Quantum Hamiltonian Truncation

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Outline

1. Introduction
2. Hamiltonian Truncation
3. Schwinger Model
4. Time Evolution
5. Summary

Collaborators

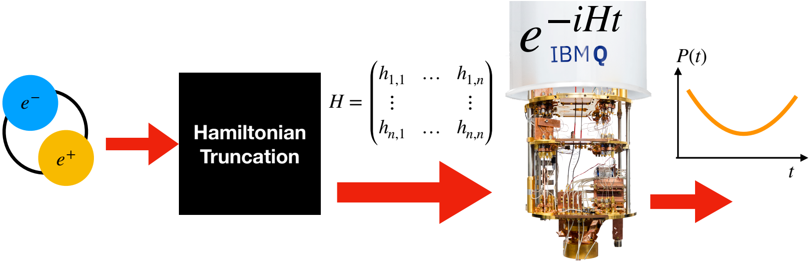
My talk will be based on [PRD **110**, 9 (2024), arXiv:2407.190222] with

Michael Spannowsky

Timur Sypchenko

Simon Williams

Outline



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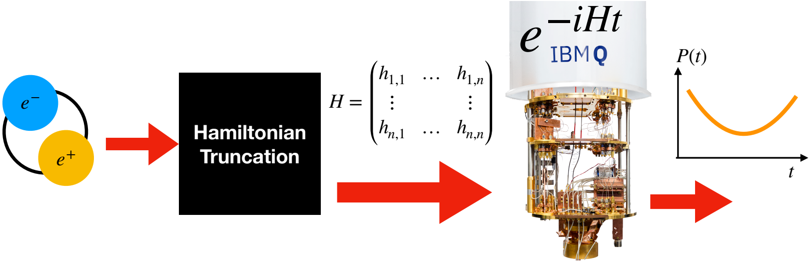

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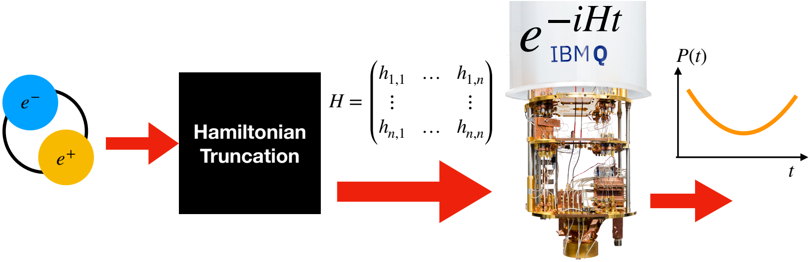

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2. We use *Hamiltonian Truncation* to generate an approximate Hamiltonian for our system of low dimensionality. I will be explicit about the truncation we use.
3. We use an IBM quantum device to determine how this probability evolves with time.

Method Overview

Hamiltonian Setup

$$H = H_0 + V \tag{1}$$

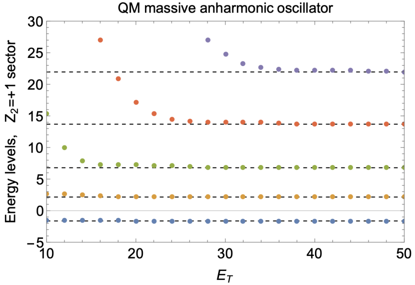
- H_0 is an exactly solvable Hamiltonian
- V represents a new interaction, which may be strong.
- Work in the eigenbasis of H_0. Truncate so that only a finite number of states with $E_0 \leq E_T$ are included in the basis.
- Diagonalize numerically to calculate spectrum and wavefunctions.
- Has been applied to a variety of QFTs including 2d QCD. See [Konik et al '17], [Katz, Fitzpatrick '22] for overviews.

A Simple Example: The Anharmonic Oscillator

Take the quantum mechanical model

$$H = \frac{p^2 + x^2}{2} + \lambda x^4 \,. \tag{2}$$

Decompose the Hamiltonian so that H_0 is the SHO and $V = \lambda x^4$. Work in the SHO eigenbasis: $H_0 \left|n\right\rangle = (n + 1/2) \left|n\right\rangle$



- Truncate basis to include states $\left|n\right\rangle$ for $n + 1/2 \leq E_T$.
- All energy eigenvalues are upper bounds for the true energies due to min-max theorem.
- Method generalises to QFTs.

Schwinger Model

QED in 1+1 dimensions

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ - gA\!\!\!/ - m\right)\psi \ , \tag{3}$$

- Shares qualitative features with QCD including confinement, chiral symmetry breaking, $U(1)_A$ anomaly.
- We take there to be only 1 Dirac fermion.
- Put on a circle of circumference L and use periodic boundary conditions.
- Studied extensively using lattice gauge theory on a variety of quantum computing platforms e.g. [P. Hauke et al '13].

Bosonisation

The $m = 0$ theory was solved exactly by Schwinger. It is a theory of confined, noninteracting, pseudoscalar mesons.

$$H_0 = \frac{1}{2}\int_0^L dx \ :\Pi^2 + (\partial_x\phi)^2 + \frac{g^2}{\pi}\phi^2: \ , \tag{4}$$

The scalar has mass $M = g/\sqrt{\pi}$. Bosonisation helpfully removes gauge redundant d.o.fs. Normal ordering in (4) removes UV divergences.

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When $m \neq 0$, the theory becomes interacting

$$V = -2cmM \int_0^L dx \; :\cos\left(\sqrt{4\pi}\phi + \theta\right): , \tag{5}$$

chiral symmetry is broken, and the θ parameter becomes physical, but we only consider $\theta = 0$ here.

Basis States

Quantise the massive scalar field on the circle

$$\phi(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2LE_n}} \left(a_n \, e^{ik_n x} + a_n^\dagger \, e^{-ik_n x} \right) \ . \tag{6}$$

where the n represent the different momentum modes on the circle $k_n = 2\pi n/L$.

Work in eigenbasis of H_0

$$|\{\mathbf{r}\}\rangle = \prod_{n=-\infty}^{n=\infty} \frac{1}{\sqrt{r_n!}} \left(a_n^\dagger \right)^{r_n} |0\rangle \ , \tag{7}$$

which is the usual Fock basis.

Truncation

List the states in order of increasing H_0 eigenvalue and take the first 2^{n_q} states from this list.

This is not a *local* truncation - different from the lattice.

For instance, with $n_q = 2$ and $gL = 8$, the states we would retain are

$$|0\rangle\,, \quad \frac{1}{\sqrt{2}}\left(a_0^\dagger\right)^2|0\rangle\,, \quad a_1^\dagger a_{-1}^\dagger|0\rangle\,, \quad \frac{1}{\sqrt{4!}}\left(a_0^\dagger\right)^4|0\rangle\,. \tag{8}$$

These states form our computational basis for quantum computing. Calculate matrix elements

$$V_{\mathbf{r},\mathbf{r}'} = \left\langle\{\mathbf{r}'\}\right| :\cos\left(\sqrt{4\pi}\phi\right): \left|\{\mathbf{r}\}\right\rangle \tag{9}$$

between these states. Gives H as a $2^{n_q} \times 2^{nq}$ matrix

Qubit Resources for Simulating Scalar Field Theory

To simulate a collision with energy $\sqrt{s}$, the max energy state in the truncated basis in HT E_{max}, or the lattice spacing should be

$$\sqrt{s} \ll E_{\text{max}} \approx 1/a$$

The number of qubits needed for lattice formulation:

$$N_q^{\text{lattice}} = n_q^{\text{lattice}}(L/a)$$

[Klco and Savage '18]

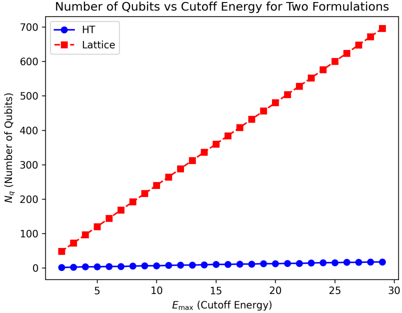
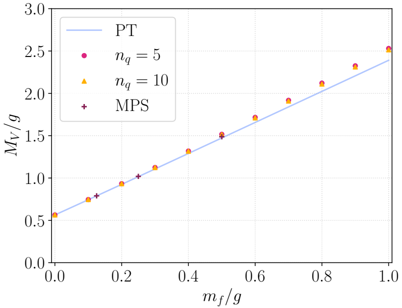


Figure: Comparing qubits needed for the lattice and HT formulations of scalar field theory, with $n_q^{\text{lattice}} = 3$, $ML = 8$.

Sanity Check

Numerical estimates for particle masses converge to known results as (qubit number n_q) is increased



HT data taken at $gL = 8$. PT = second order perturbation theory in infinite volume. MPS = matrix product states M. Bañuls et al '13.

Quantum Quench

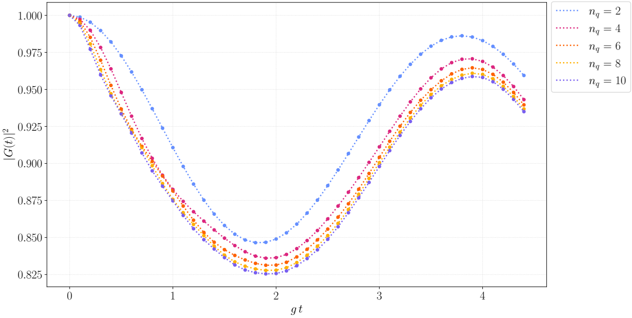
We consider the time dependence of the probability that the Schwinger model stays in its $m = 0$ vacuum state, following a quantum quench to $m/g = 0.2$.

$$G(t) = \left\langle 0 \middle| e^{-iHt} \middle| 0 \right\rangle \,, \qquad P(t) = |G(t)|^2 \,. \tag{10}$$

This particular probability cannot be computed without state preparation in Kogut-Susskind lattice formulation of the Schwinger model.

These routines can be extremely costly. The resources required to implement the state-preparation for an arbitrary state can scale exponentially [Sun et al '23].

Time Evolution Converges



- The vacuum survival probability converges as $n_q \to \infty$.
- Already at $n_q = 2$, we get a reasonable approximation to the continuum time evolution. We are within 5% of the $n_q = 10$ result.
- This is a classical calculation.

Pauli Decomposition

To do the calculation on a NISQ device, we decompose the Hamiltonian as

$$H = \sum_{i_1 \ldots i_{n_q}=0}^{3} \alpha_{i_1 \ldots i_{n_q}} \left(\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_{n_q}} \right) \tag{11}$$

Any Hermitian matrix can be decomposed this way to yield real coefficients $\alpha_{i_1 \ldots i_{n_q}}$.

For a generic dense Hamiltonian matrix, there will be $\sim 4^{n_q}$ nonzero coefficients in this decomposition.

Trotterisation

We use the Trotter-Suzuki approximation to first order. Error $\sim O(t^2/n)$.

$$|\psi(t)\rangle = e^{-iHt}\,|\psi(0)\rangle \approx \left[\prod_{i_1,\ldots,i_{n_q}} e^{-i\frac{t}{n}\alpha_{i_1,\ldots,i_{n_q}}\left(\sigma_{i_1}\otimes\cdots\otimes\sigma_{i_{n_q}}\right)}\right]^n |\psi(0)\rangle \ . \quad (12)$$

The exponential of each Pauli term can be implemented on a qubit-based quantum device through a *short* sequence of single-qubit rotation gates and CNOT gates.

The number of gates needed per trotter step grows with the number of nonzero $\alpha_{i_1\ldots i_{n_q}}$ coefficients. This is *exponential* growth.

Trotter Error

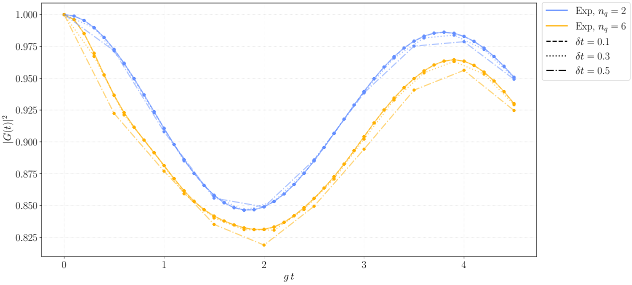


Figure: Blue curves are for $n_q = 2$ and yellow for $n_q = 6$.

We will use $gt/n = g\delta t = 0.3$ for $n_q = 2$ on the quantum device.

Quantum Hamiltonian Truncation

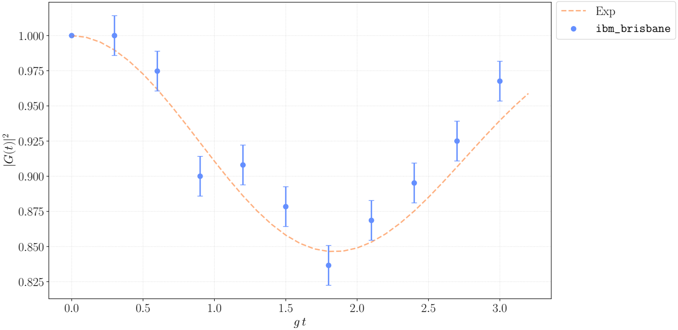


Figure: Time evolution of the Schwinger model via HT run on the ibm brisbane 127-qubit quantum computer (though we only use 2 of them). The results are enhanced using error mitigation and suppression routines through QISKIT and Q-CTRL.

Summary and Conclusion

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3. HT was able to give fairly accurate results with a very small Hamiltonian.
4. Our approach did not require initial state prep, because HT gave us the freedom to pick a 'good' computational basis.
5. The tools we used could be applied to many other QFTs and observables - there are many other exciting applications to explore!

Thank you!

What QFTs Have Been Studied Using HT?

An incomplete selection of studies, with an hep-th focus: Please see [Konik et al '17], [Katz, Fitzpatrick '22] for a more complete review.

In 2 dimensions

- Minimal model CFT deformed with relevant primary operator [Yurov, Zamolodchikov '89]...
- $\mathrm{SU}(3)$ gauge theory with fundamental Dirac fermions on the lightcone [Hornbostel, Brodsky, Pauli '90]...
- ϕ^4 deformation of massive scalar field [Rychkov, Vitale '14]...

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In 3 dimensions

- $\phi^2 + i\phi^3$ deformation of free scalar CFT on S^3 [Hogervorst '18]...
- ϕ^4 deformation of massive scalar on $\mathbb{R} \times T^2$ [Elias-Miró, Hardy '18]...
- ϕ^4 deformation of scalar CFT on the lightcone [Anand, Katz, Khandker, Walters '18]...