Complementary polynomials in quantum signal processing

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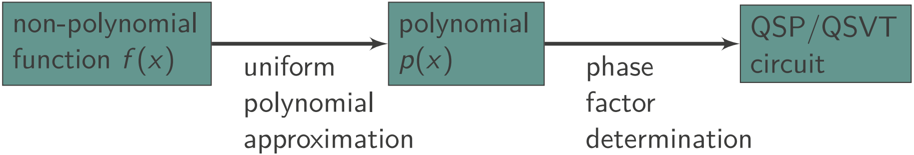
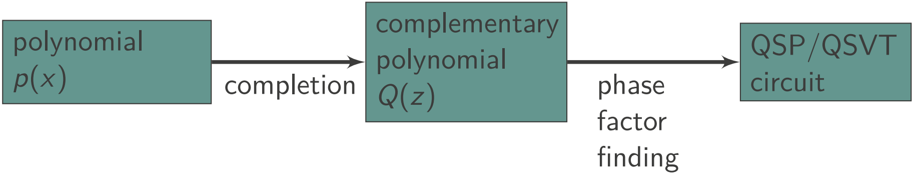
- classical pre-processing workflow:



Quantum signal processing II: result



Result. Exact contour integral and Fourier series representations for the complementary polynomial $Q(z)$, together with an efficient Fast Fourier Transform (FFT)-based algorithm for computing $Q(z)$ in the monomial basis.

Complementary polynomials I

- polynomials $p(x) \in \mathbb{R}[x]$ can be expanded in the Chebyshev basis,

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- Chebyshev polynomials correspond to Laurent polynomials,

$$T_n(x) = \frac{1}{2}(z^n + z^{-n}) \quad (n \in \mathbb{N} \cup \{0\}, x = \operatorname{Re} z, z \in \mathbb{T}),$$

where $\mathbb{T} \coloneqq \{z \in \mathbb{C} : |z| = 1\}$, i.e., $p(x)$ maps to some $F(z) \in \mathbb{R}[z, z^{-1}]$

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  where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, i.e., $p(x)$ maps to some $F(z) \in \mathbb{R}[z, z^{-1}]$
- work with (Laurent) polynomials on $\mathbb{T} \subset \mathbb{C}$, use complex analysis; Laurent polynomials on $\mathbb{T}$ may be identified with trigonometric polynomials, allowing for Fourier analytic interpretation of results

Complementary polynomials II

Theorem (Laurent QSP, Haah 2019 [1])

Let $F(z) \in \mathbb{R}[z, z^{-1}]$ *with* $\deg F = d \in \mathbb{Z}_{\geq 1}$ *and parity* $d \bmod 2$ *such that* $|F(z)| < 1$ *on* $\mathbb{T}$*. Then, there exists a complementary polynomial* $G(z) \in \mathbb{R}[z, z^{-1}]$ *and phase factors* $\phi = (\phi_j)_{j=0}^{d} \in (-\pi, \pi]^{d+1}$ *such that*

$$\begin{pmatrix} F(z) & \mathrm{i}G(z) \\ \mathrm{i}G(z^{-1}) & F(z^{-1}) \end{pmatrix} = \begin{pmatrix} \cos\phi_0 & \mathrm{i}\sin\phi_0 \\ \mathrm{i}\sin\phi_0 & \cos\phi_0 \end{pmatrix} \times \left[\prod_{j=1}^{d} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} \cos\phi_j & \mathrm{i}\sin\phi_j \\ \mathrm{i}\sin\phi_j & \cos\phi_j \end{pmatrix} \right]$$

holds for all $z \in \mathbb{T}$.

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Complementary polynomials problem

Given $P(z) \in \mathbb{C}[z]$ satisfying $|P(z)| < 1$ on $\mathbb{T}$, find $Q(z) \in \mathbb{C}[z]$ such that

$$|P(z)|^2 + |Q(z)|^2 = 1 \quad (z \in \mathbb{T}).$$

Complementary polynomials III

Define a Fourier multiplier Π acting on $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ by

$$\Pi[e^{in\theta}] = \begin{cases} e^{in\theta} & n > 0 \\ \frac{1}{2} & n = 0 \\ 0 & n < 0. \end{cases}$$

Theorem (Fourier analytic representation of complementary polynomials)

Suppose that $P(z) \in \mathbb{C}[z]$ satisfies the conditions of the complementary polynomials problem. Then,

$$Q(e^{i\theta}) = \exp\left(\Pi\left[\log\left(1 - |P(e^{i\theta})|^2\right)\right]\right) \quad (\theta \in (-\pi, \pi])$$

solves the complementary polynomials problem for $z = e^{i\theta} \in \mathbb{T}$.

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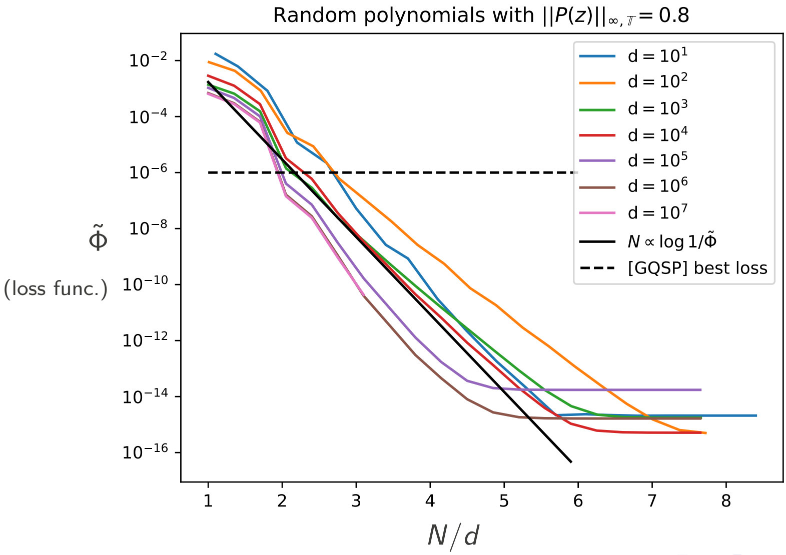
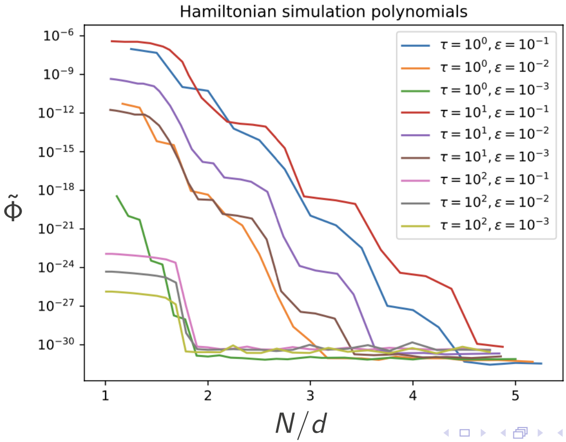
- algorithm compares favorably, with respect to both accuracy and runtime, to previous state-of-the-art optimization-based method developed for generalized QSP (GQSP) [2]
- GQSP removes the reality/parity constraints of standard QSP: arbitrary complex polynomials $P(z)$ with $\|P(z)\|_{\infty,\mathbb{T}} < 1$ may be represented

Numerical results I: random polynomials

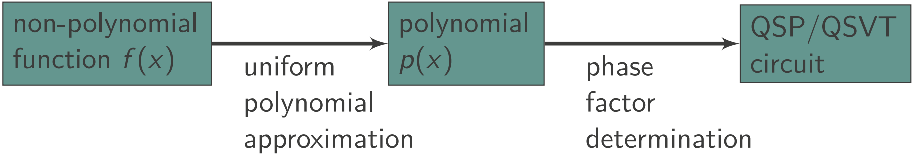


Numerical results II: Hamiltonian simulation

$$e^{i\tau x} = J_0(\tau) + \sum_{n=1}^{d} i^n J_n(\tau) T_n(x) + \mathrm{err}(\tau, d), \quad \varepsilon = |\mathrm{err}(\tau, d)|$$



Summary and outlook



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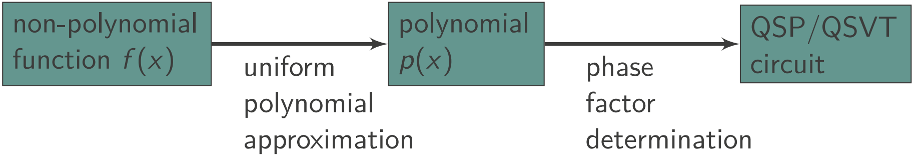

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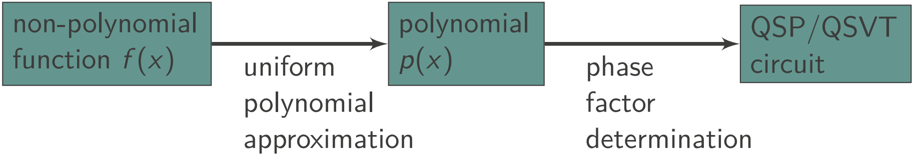

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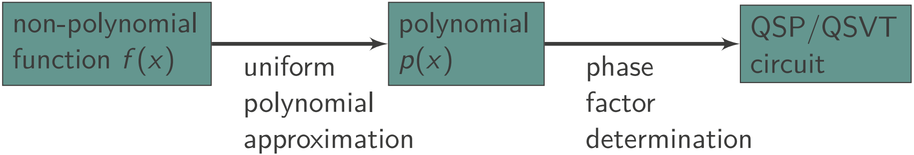

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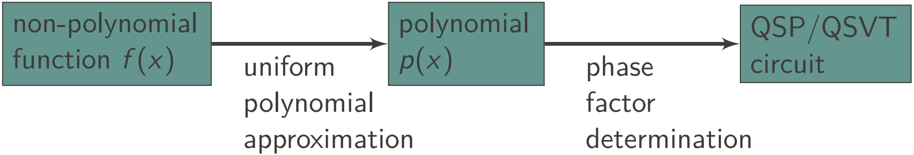

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- recent work of Alexis, et. al. [3]: phase factors also computable via FFTs
- future work: complementary polynomials for state-of-the-art GQSP-based Hamiltonian simulation [4], complementary polynomials for MQSP

References

[1] J. Haah. Product Decomposition of Periodic Functions in Quantum Signal Processing. Quantum 3, 190 (2019).

[2] D. Motlagh and N. Wiebe. Generalized Quantum Signal Processing. PRX Quantum 5, 020368 (2004).

[3] M. Alexis, L. Lin, G. Mnatsakanyan, C. Thiele, and J. Wang. Infinite quantum signal processing for arbitrary Szegö functions. arXiv:2407.05634 [quant-ph] (2024).

[4] D.W. Berry, D. Motlagh, G. Panaleoni, and N. Wiebe. Doubling the efficiency of Hamiltonian simulation via generalized quantum signal processing. Phys. Rev. A 110, 012612 (2024).