On multivariate polynomials achievable with quantum signal processing

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- Parameterized single-qubit unitary ($z \in \mathbb{T}$)

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- $P(z), Q(z)$ are polynomials of degree n.

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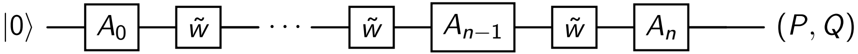

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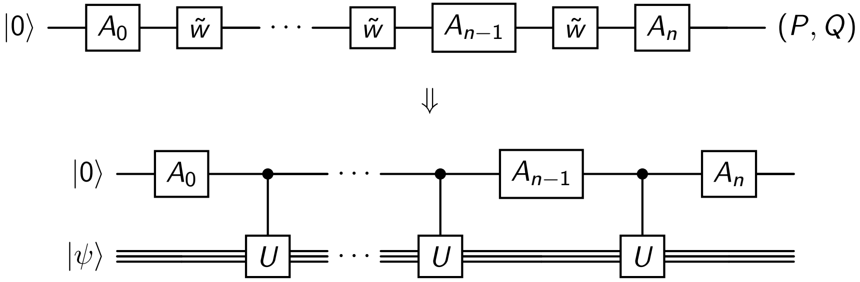

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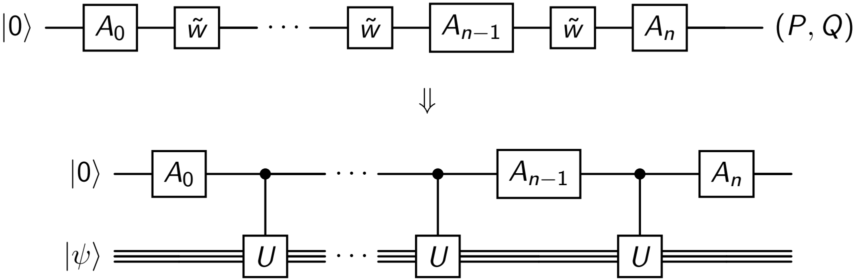

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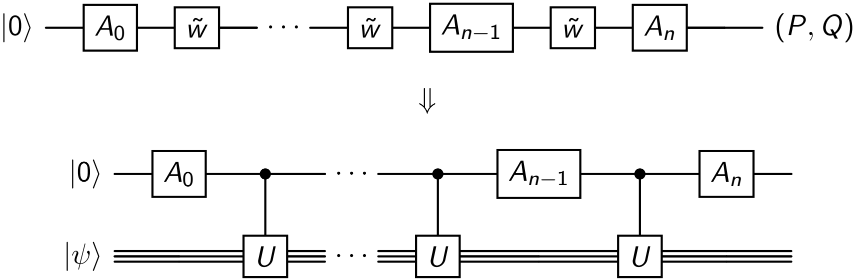

- **Spectral theorem** $\Rightarrow$ This constructs $|0\rangle P(U)|\psi\rangle + |1\rangle Q(U)|\psi\rangle$;
- *QSP* $\rightarrow$ *Quantum eigenvalue transformation of unitaries* *[Dong, Lin, Tong '22]*

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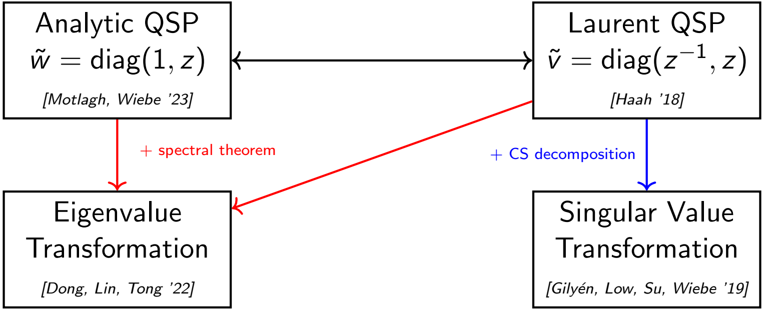
Analytic QSP
$\tilde{w} = \mathrm{diag}(1, z)$
[Motlagh, Wiebe '23]
Laurent QSP
$\tilde{v} = \mathrm{diag}(z^{-1}, z)$
[Haah '18]
+ spectral theorem
+ CS decomposition
Eigenvalue Transformation
[Dong, Lin, Tong '22]
Singular Value Transformation
[Gilyén, Low, Su, Wiebe '19]

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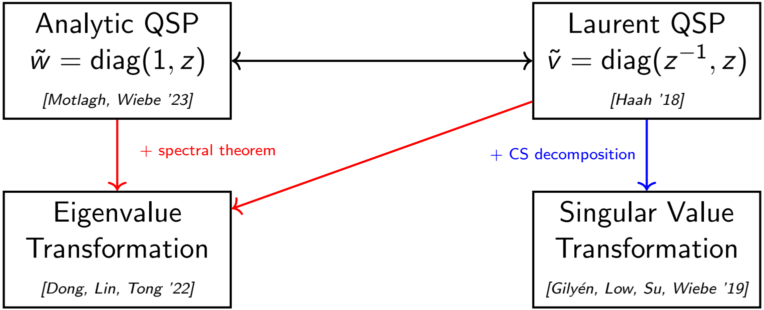

- Phase estimation/detection
- Ground-state preparation
- ...

- Matrix inversion
- Hamiltonian simulation
- ...

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Univariate case $U \to P(U)$

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Multivariate case $U_1, \dots, U_m \to P(U_1, \dots, U_m)$

- This work

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Classical choices *[Rossi, Chuang '22]*

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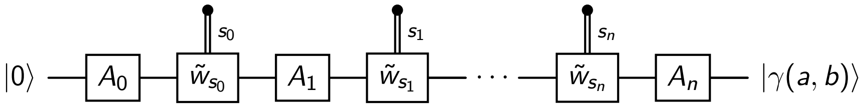

with $A_k \in SU(2), s_k \in \{a, b\}$.

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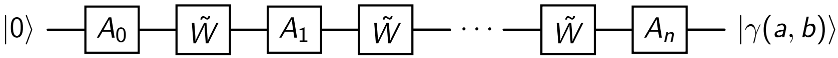

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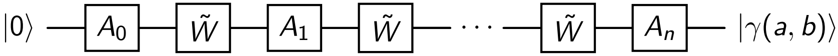

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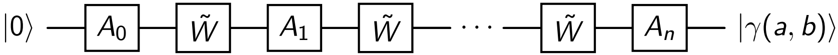

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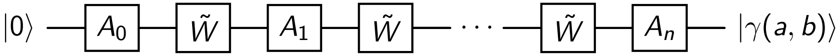

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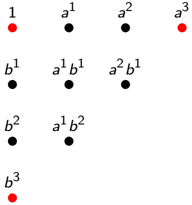
- **Conjecture**: 3D protocol $\subseteq$ Rossi-Chuang protocol when $R \equiv 0$.

Constructive result

Theorem

Any triple $|\gamma(a,b)\rangle = (P, Q, R)$ *of degree* n *whose coefficient vectors of* $1, a^n, b^n$ *are non-zero admits a sequence* $\{A_k\}_k \in SU(3)$ *such that:*

$$A_n \tilde{W} A_{n-1} \tilde{W} \cdots \tilde{W} A_0 |0\rangle = |\gamma(a,b)\rangle$$



- Analogous to univariate QSP decomposition of [Motlagh, Wiebe '23]

A necessary condition for two-dimensional polynomials

$$|\gamma(a,b)\rangle = P(a,b)|0\rangle + Q(a,b)|1\rangle = \sum_{k,h} |\gamma_{k,h}\rangle a^k b^h$$

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Example: [Németh et al. '23]

$$P(a,b) = a^2b^2 + 1 - \frac{122+8i}{37}(ab^2+a) + \frac{114+56i}{37}(a^2+b^2) + \frac{362-248i}{111}(a^2b+b) + \left(\frac{692}{111} - \frac{719i}{222}\right)ab$$

$$Q(a,b) = a^2b^2 - 1 - \frac{122+66i}{37}(ab^2-a) - \frac{56+114i}{37}(a^2-b^2) + \frac{362-418i}{111}(a^2b-b)$$

Corollary: inapproximability

Define the following quantity:

$$q(\gamma) = \min \left\{ \max_{x,y} \left| \det \left[|\gamma_{x,0}\rangle \; |\gamma_{y,0}\rangle \right] \right|, \; \max_{x,y} \left| \det \left[|\gamma_{0,x}\rangle \; |\gamma_{0,y}\rangle \right] \right| \right\}$$

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Corollary (Inapproximability)

If a polynomial $|\gamma'(a,b)\rangle$ *satisfies*

$$\left\| |\gamma'(a,b)\rangle - |\gamma(a,b)\rangle \right\| < q(\gamma)/4,$$

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- Previous example cannot be approximated within $\epsilon \simeq 0.013$.

Conclusions and outlook

Constructive result:

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  - Can be extended to m variables using $m+1$ dimensions;
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Thank you for your attention!