

# Quantum Hamiltonian Truncation

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# Outline

- 1 Introduction
- 2 Hamiltonian Truncation
- 3 Schwinger Model
- 4 Time Evolution
- 5 Summary

# Collaborators

My talk will be based on [PRD **110**, 9 (2024), arXiv:2407.190222] with



Michael Spannowsky



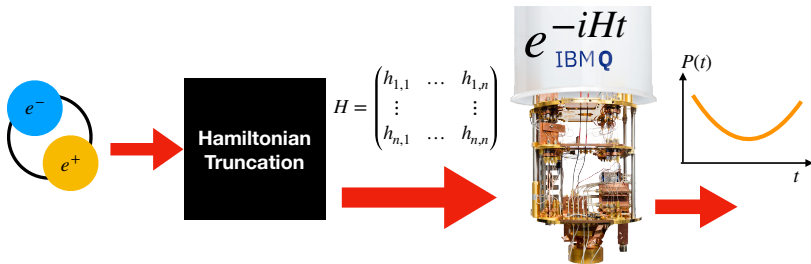
Timur Sypchenko



Simon Williams

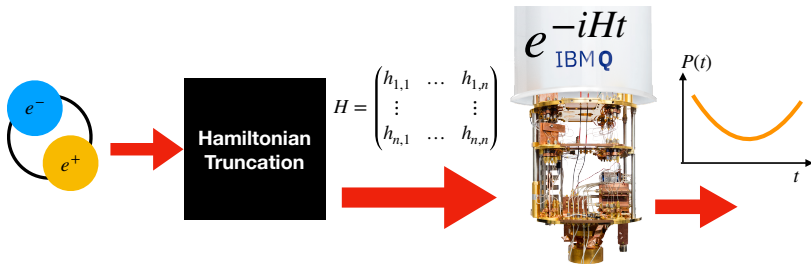


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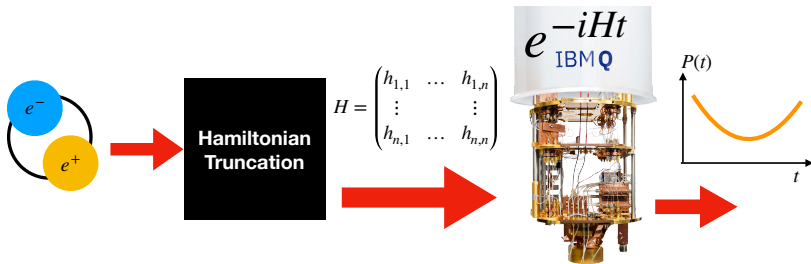
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- 2 We use *Hamiltonian Truncation* to generate an approximate Hamiltonian for our system of low dimensionality. I will be explicit about the truncation we use.
- 3 We use an IBM quantum device to determine how this probability evolves with time.

# Method Overview

## Hamiltonian Setup

$$H = H_0 + V \quad (1)$$

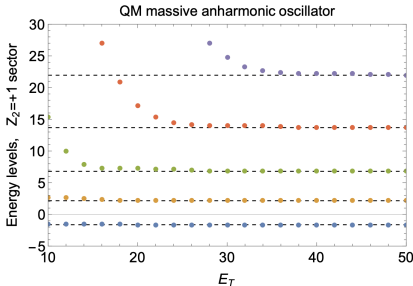
- $H_0$  is an exactly solvable Hamiltonian
- $V$  represents a new interaction, which may be strong.
- Work in the eigenbasis of  $H_0$ . Truncate so that only a finite number of states with  $E_0 \leq E_T$  are included in the basis.
- Diagonalize numerically to calculate spectrum and wavefunctions.
- Has been applied to a variety of QFTs including 2d QCD. See [Konik et al '17], [Katz, Fitzpatrick '22] for overviews.

# A Simple Example: The Anharmonic Oscillator

Take the quantum mechanical model

$$H = \frac{p^2 + x^2}{2} + \lambda x^4. \quad (2)$$

Decompose the Hamiltonian so that  $H_0$  is the SHO and  $V = \lambda x^4$ . Work in the SHO eigenbasis:  $H_0 |n\rangle = (n + 1/2) |n\rangle$



- Truncate basis to include states  $|n\rangle$  for  $n + 1/2 \leq E_T$ .
- All energy eigenvalues are upper bounds for the true energies due to min-max theorem.
- Method generalises to QFTs.



# Schwinger Model

## QED in 1+1 dimensions

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i\not{\partial} - g\not{A} - m) \psi , \quad (3)$$

- Shares qualitative features with QCD including confinement, chiral symmetry breaking,  $U(1)_A$  anomaly.
- We take there to be only 1 Dirac fermion.
- Put on a circle of circumference  $L$  and use periodic boundary conditions.
- Studied extensively using lattice gauge theory on a variety of quantum computing platforms e.g. [P. Hauke et al '13].

# Bosonisation

The  $m = 0$  theory was solved exactly by Schwinger. It is a theory of confined, noninteracting, pseudoscalar mesons.

$$H_0 = \frac{1}{2} \int_0^L dx : \Pi^2 + (\partial_x \phi)^2 + \frac{g^2}{\pi} \phi^2 : , \quad (4)$$

The scalar has mass  $M = g/\sqrt{\pi}$ . Bosonisation helpfully removes gauge redundant d.o.fs. Normal ordering in (4) removes UV divergences.

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When  $m \neq 0$ , the theory becomes interacting

$$V = -2cmM \int_0^L dx : \cos(\sqrt{4\pi}\phi + \theta) : , \quad (5)$$

chiral symmetry is broken, and the  $\theta$  parameter becomes physical, but we only consider  $\theta = 0$  here.

# Basis States

Quantise the massive scalar field on the circle

$$\phi(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2LE_n}} \left( a_n e^{ik_n x} + a_n^\dagger e^{-ik_n x} \right) . \quad (6)$$

where the  $n$  represent the different momentum modes on the circle  
 $k_n = 2\pi n/L$ .

Work in eigenbasis of  $H_0$

$$|\{\mathbf{r}\}\rangle = \prod_{n=-\infty}^{n=\infty} \frac{1}{\sqrt{r_n!}} \left( a_n^\dagger \right)^{r_n} |0\rangle , \quad (7)$$

which is the usual Fock basis.

# Truncation

List the states in order of increasing  $H_0$  eigenvalue and take the first  $2^{n_q}$  states from this list.

This is not a *local* truncation - different from the lattice.

For instance, with  $n_q = 2$  and  $gL = 8$ , the states we would retain are

$$|0\rangle, \quad \frac{1}{\sqrt{2}} \left(a_0^\dagger\right)^2 |0\rangle, \quad a_1^\dagger a_{-1}^\dagger |0\rangle, \quad \frac{1}{\sqrt{4!}} \left(a_0^\dagger\right)^4 |0\rangle. \quad (8)$$

These states form our computational basis for quantum computing.  
Calculate matrix elements

$$V_{\mathbf{r}, \mathbf{r}'} = \langle \{\mathbf{r}'\} | : \cos(\sqrt{4\pi}\phi) : | \{\mathbf{r}\} \rangle \quad (9)$$

between these states. Gives  $H$  as a  $2^{n_q} \times 2^{n_q}$  matrix

# Qubit Resources for Simulating Scalar Field Theory

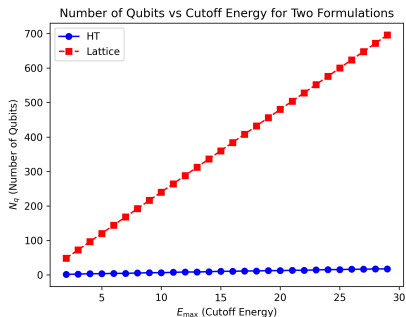
To simulate a collision with energy  $\sqrt{s}$ , the max energy state in the truncated basis in HT  $E_{\max}$ , or the lattice spacing should be

$$\sqrt{s} \ll E_{\max} \approx 1/a$$

The number of qubits needed for lattice formulation:

$$N_q^{\text{lattice}} = n_q^{\text{lattice}} (L/a)$$

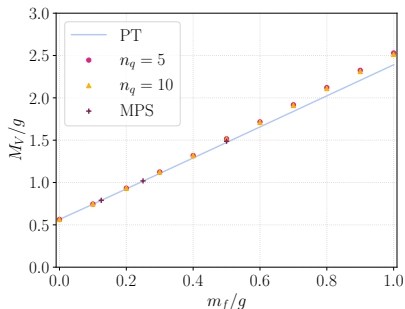
[Klco and Savage '18]



**Figure:** Comparing qubits needed for the lattice and HT formulations of scalar field theory, with  $n_q^{\text{lattice}} = 3$ ,  $ML = 8$ .

# Sanity Check

Numerical estimates for particle masses converge to known results as (qubit number  $n_q$ ) is increased



HT data taken at  $gL = 8$ . PT = second order perturbation theory in infinite volume. MPS = matrix product states M. Bañuls et al '13.

# Quantum Quench

We consider the time dependence of the probability that the Schwinger model stays in its  $m = 0$  vacuum state, following a quantum quench to  $m/g = 0.2$ .

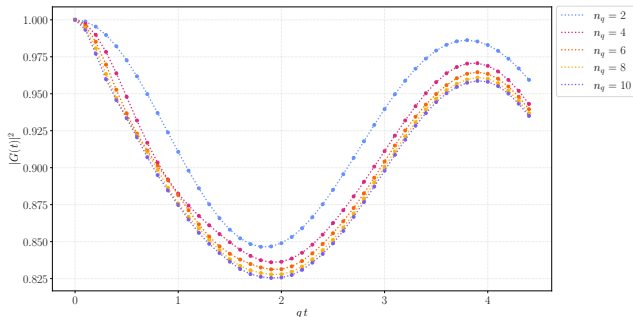
$$G(t) = \langle 0 | e^{-iHt} | 0 \rangle, \quad P(t) = |G(t)|^2. \quad (10)$$

This particular probability cannot be computed without state preparation in Kogut-Susskind lattice formulation of the Schwinger model.

These routines can be extremely costly. The resources required to implement the state-preparation for an arbitrary state can scale exponentially [Sun et al '23].



# Time Evolution Converges



- The vacuum survival probability converges as  $n_q \rightarrow \infty$ .
- Already at  $n_q = 2$ , we get a reasonable approximation to the continuum time evolution. We are within 5% of the  $n_q = 10$  result.
- This is a classical calculation.

# Pauli Decomposition

To do the calculation on a NISQ device, we decompose the Hamiltonian as

$$H = \sum_{i_1 \dots i_{n_q} = 0}^3 \alpha_{i_1 \dots i_{n_q}} \left( \sigma_{i_1} \otimes \dots \otimes \sigma_{i_{n_q}} \right) \quad (11)$$

Any Hermitian matrix can be decomposed this way to yield real coefficients  $\alpha_{i_1 \dots i_{n_q}}$ .

For a generic dense Hamiltonian matrix, there will be  $\sim 4^{n_q}$  nonzero coefficients in this decomposition.

# Trotterisation

We use the Trotter-Suzuki approximation to first order. Error  $\sim O(t^2/n)$ .

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle \approx \left[ \prod_{i_1, \dots, i_n} e^{-i \frac{t}{n} \alpha_{i_1, \dots, i_n} (\sigma_{i_1} \otimes \dots \otimes \sigma_{i_n})} \right]^n |\psi(0)\rangle . \quad (12)$$

The exponential of each Pauli term can be implemented on a qubit-based quantum device through a *short* sequence of single-qubit rotation gates and CNOT gates.

The number of gates needed per trotter step grows with the number of nonzero  $\alpha_{i_1 \dots i_n}$  coefficients. This is *exponential* growth.

# Trotter Error

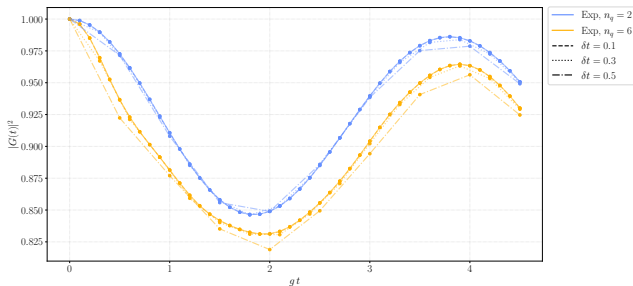
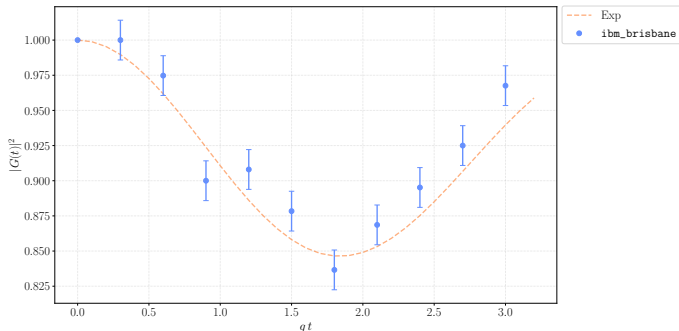


Figure: Blue curves are for  $n_q = 2$  and yellow for  $n_q = 6$ .

We will use  $gt/n = g\delta t = 0.3$  for  $n_q = 2$  on the quantum device.

# Quantum Hamiltonian Truncation



**Figure:** Time evolution of the Schwinger model via HT run on the ibm brisbane 127-qubit quantum computer (though we only use 2 of them). The results are enhanced using error mitigation and suppression routines through QISKIT and Q-CTRL.

# Summary and Conclusion

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- 3 HT was able to give fairly accurate results with a very small Hamiltonian.
- 4 Our approach did not require initial state prep, because HT gave us the freedom to pick a 'good' computational basis.
- 5 The tools we used could be applied to many other QFTs and observables - there are many other exciting applications to explore!

Thank you!

# What QFTs Have Been Studied Using HT?

An incomplete selection of studies, with an hep-th focus: Please see [Konik et al '17], [Katz, Fitzpatrick '22] for a more complete review.

## In 2 dimensions

- Minimal model CFT deformed with relevant primary operator [Yurov, Zamolodchikov '89]...
- SU(3) gauge theory with fundamental Dirac fermions on the lightcone [Hornbostel, Brodsky, Pauli '90]...
- $\phi^4$  deformation of massive scalar field [Rychkov, Vitale '14]...

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- $\phi^2 + i\phi^3$  deformation of free scalar CFT on  $S^3$  [Hogervorst '18]...
- $\phi^4$  deformation of massive scalar on  $\mathbb{R} \times T^2$  [Elias-Miró, Hardy '18]...
- $\phi^4$  deformation of scalar CFT on the lightcone [Anand, Katz, Khandker, Walters '18]...