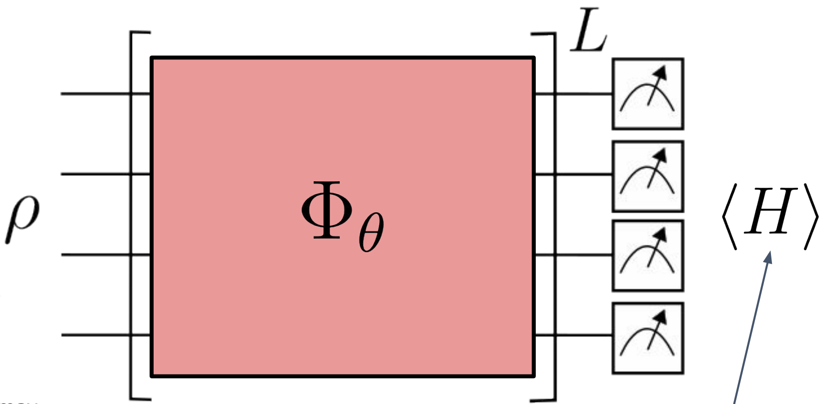

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Compute the expectation value

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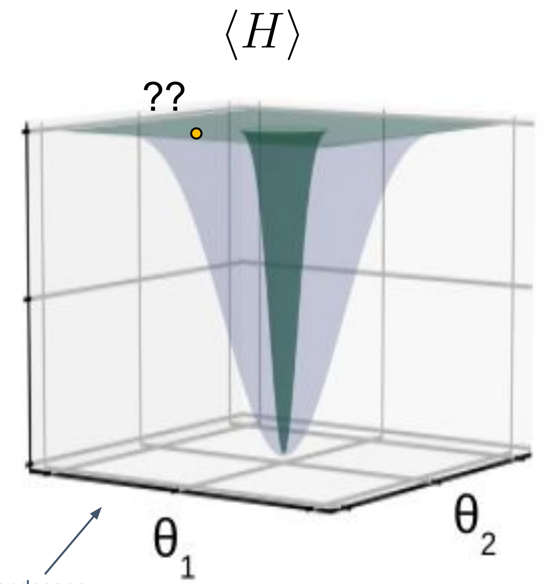

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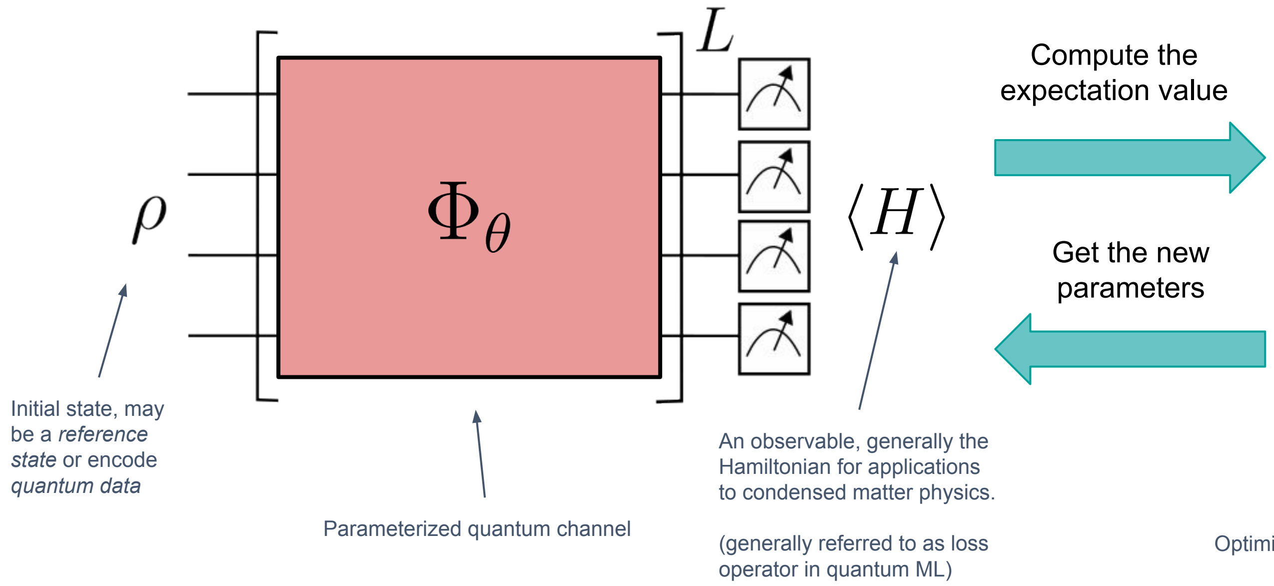

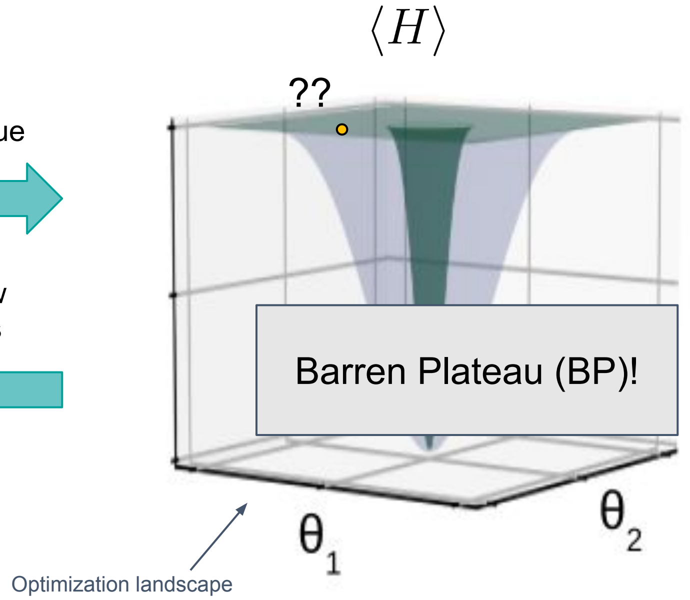

Barren Plateaus: a curse of dimensionality

The **exponential** concentration of the loss function, dubbed **barren plateau**, is a major **threat** to the trainability of Parameterized Quantum Circuits (PQCs), and is tightly linked to the **dimensionality** of the **space explorable** by the circuit

Space of **bounded operators** on H

Space of **bounded, traceless operators invariant** under the action of the PQC

$$\mathcal{B} = \bigoplus_{\kappa \in \{0,1\}^M} \mathcal{B}_\kappa$$

E. Fontana et al., Nat. Comm. 2024, M. Ragone et al., Nat. Comm. 2024

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Variance wrt θ of the **loss function**

Each subspace contributes **independently** to the variance

$$\mathbb{V}^\infty\{\langle H \rangle_\theta\} = \sum_\kappa \mathbb{V}_\kappa^\infty$$

Each contribution is **inversely proportional** to the dimension of the **corresponding subspace**

$$\mathbb{V}_\kappa^\infty \propto \frac{1}{\dim(\mathcal{B}_\kappa)}$$

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Typically, such dimension **scales exponentially** in the number *n* of qubits. However, in the presence of **symmetry**, some of such contributions might be **constant** or **scale polynomially** ⇒ No BP!

E. Fontana et al., Nat. Comm. 2024, M. Ragone et al., Nat. Comm. 2024

Barren Plateaus in a picture

One can visualize what's going on here using a **graph**. The space of bounded operators is represented here by a set of points, and the action of the PQC by arrows connecting some.

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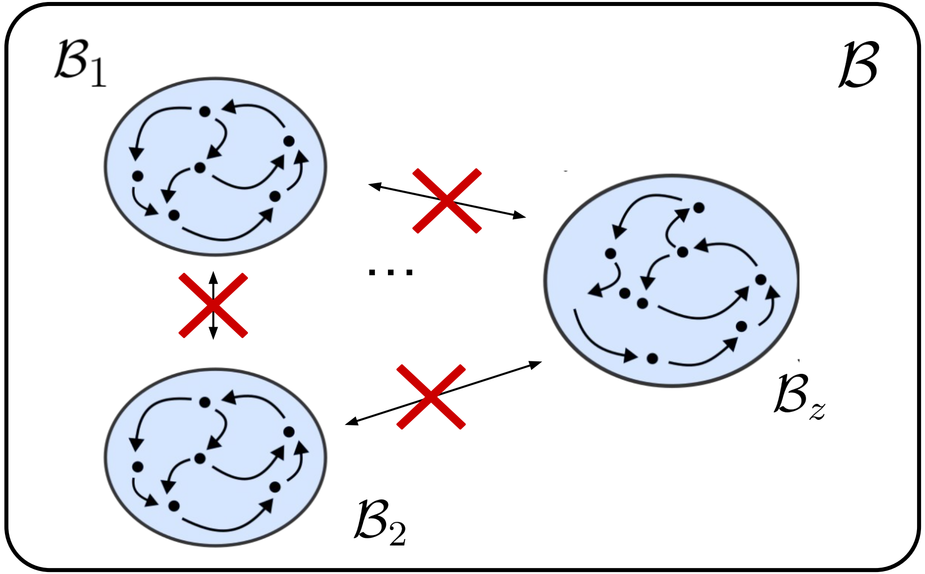

These are generally computed using an **algebraic approach** (Dynamical Lie Algebra)

The emergence of Noise-Induced Barren Plateaus (NIBP)

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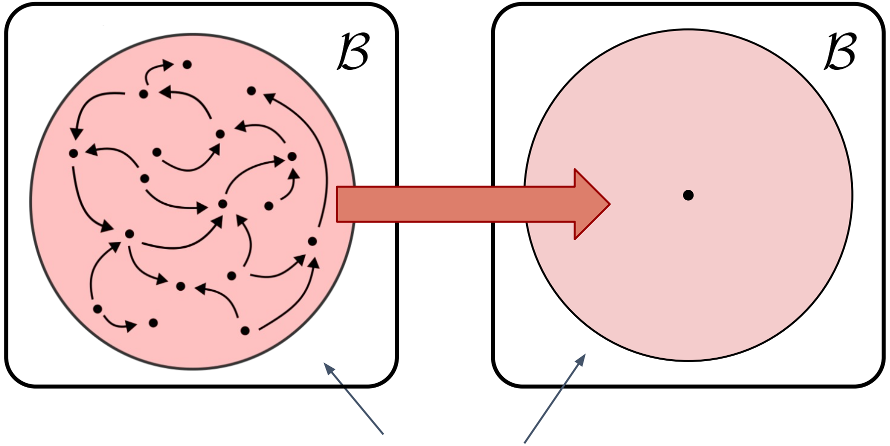

As the quantum information is lost, everything becomes **less and less distinguishable**. In the end, all points collapse into the same one, and all parameters inevitably produce the same output, i.e. the **variance vanishes**.

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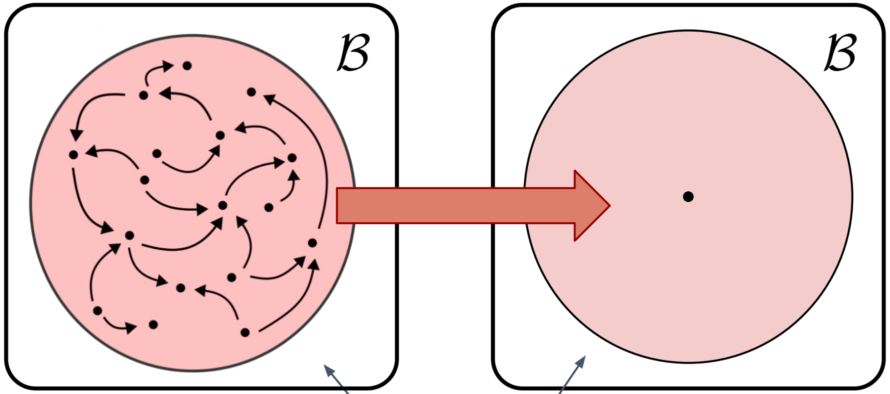

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NIBP phenomenon,

$$|\mathbb{V}^L| \in O(e^{\beta L})$$

$$\Downarrow$$

$$\mathbb{V}^{\infty} = 0$$

NIBP phenomenon, expressed in terms of the variance in the **deep circuit limit**

S. Wang et al., Nat. Comm. 2021

A rich spectrum of quantum noises

Generally, there exist a wide variety of **quantum channels**, but we only have rigorous result on the **‘extreme’** cases.

NIBP phenomenon due to a too **strong interaction** with the environment

?

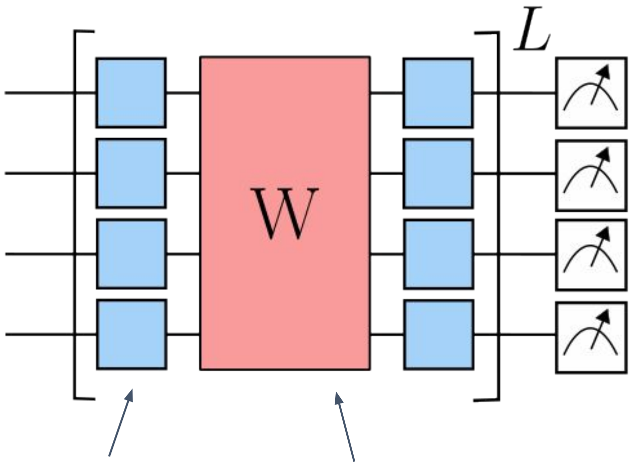
Non-unital noise? More?

BP phenomenon due to an **exponential growth** of the computational space

‘Too much’ noise ← Noise properties → No noise

How we plan to study it?

To investigate the main mechanisms at play in this general scenario, we devise a simple yet general model, comprised of **local unitary designs** and **general quantum channels**



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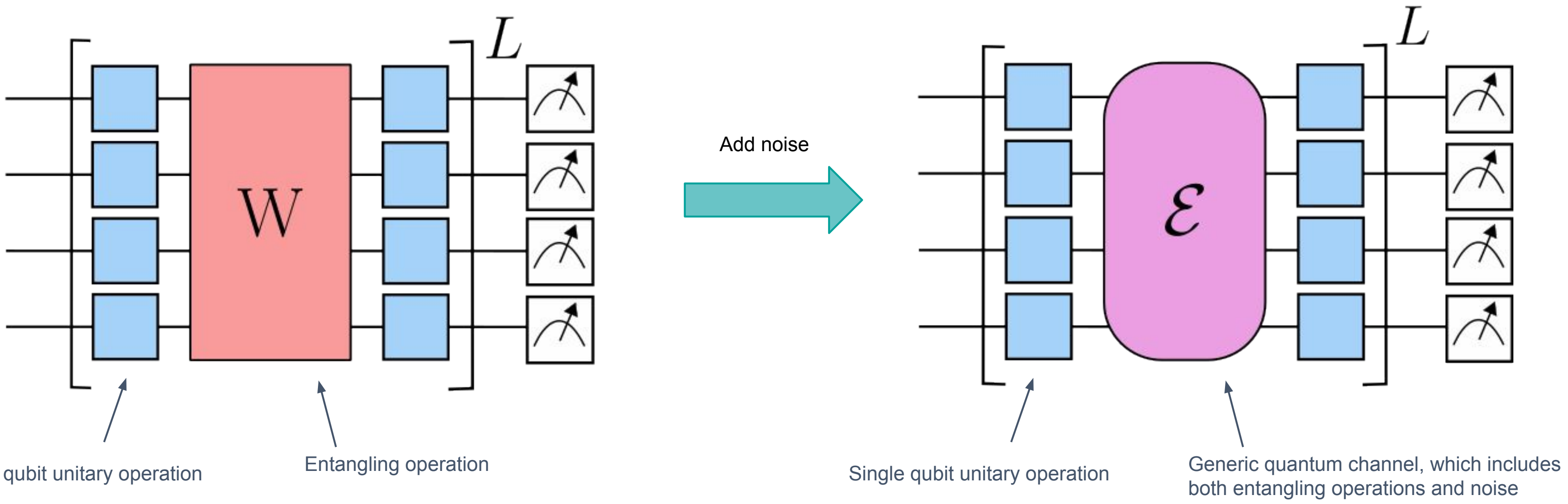
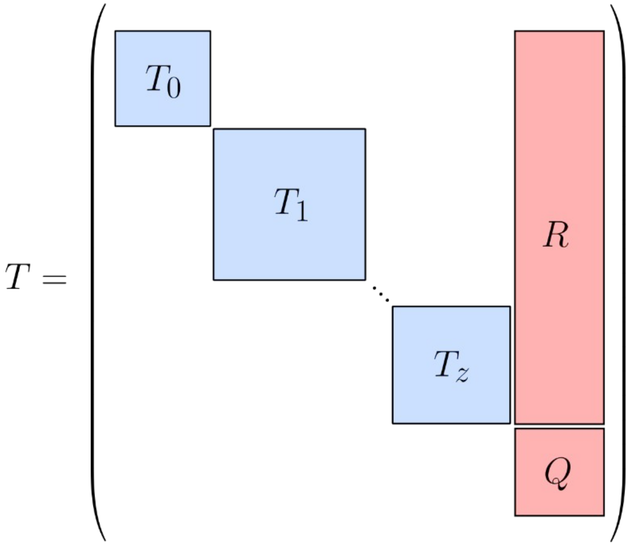


Two main applications

In this way, we can associate a **non-negative** matrix to each **layer** of the channel.
Interestingly, the **spectral properties** of T can give valuable information on the variance of the **full model**.



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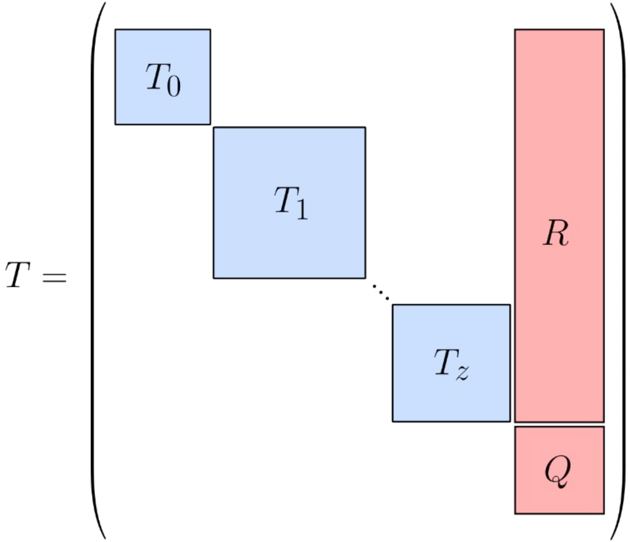

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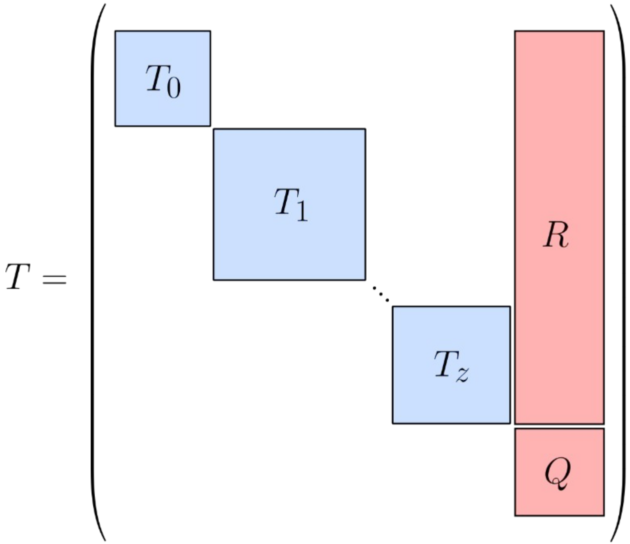

- **Analytical variance** computation in the **deep** circuit limit (L>>1)

- General formalization of **BP free**, smart initializations as **stochastic unravelling** of **sufficiently weak** noise maps

Analytical variance for general channels

*Holds rigorously for aperiodic circuits. For periodic circuits only holds for the Cesàro sum of all depths L.

For deep circuits, we have **Theorem 1 (Deep circuits):**

The variance converges exponentially fast in the number of layers L to a limiting value, i.e.*

$$\left|\mathbb{V}^{L}_{\rho,H} - \mathbb{V}^{\infty}_{\rho,H}\right| \in O\left(e^{-\beta L}\|H\|_2^2\right)$$

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Variance of loss function in the limit of **large number of layers**

Locality (= purity) of the initial state and observable

Absorption matrix, represents the absorption to the invariant subspaces

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Dimension of such subspaces

Sum over suitable **invariant subspaces of both** the single qubit blocks and the intermediate channel, based on the **largest block eigenvalue**

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Unitary contribution

Noise contribution

Dimension of such subspaces

Examples: Unitary circuits

The absorption term A vanishes in this case because unitary dynamics is **reversible**.

Corollary III.1.1 (Deep, unitary circuits).

If the channel is unitary, then the absorption terms vanish.

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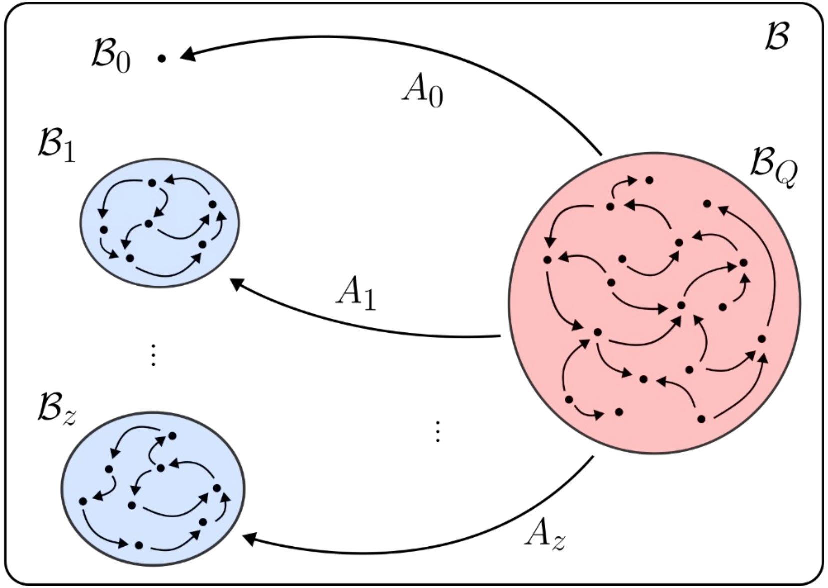

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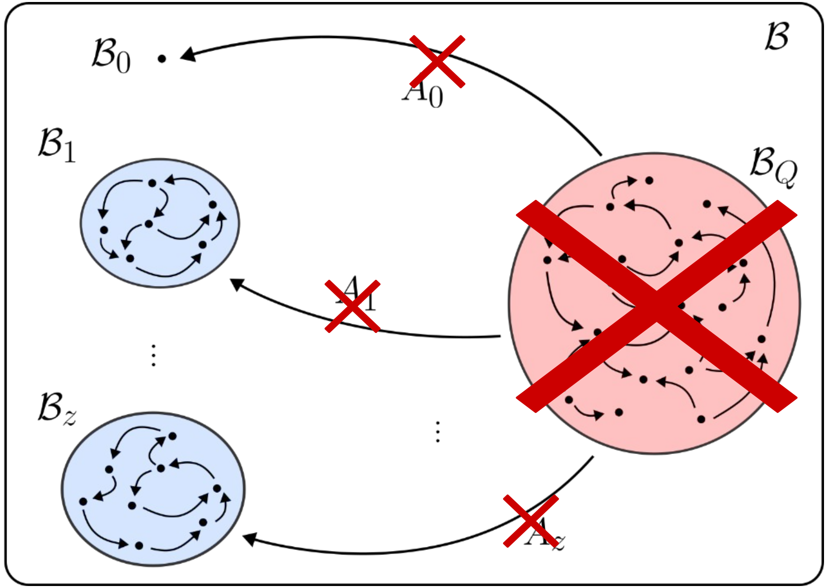

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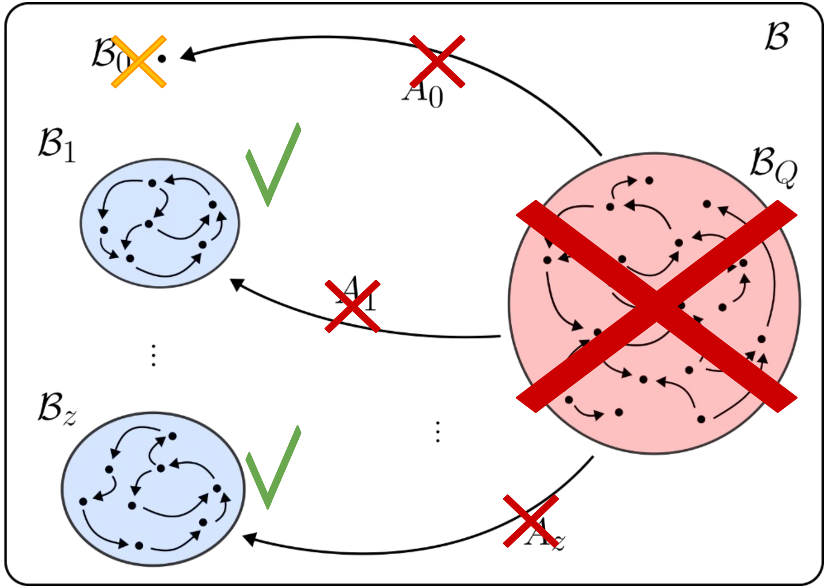

Examples: Contractive circuits

Conversely, since each invariant subspace norm-preserving to '*survive*' the deep circuit limit, in the strictly contractive case **only the absorption terms** remain.

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If the channel has no unitary subspaces, then only the absorption term survives.

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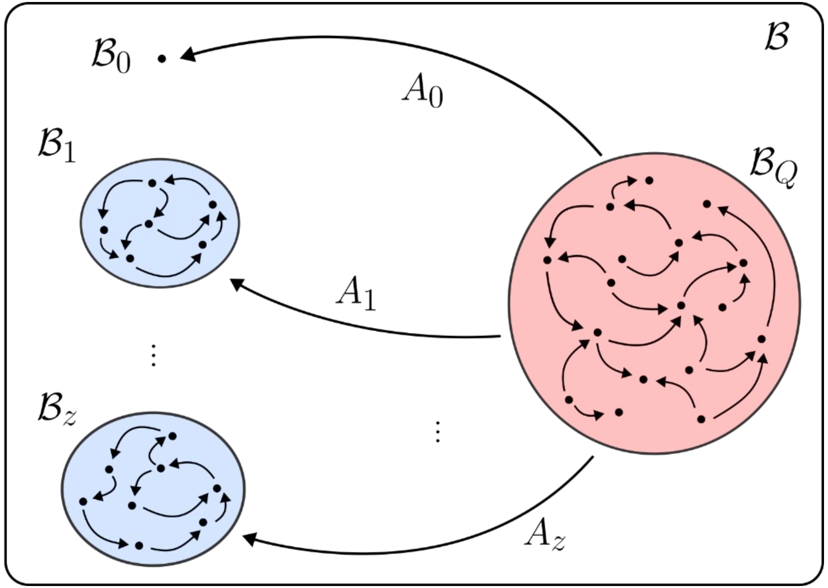
In particular, if the channel is unital, then $\mathbb{V}^{\infty}_{\rho,H} = 0$.



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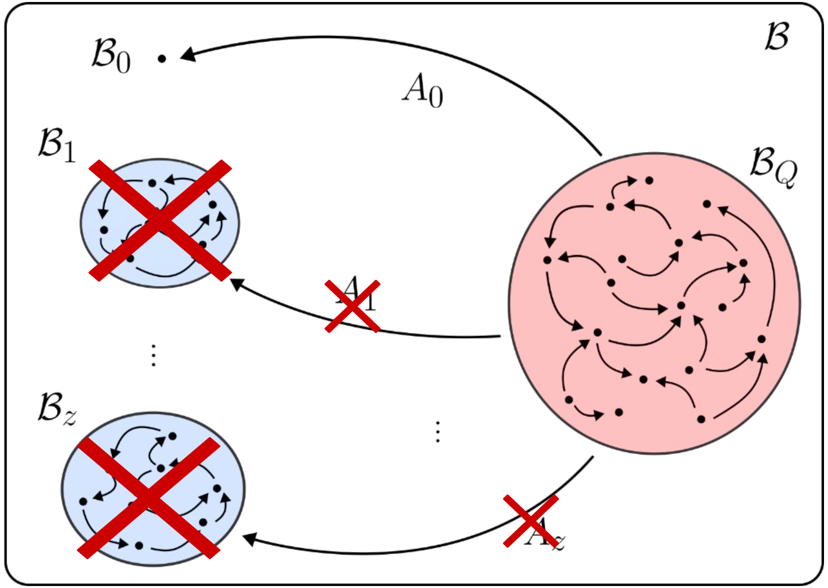

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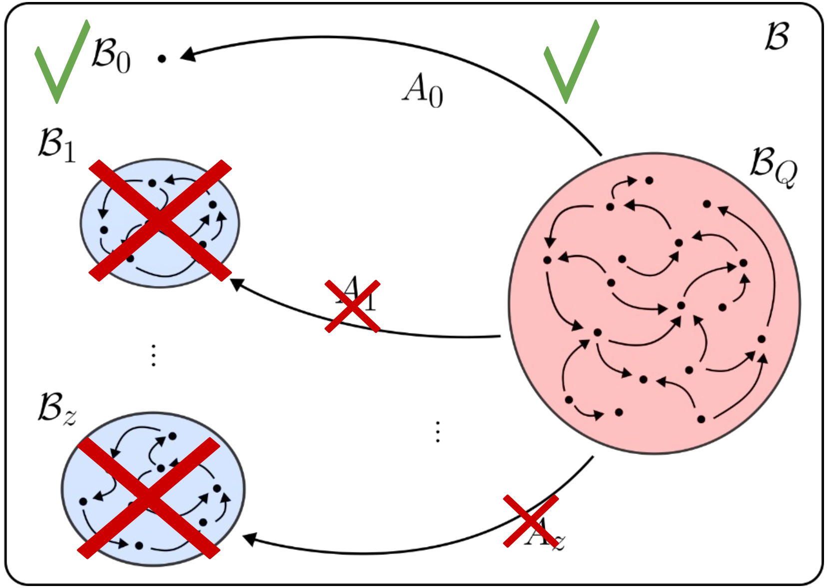

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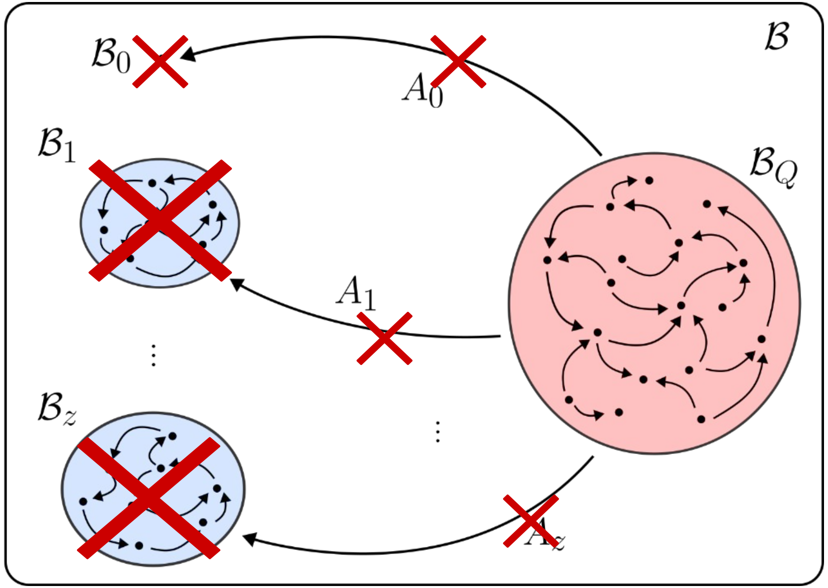

Lower-bounding the variance

Recalling that:

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This regime is achieved easily by constraining the **second largest eigenvalue** of relevant blocks

Here, lower-bounds can actually be found.

Meaning in terms of channel property

Theorem 2 (Shallow circuit).
If the diagonal entries of T scale as T_{ij}~1-log(n)/L, then β~log(n)/L and the variance is lower bounded by

$$\mathbb{V}^{L}_{\rho,H} \geq F(n)(\ell_{\rho}, \ell_{H})$$

where F(n) decays polynomially with n.

Orthogonality measure between the initial state and observable.

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Basically, **weak enough** noise won't produce NIBP

Stochastic unravelings of channels and QResNets

For any given noise map, we are guaranteed to be able to express it a **stochastic unraveling**, i.e. as the **expectation value** of some simpler operators. For maps as before, such operators can even be **unitary**!

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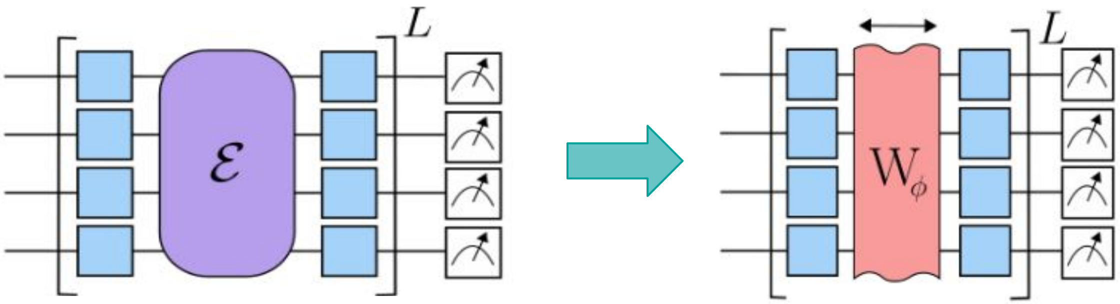
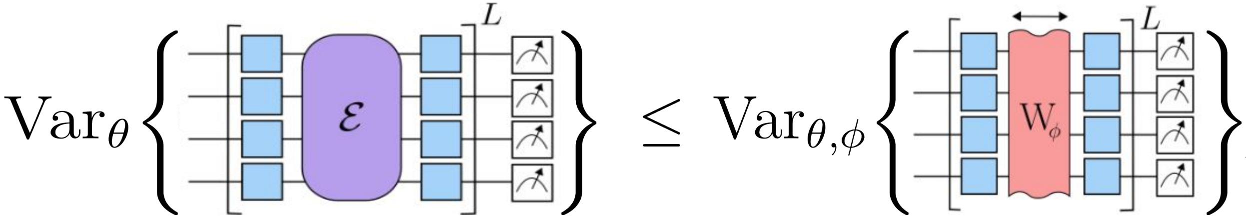


We call this a **QResNet!**

Stochastic unravelings of channels and QResNets

Proposition 4 (informal)
The variance of a channel is always smaller than the variance of the corresponding stochastic unravelling, with equality holding if and only if the channel is unitary.

Quantum channel is **still interpreted as noise**



Quantum channel is used to derive the corresponding **QResNet**, and the used as **trainable anstaz**

Where the probability distribution over ϕ is **defined by the unraveling**.

Conclusions

We introduced a new tool for the analysis of concentration in noisy variational circuits.

In particular, we observed:

- **Concentration** for noisy circuits is more **intricate** than one can naively expect.
- Noise can **not only induce** barren plateaus, but **also prevent it** if engineered correctly.
- **Weak noise** maps are closely related to **small angle-like initialization** strategies.

QUESTIONS?

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