Carleman- Lattice Boltzmann Approach to the Quantum Simulation of Fluids

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QT4HEP @ CERN

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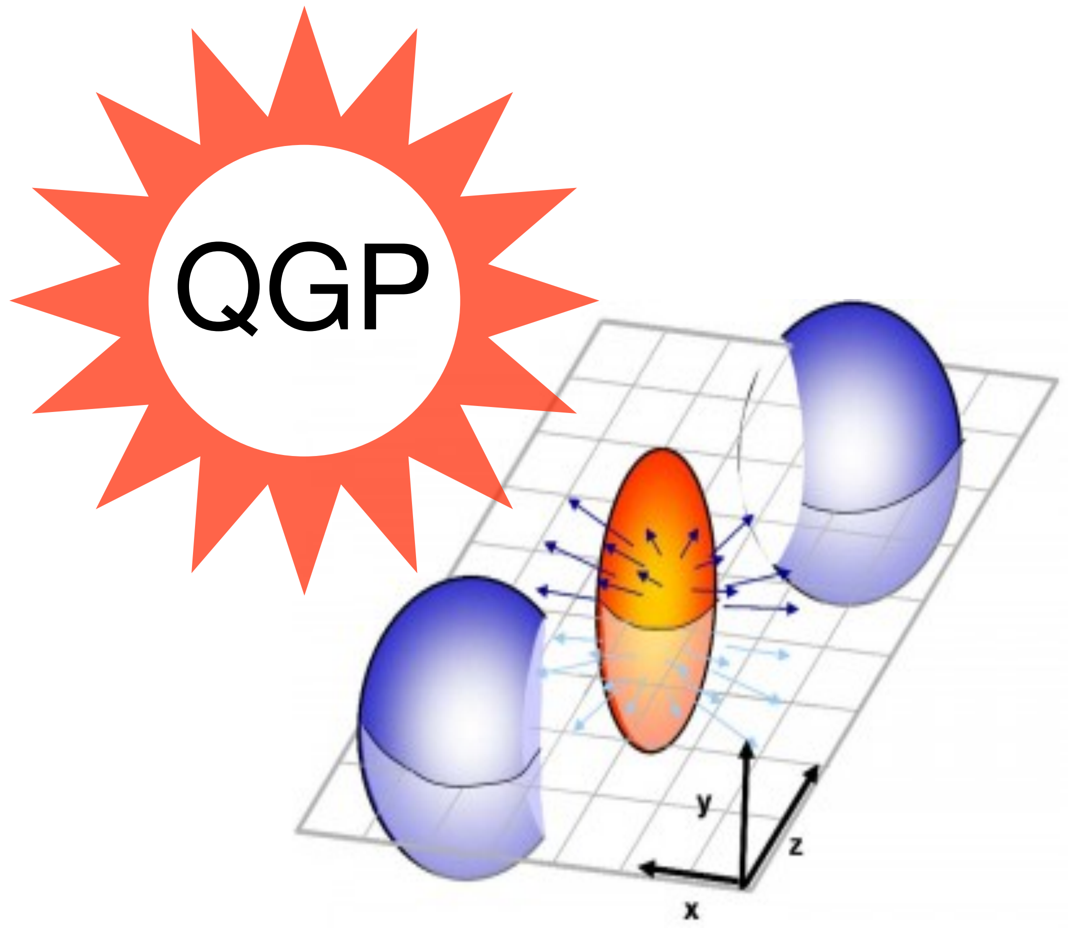
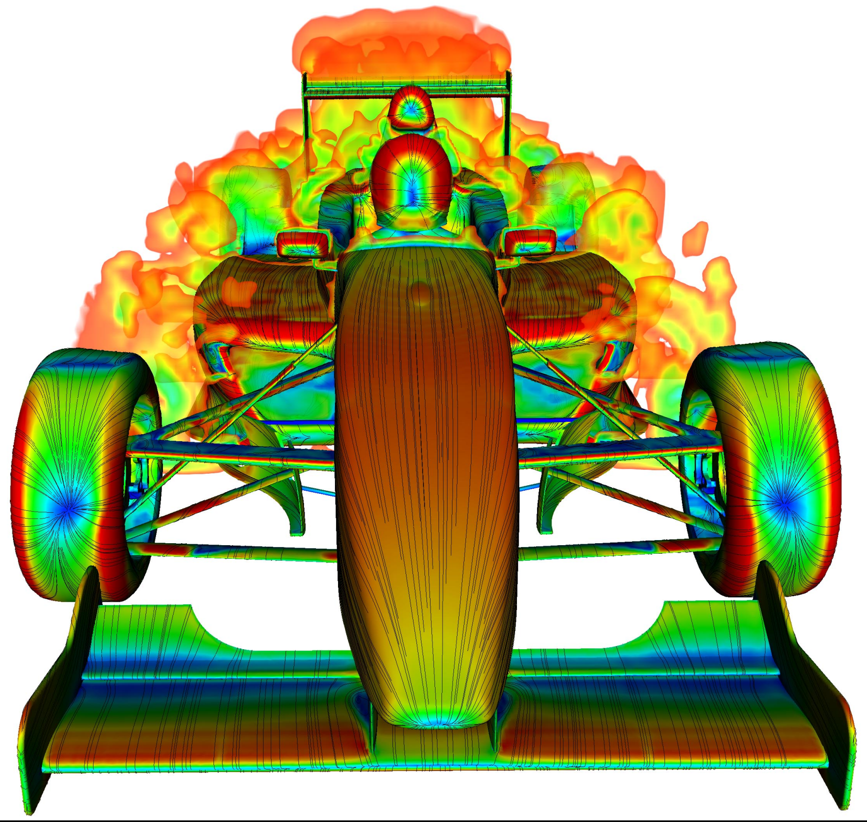
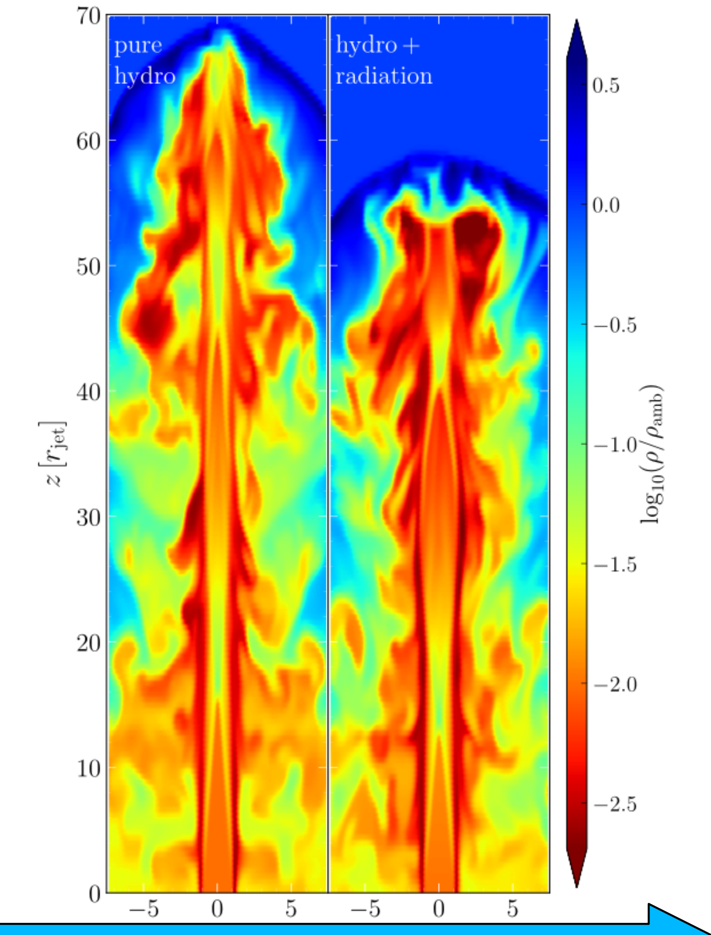
Non-linearity in Fluid dynamics







Re 10^2 10^7 10^{35}

DoF~ $\mathrm{Re}^{\frac{9}{4}}$

CC~ Re^3

Note: quantum computing

Exponential scaling of Hilbert space in QM

q	$\log_{10}$ ***Re***	
30	3	
70	7	car
----------	Exascale	
120	16	Weather
240	32	
480.	64	
960	128	

Logical qubits

$q_{QGP} \approx 30$

Lattice Boltzmann method

Boltzmann equation p.d.f. $f(x, v, t)$

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{F}{m}\frac{\partial f}{\partial v} = \Omega(f)$$

collision term:

Mass conservation

Momentum conservation

Energy conservation

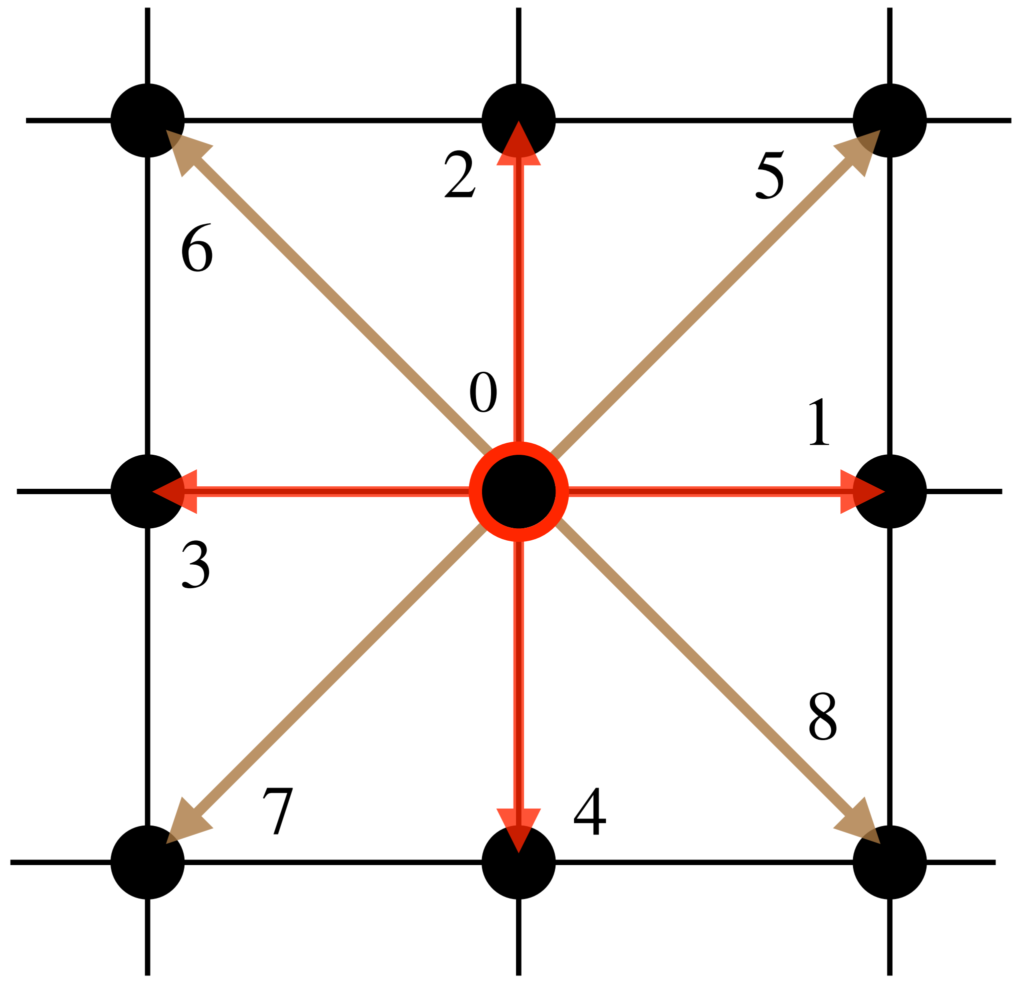
$f(x, v, t) \rightarrow f_i(x_k, t)$ N lattice sites x_k + Q discrete velocities c_i

$$c_l \delta t = \delta x$$

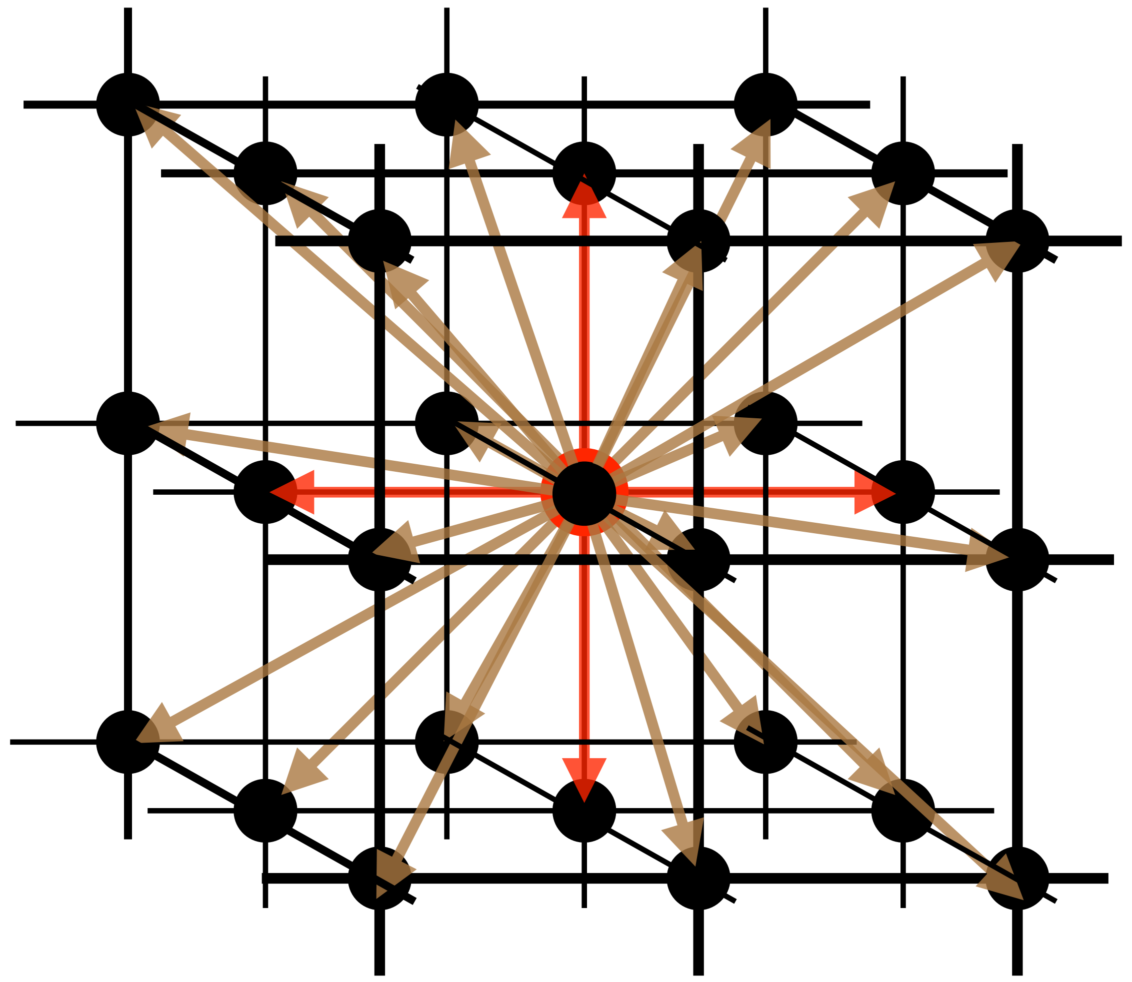
BGK relaxation

$$-\omega(f - f_{eq})$$

D2Q9



D3Q27



Lattice Boltzmann equation

$$f_i(x + c_i, t + \Delta t) = f_i(x, t) + \Omega_i(x, t)$$

For each i, **parallelisation**

Relaxation $$f_i'(x, t) = (1 - \omega)f_i(x, t) + \omega f_i^{eq}$$

Local — non linear

$$\omega = \Delta t / t_r$$

Streaming $$f_i(x + c_i, t + \Delta t) = f_i'(x, t)$$

Exact-Unitary operation!

Non local — linear

Carleman linearisation

The logistic equation

$$\dot{x} = -ax + bx^2$$

Solution $\quad R = b/a$

$$x(t) = \frac{x_0 e^{-at}}{1 - Rx_0\left(1 - e^{-at}\right)}$$

CL

$$x^2 \to x^{(2)} \qquad \dot{x} = -ax + bx^{(2)}$$

$$x^3 \to x^{(3)} \qquad \dot{x}^{(2)} = 2x\dot{x} = -a2x^{(2)} + 2bx^{(3)}$$

$$\vdots \qquad \vdots$$

$$x^K \to x^{(K)} \qquad \dot{x}^{(K)} = -aKx^{(K)} + bKx^{(K+1)}$$

$$x_K(t) = x_0 e^{-at} \sum_{k=0}^{K} \left[Rx_0\left(1 - e^{-at}\right)\right]^k$$

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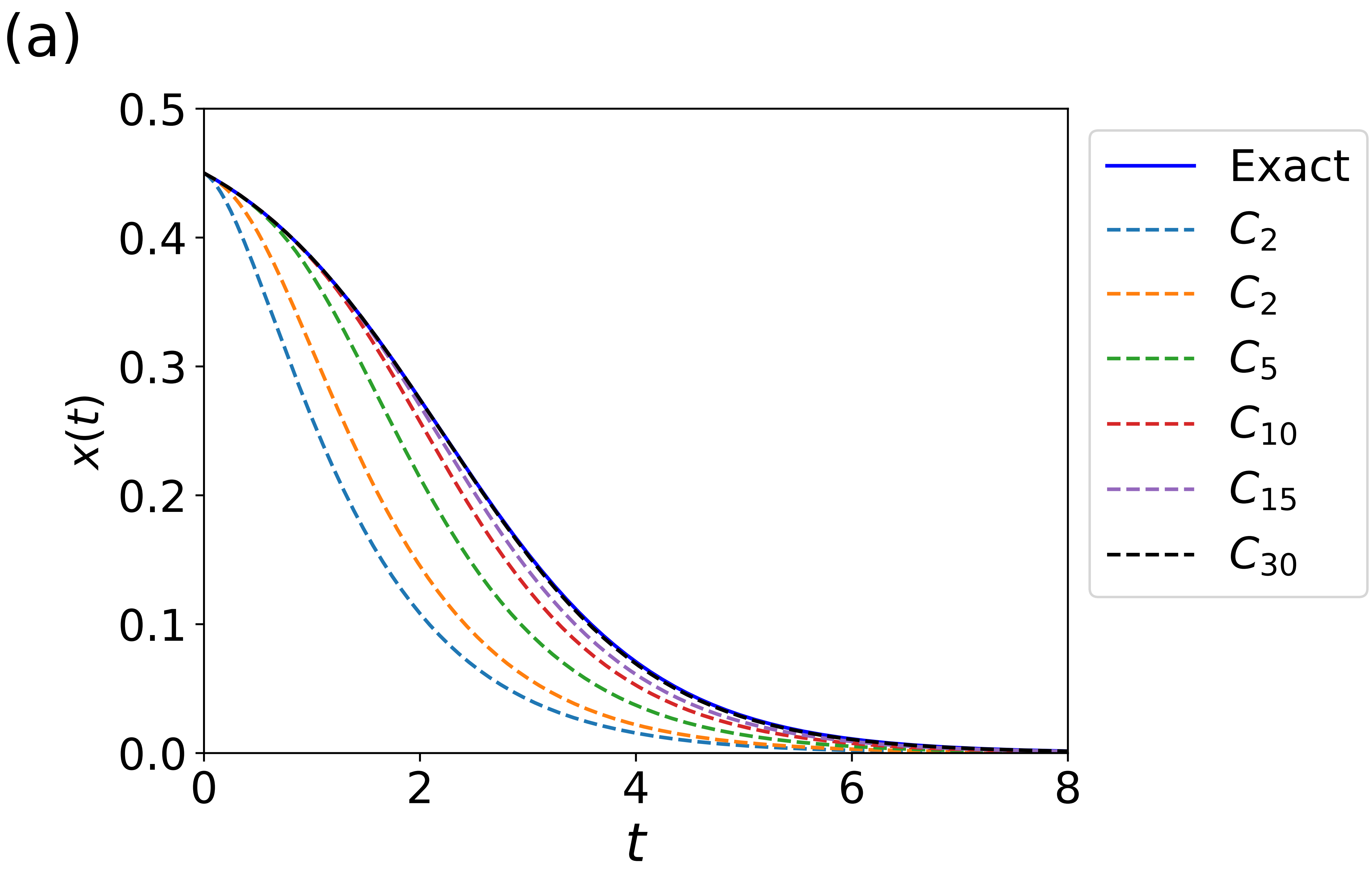
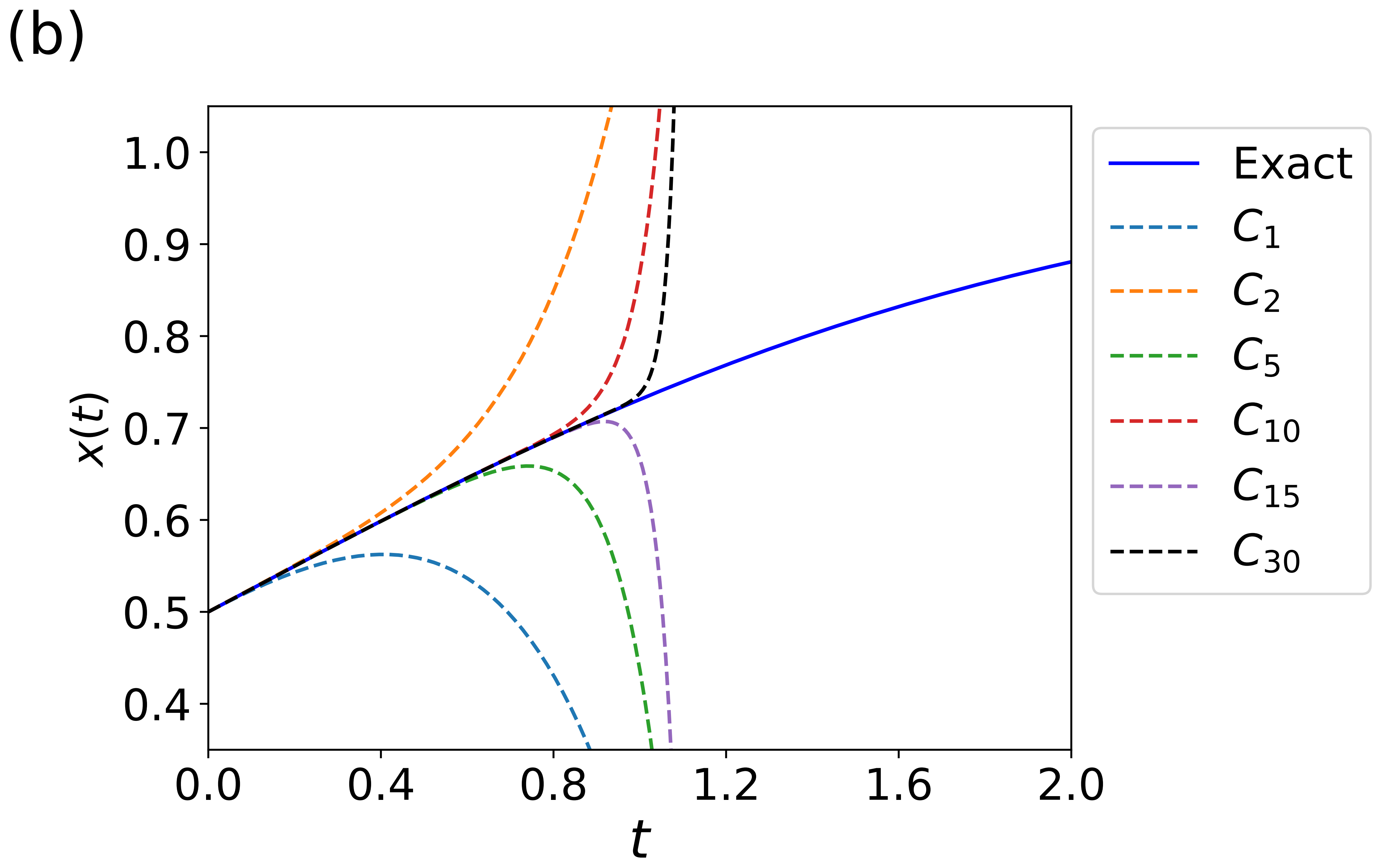
Carleman linearisation

$a, b < 0$

$a, b > 0$





Carleman Linearization

$$f_i'(x,t) = (1-\omega)f_i(x,t) + \omega f_i^{eq} \longrightarrow f_i' = A_{ij}f_j + B_{ijk}f_jf_k + \dots$$

$$f_i \to f_i \qquad f_if_j \to f_{ij}^{(2)}$$

$$\Rightarrow$$

$$f_i' \qquad f_{ij}' = f_i'f_j'$$

Carleman vector

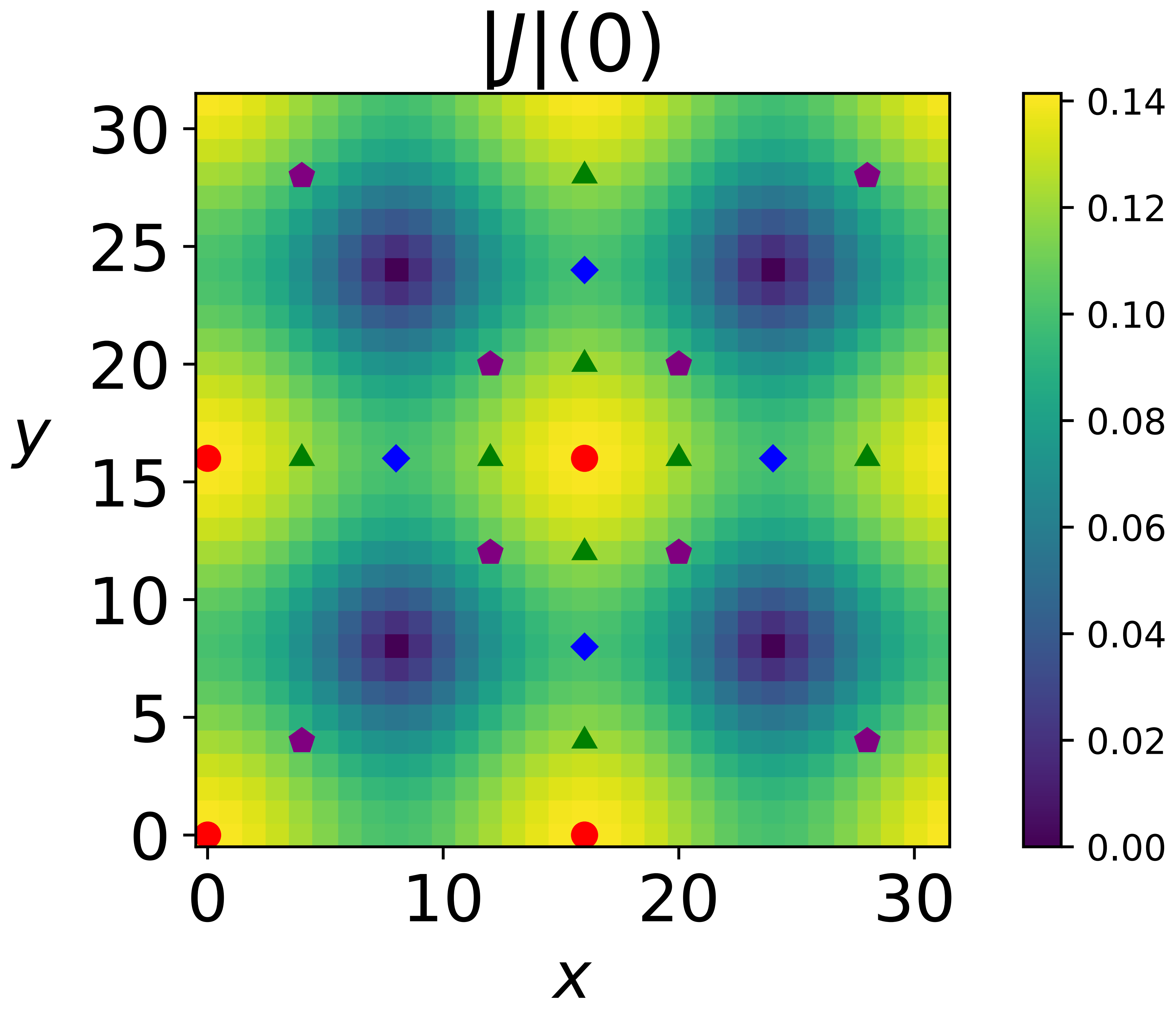
$$V = \begin{pmatrix} f \\ f^{(2)} \end{pmatrix} \qquad V' = \mathcal{R}V$$

Exponential increase of Carleman variables with τ

$$N = 2^{10}, Q = 9 \qquad \tau = 2, \quad \# \sim 2^{26}$$

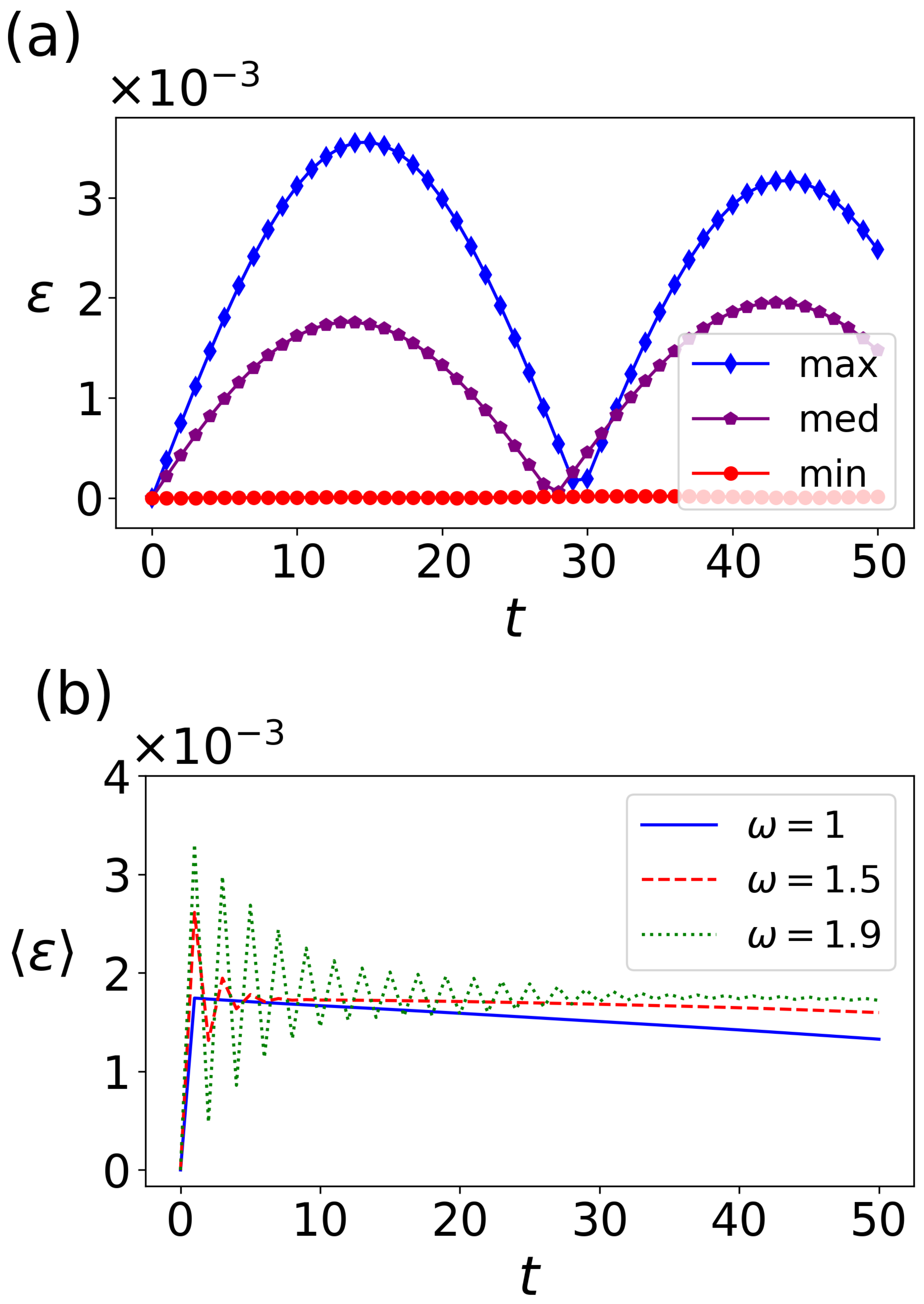
$$\sum_{k=1}^{\tau} (N \times Q)^k, \qquad n_q \sim 26$$

Kolmogorov flow



$$\langle \text{RMSE} \rangle = \sqrt{\sum_{i=1}^{N} \frac{1}{N} \left(\frac{f_i - f_i^{CL}}{f_i} \right)^2}$$

Low relative error ϵ at $\tau = 2$



Re~100

CS, SS. AVS Quantum Science. 2024 Apr 22;6(2):023802.
CS, SS et al.: http://arxiv.org/abs/2501.02582

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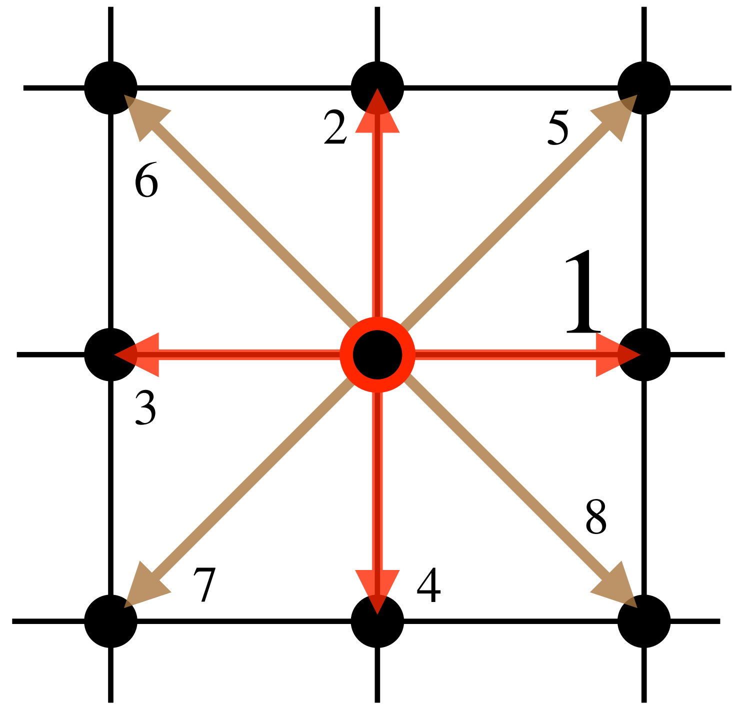
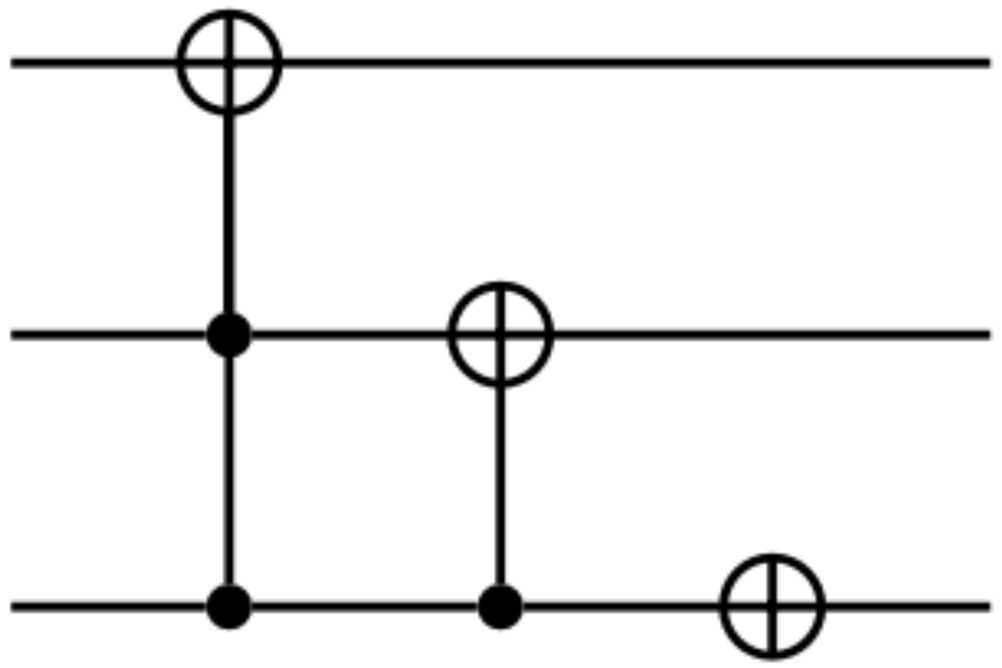
Streaming operator



2D system $|p\rangle_p = |x_p\rangle |y_p\rangle$

Streaming S_1 $S_1 |p\rangle_p = |x_p + 1\rangle |y_p\rangle$ $S_1 = S_+ \otimes \mathbf{1}$

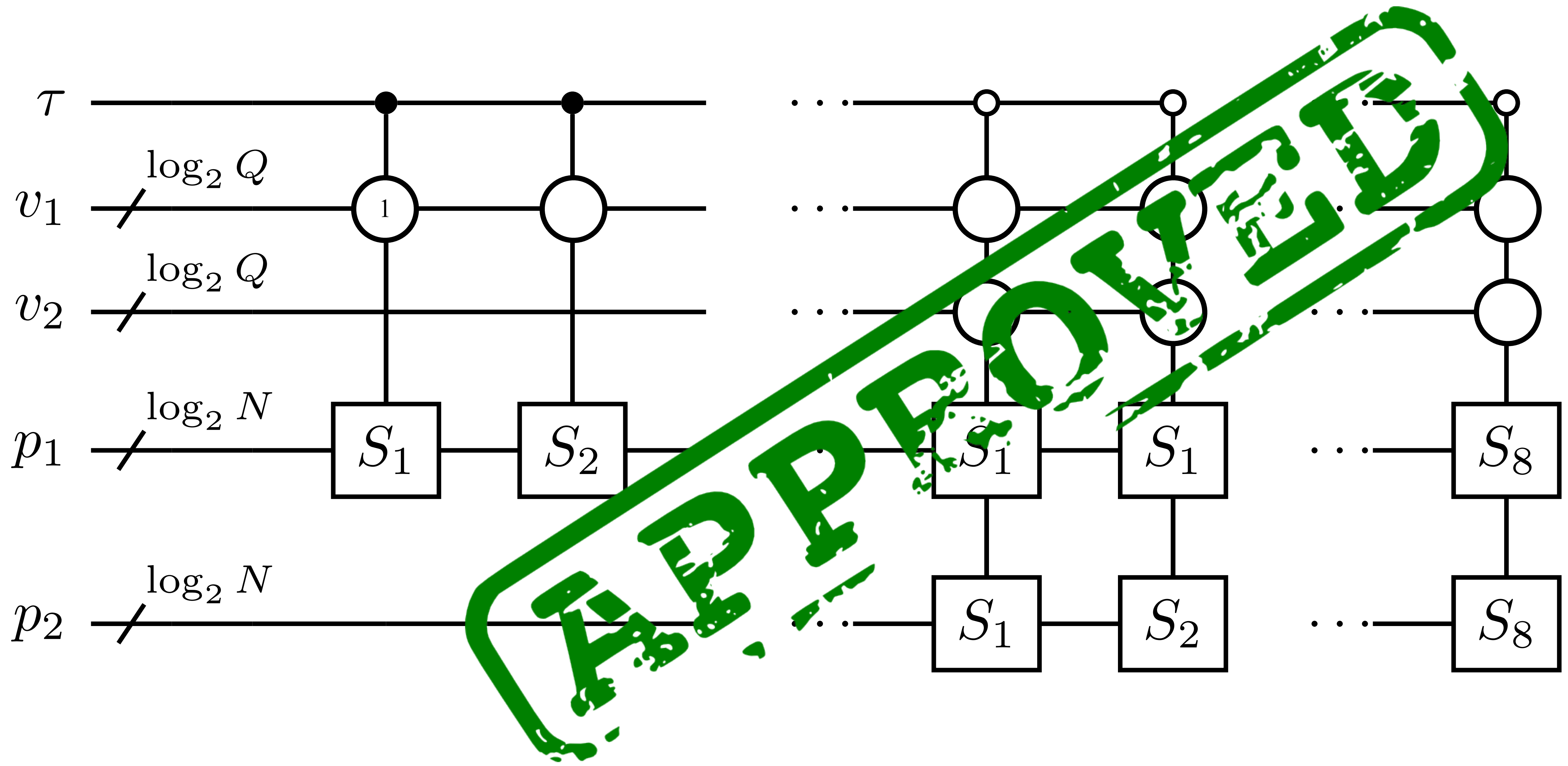
$S_+ =$



x-direction

$\mathcal{O}(q^2)$

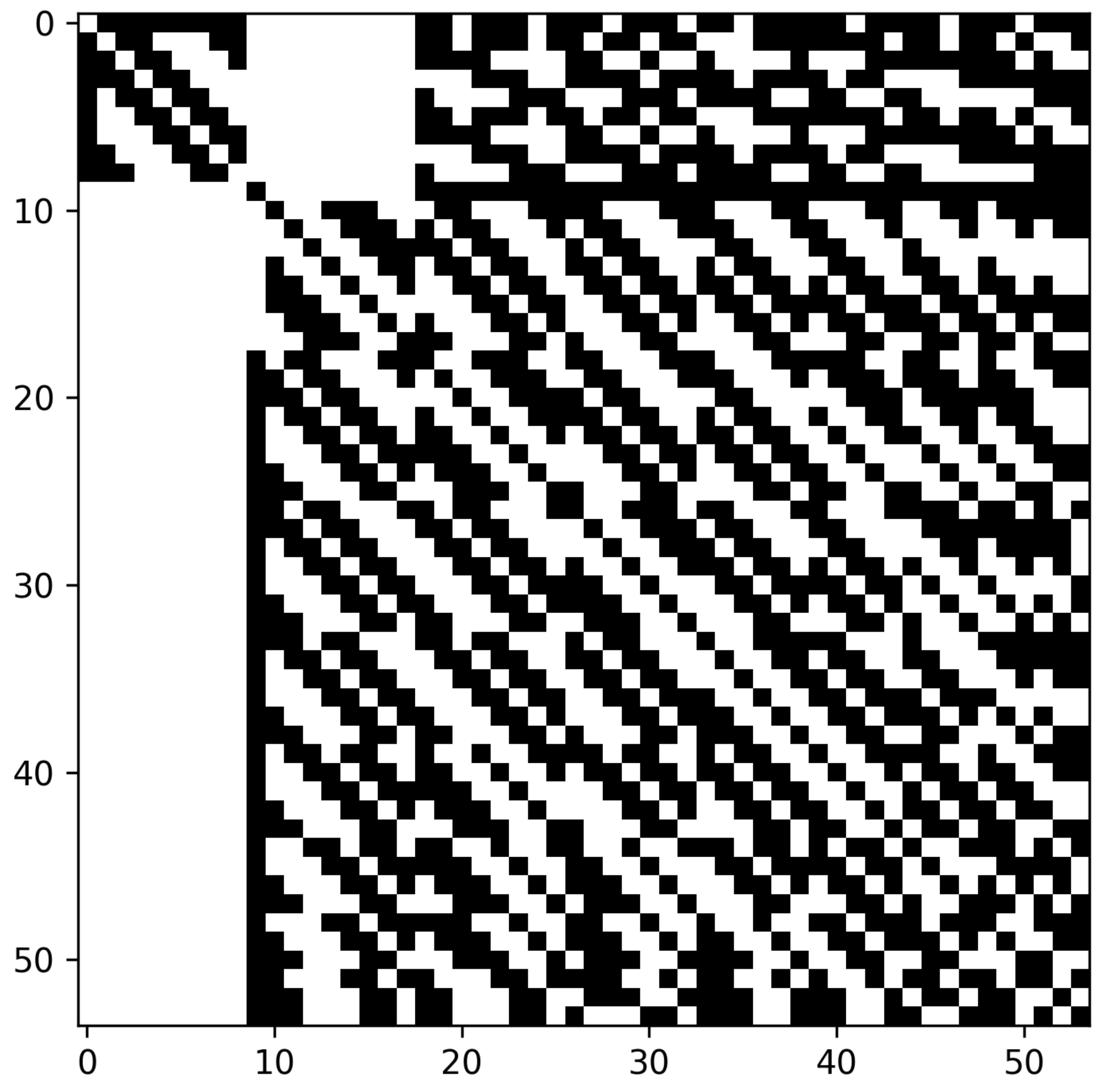
$\mathcal{S} =$



Relaxation operator

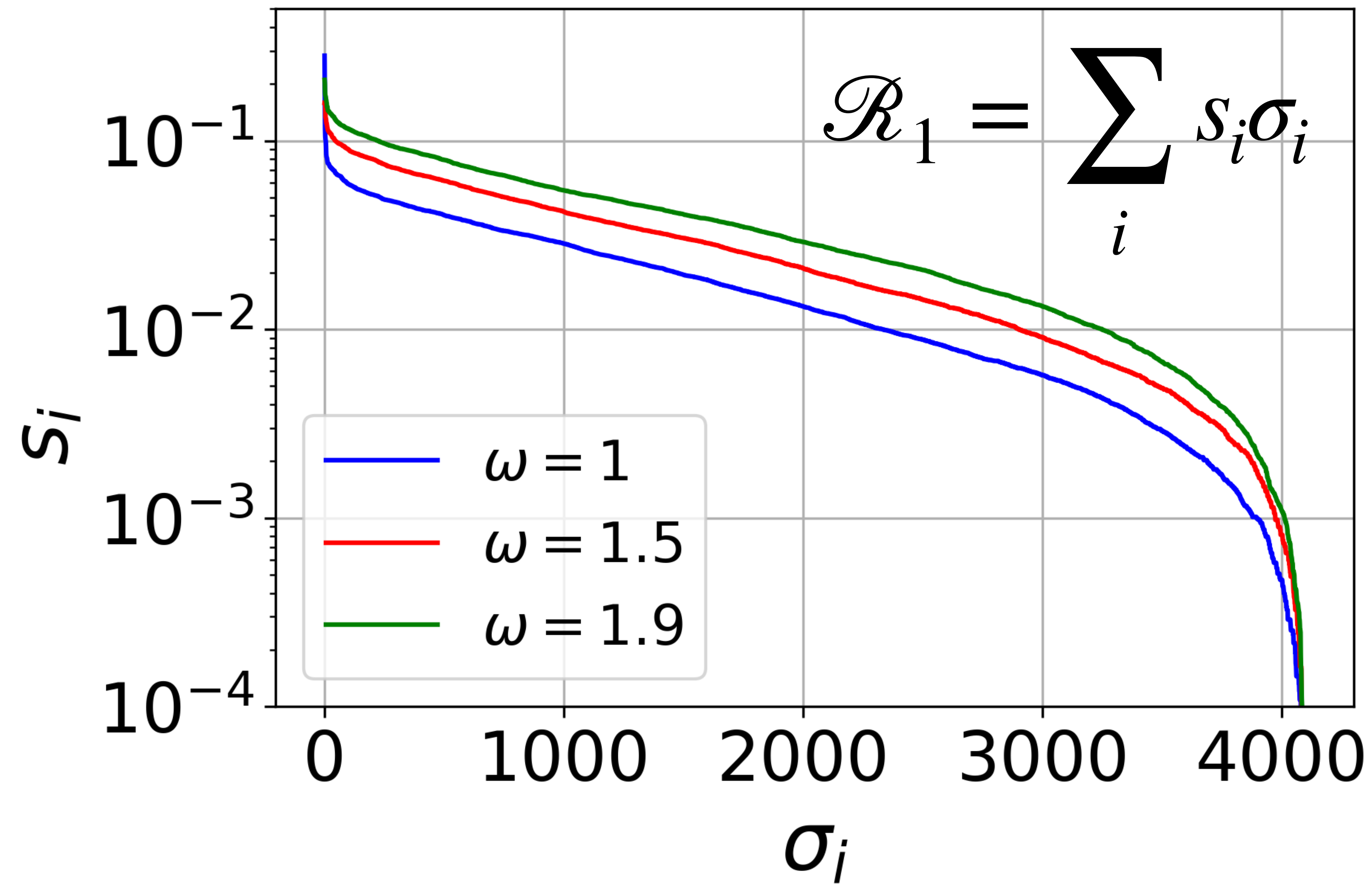
Operator for single site $N = 1$

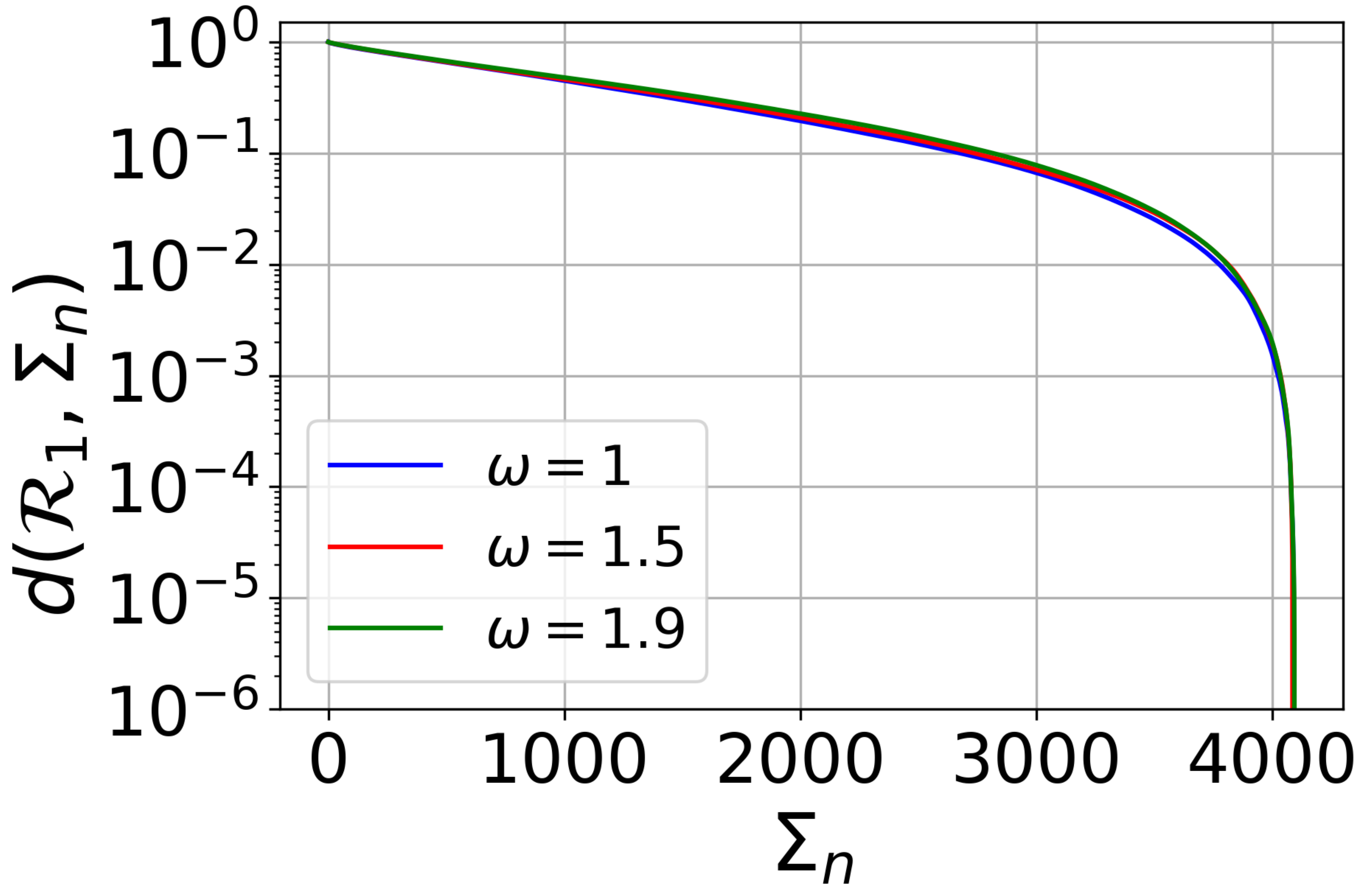
$\mathcal{R}_1 =$



Expansion over Pauli basis

$$\sigma = (I_q, \sigma_x \otimes I \ldots \otimes I, \ldots, \sigma_z \otimes \ldots \otimes \sigma_z)$$





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Efficient circuit for $\mathscr{R}$

Block-encoding with matrix access oracles

↓

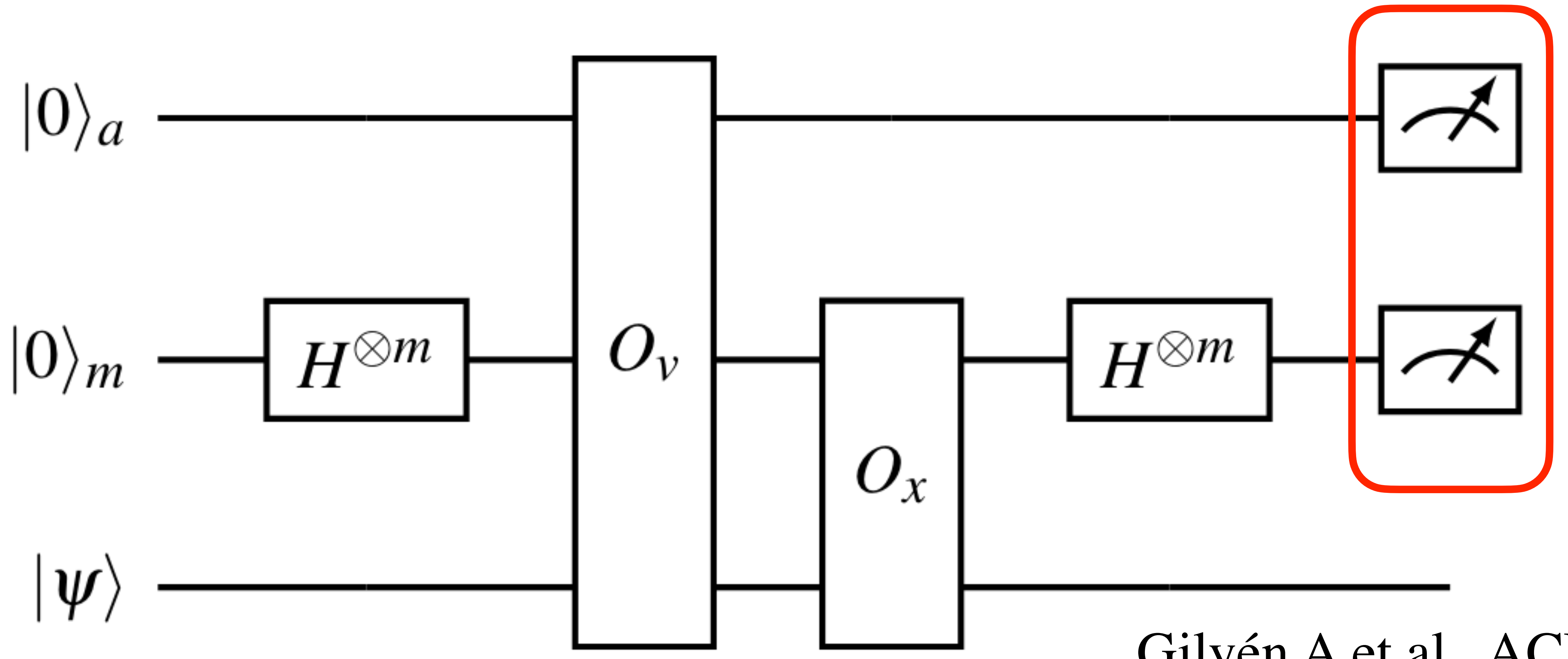
Relaxation is a local process: $$\mathscr{R}_N = \begin{pmatrix} 1_N \otimes A & \Delta \otimes B \\ 0 & 1_{N^2} \otimes A^{\otimes 2} \end{pmatrix},$$

Sparsity is fixed for any N, $s = Q^2$

For sparse matrices we can save the triplet $M : (i, j, v_{ij})$

Oracle $\hat{O}_v$ assigns the value

Oracle $\hat{O}_x$ assigns the position



Probabilistic result

Gilyén A et al., ACM p. 193–204. (STOC 2019).

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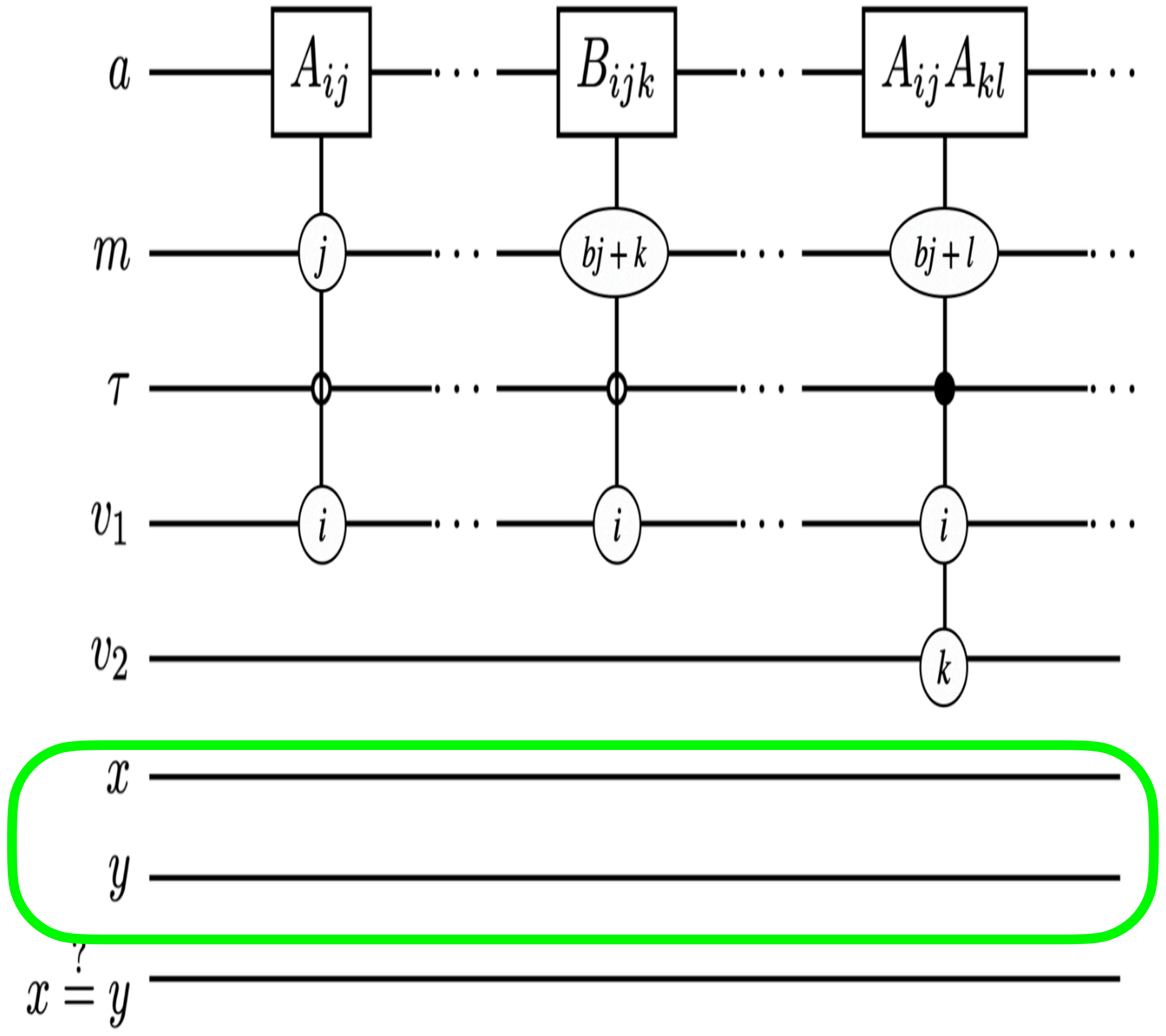
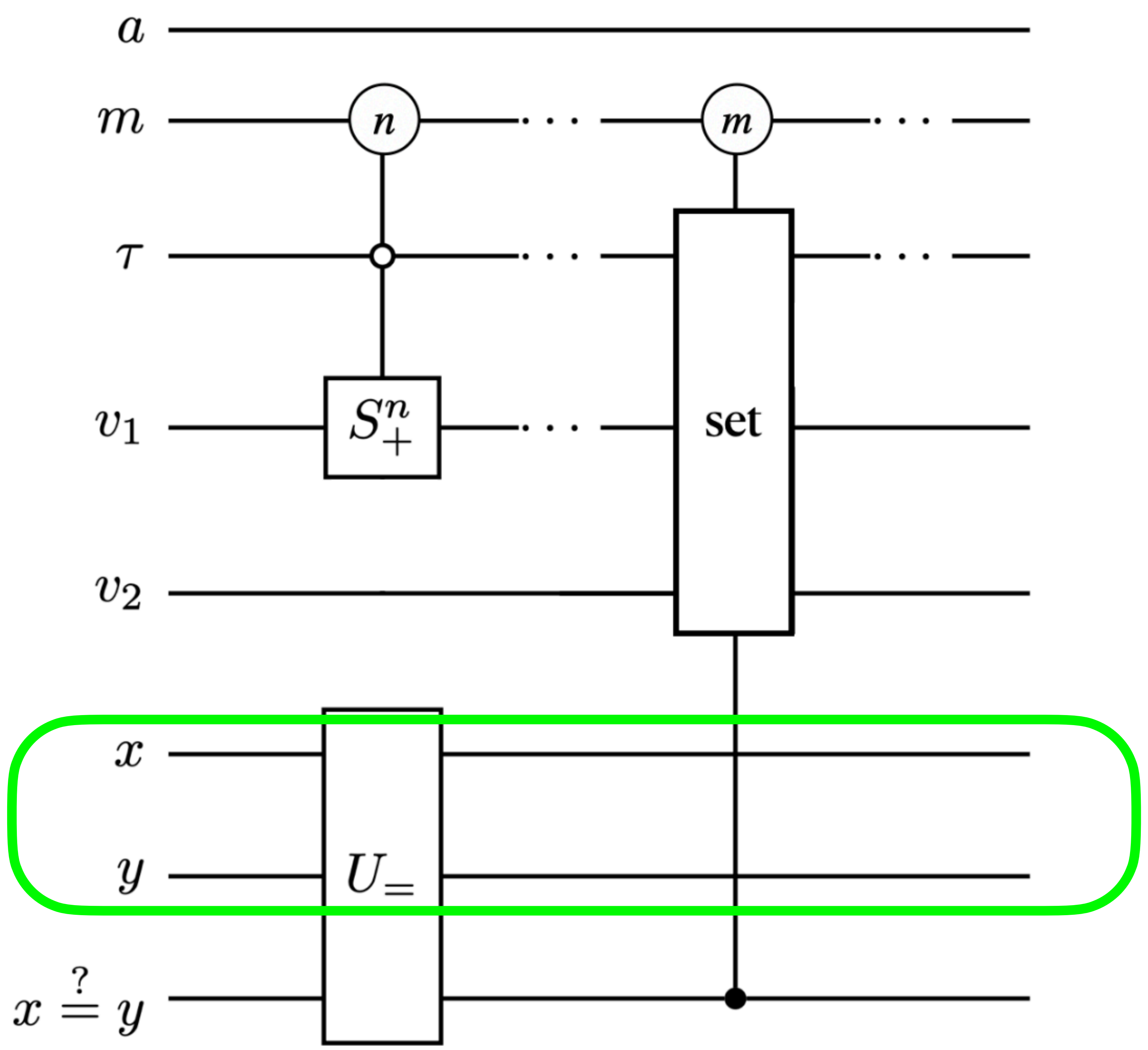
Efficient circuit for $\mathscr{R}$

$\hat{O}_v =$

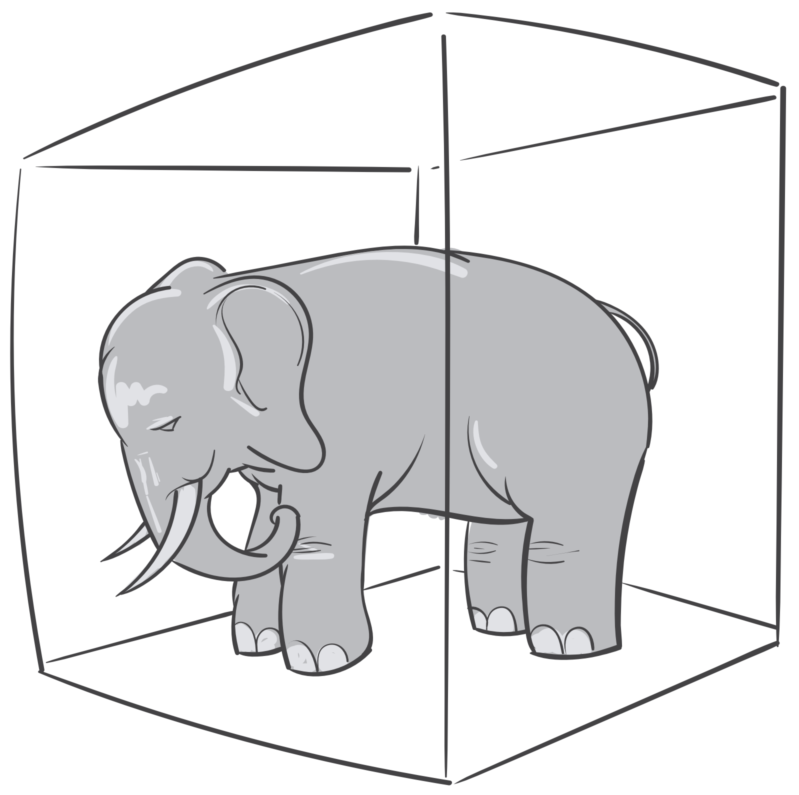


$\hat{O}_x =$



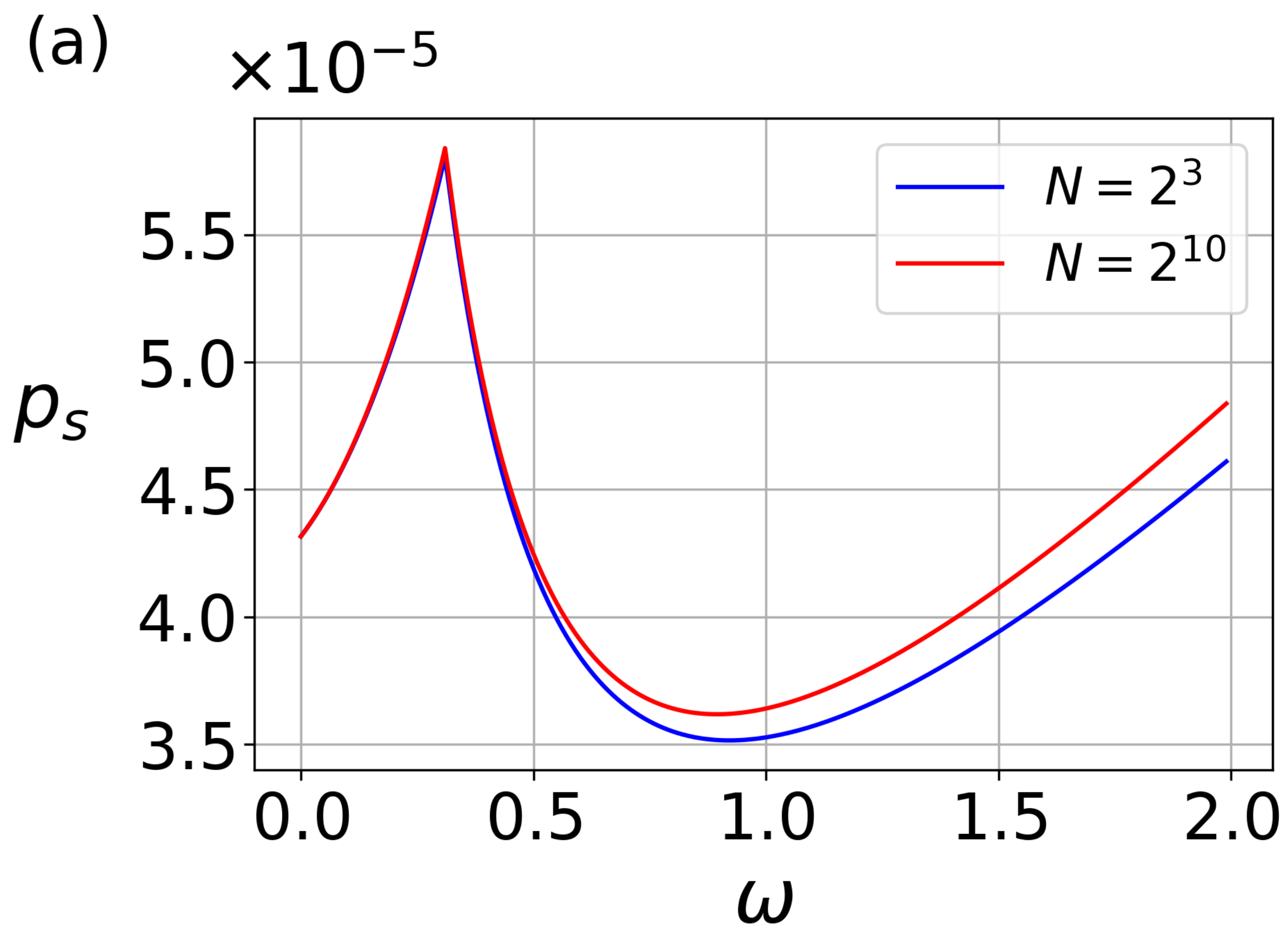
1. Camps D, Lin L, Van Beeumen R, Yang C.SIAM J Matrix Anal Appl. 2024 Mar 31;45(1):801–27.
CS, SS, et al. http://arxiv.org/abs/2501.02582

Problems / solutions(?)



Algorithm is **efficient** in terms of gate complexity $\mathcal{O}(Pol(\log_2 N))$

Success probability is very low!



- ★ Oblivious amplitude amplification for non-unitary dynamics (TBD)
- ★ Probabilistic behaviour restrained by logarithmic embedding of time (modified algorithm !)
- ★ Truncation at second order is enough ($\tau = 3$ optimal ?) (Li, et al.: arXiv.2303.16550.)

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Thanks!

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22/01/2025

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