IR finite correlation functions in de Sitter space, a smooth massless limit, and an autonomous equation

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Based on A. Kamenshchik & P. P., arXiv: [2410.16226]

CERN TH Workshop: Looping in the Primordial Universe
31 October 2024

Istituto Nazionale di Fisica Nucleare
Sezione di Bologna

Theory and Phenomenology of Fundamental Interactions
UNIVERSITY AND INFN · BOLOGNA

◯ Scalar fields might play a central role during the inflationary (quasi-)de Sitter stage and be responsible for the observable large-scale structure formation.

◯ Quantum scalar fields in de Sitter (dS) space:

· The choice of the vacuum becomes a non-trivial task in a curved background.

· In the mainstream, people choose in dS set up the so-called Bunch-Davies vacuum, where the short-wavelength modes behave as the corresponding modes in Minkowski space.

· For a massive scalar field, there exists a one-parameter family of dS-invariant vacua states (Allen,Phys.Rev.D,1985), while for a massless one does not (Allen & Folacci,Phys.Rev.D,1987).

◯ Consequently, the abyss between computation's results for massive and massless scalar fields in dS appears: there is no regular massless limit.

IN THIS TALK

◊ In contrast to the standard massive scalar field's theory based on dS-invariant vacuum, we develop a vacuum-independent reasoning that may not possess dS invariance but results in a smooth massless limit of the correlation function's infrared (IR) part. Our elaboration can be considered a theory of a massive scalar field with the vacuum «inherited» from a massless one.

· We employ Yang-Feldman formalism (Yang & Feldman, Phys.Rev., 1950) that recursively defines the interacting field as a coupling constant's formal power series via the free one.
· Such a formalism in dS appears to be rather convenient for the leading infrared logarithm approximation (Woodard, Nucl. Phys. B, 2005; Tsamis & Woodard, Nucl. Phys. B, 2005, etc.).

♢ We propose a trick to «hang up» the mass that affords to calculate a correlation function of a free massive scalar field and proceed with quantum corrections relying only on the known correlation function's IR part of a free massless one. The free massive correlation function coincides with the Ornstein-Uhlenbeck mean-reverting stochastic process' one. By virtue, it over time forcibly tends to the equilibrium state, which is dS-invariant.

♢ Through our Yang-Feldman-type equation up to the three-loop quantum corrections for the long-wavelength modes' two- & four-point correlation functions have been calculated.
· We are in agreement with the Schwinger-Keldysh technique's results at the late-time limit (Kamenshchik, Starobinsky, Vardanyan, Eur.Phys.J.C, 2022) and Starobinsky's stochastic approach (Starobinsky, 1986; Starobinsky & Yokoyama, Phys.Rev.D, 1994).
· We compared outcomes with the Hartree-Fock approximation (leaves aside «Sunset»).

♢ At last, we constructed an autonomous equation for the two-point function. Integrating its approximate version, one obtains a non-analytic expression with respect to λ that reproduces the correct perturbative series up to the two-loop level.

Yang-Feldman formalism

The Yang-Feldman formalism recursively defines the interacting field as a coupling constant's formal power series through the free field (Yang & Feldman, Phys.Rev., 1950).

In this approach, a solution to the Klein-Gordon equation for a scalar field is placed by

$$\phi(t,\vec{x}) = \phi_0(t,\vec{x}) - \int d^4x' \sqrt{-g(x')}\, G_R(t,\vec{x};t',\vec{x}')\, V'_\phi(\phi(t',\vec{x}')),$$

where $\phi_0(t,\vec{x})$ is the solution for the homogeneous equation, $\Box\phi_0(t,\vec{x}) = 0$, and the Green's function is any solution to $\Box\, G_R(t,\vec{x};t',\vec{x}') = \delta(t-t')\,\delta(\vec{x}-\vec{x}')/\sqrt{-g(x')}$ with retarded b.c., $G_R(t,\vec{x};t',\vec{x}') = 0$ for $t \leq t'$. One can express this solution as

$$G_R(t,\vec{x};t',\vec{x}') = i\Theta(t-t')\,\langle[\phi_0(t,\vec{x}),\phi_0(t',\vec{x}')]\rangle.$$

The representation of the scalar field in the commutator above upon canonically normalized creation and annihilation operators is the following:

$$\phi_0(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \left(u_k(t)\, e^{i\vec{k}\vec{x}}\, \hat{a}_{\vec{k}} + u_k^*(t)\, e^{-i\vec{k}\vec{x}}\, \hat{a}^\dagger_{\vec{k}}\right),$$

here modes $u_k(t)$ in the de Sitter space, $ds^2 = dt^2 - e^{2Ht}d\vec{x}^2$, being the solution to

$$\ddot{u}_k + 3H\dot{u}_k + k^2 e^{-2Ht} u_k = 0,$$

must be normalized through the Wronskian

$$W[u_k(t), u_k^*(t)] = \dot{u}_k u_k^* - u_k \dot{u}_k^* = -ie^{-3Ht}$$

as a consequence of the canonical commutation relations. Then, straightforwardly,

$$G_R(t,\vec{x};t',\vec{x}') = i\Theta(t-t') \int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}(\vec{x}-\vec{x}')} \left(u_k(t)u_k^*(t') - u_k^*(t)\, u_k(t')\right).$$

Yang-Feldman equation for a massless scalar field in de Sitter space

We are interested in the contribution of the very soft, long-wavelength (l-w) modes, whose wave numbers are small, i.e., $k \leq He^{Ht}$.

Thus, one can neglect the last term $\sim k^2$, leading to the general solution as

$$\ddot{u}_k + 3H\dot{u}_k + k^2 e^{-2Ht} u_k = 0 \quad \Rightarrow \quad u_k^{\text{l-w}}(t) = c_1 + c_2\, e^{-3Ht}.$$

By employing Wronskian $W[u_k(t), u_k^*(t)]$, namely, $c_1^* c_2 + c_1 c_2^* = i/3H$, one gets[1]

$$G_{\text{R}}^{\text{l-w}}\left(t, \vec{x}; t', \vec{x}'\right) = \frac{\Theta(t-t')}{3H}\left(e^{-3Ht'} - e^{-3Ht}\right)\delta(\vec{x} - \vec{x}').$$

Therefore, at the leading logarithm, the Yang-Feldman equation takes a simple form

$$\phi\left(t, \vec{x}\right) = \phi_0\left(t, \vec{x}\right) - \frac{1}{3H}\int_0^t dt'\; V'_{\phi}\big(\phi\left(t', \vec{x}\right)\big).$$

Note that ***we did not use any particular choice of the vacuum***.

One can iteratively use this equation to calculate the correlation functions of the field $\phi\left(t, \vec{x}\right)$, knowing the two-point correlation function of the free massless one $\phi_0\left(t, \vec{x}\right)$.

[1] **One can arrive at the same form of the retarded Green's function owing to the explicit form for the basis functions of the chosen vacuum in the Fock space, the so-called Bunch-Davies vacuum (Woodard, Nucl. Phys. B, 2005).**

Yang-Feldman-type equation for a massive scalar field in de Sitter space

Through the use of Yang-Feldman equation, we define the free massive via massless one as

$$\tilde{\phi}(t,\vec{x}) = \phi_0(t,\vec{x}) - \frac{m^2}{3H} e^{-\frac{m^2 t}{3H}} \int_0^t dt' \, e^{\frac{m^2 t'}{3H}} \phi_0(t',\vec{x}) \, .$$

This introduced relation allows us to calculate the correlation function of a massive scalar field, relying only on the known long-wavelength infrared part of the free massless one.

Furthermore, one can find the corresponding analog to the Yang-Feldman equation:

$$\phi(t,\vec{x}) = \tilde{\phi}(t,\vec{x}) - \frac{\lambda}{3H} e^{-\frac{m^2 t}{3H}} \int_0^t dt' \, e^{\frac{m^2 t'}{3H}} \phi^3(t',\vec{x}) \, ;$$

The iterated Yang-Feldman-type equation can be written out up to a few first terms

$$\phi(t,\vec{x}) = \tilde{\phi}(t,\vec{x}) - \frac{\lambda}{3H} e^{-\frac{m^2 t}{3H}} \int_0^t dt' \, e^{\frac{m^2 t'}{3H}} \tilde{\phi}^3(t',\vec{x})$$

$$+ \frac{\lambda^2}{3H^2} e^{-\frac{m^2 t}{3H}} \int_0^t dt' \, \tilde{\phi}^2(t',\vec{x}) \int_0^{t'} dt'' e^{\frac{m^2 t''}{3H}} \tilde{\phi}^3(t'',\vec{x}) + \ldots$$

One can use it to calculate the correlation functions of the massive field $\phi(t,\vec{x})$ through the known correlation function of the free massless one $\phi_0(t,\vec{x})$.

Two-point correlation function for the free massive scalar field and a loop series

Let us calculate the two-point correlation function for the free massive field $\tilde{\phi}(t,\vec{x})$, where the spatial' spacetime points coincide while the time coordinates are different:

$$\left\langle \tilde{\phi}(t_1,\vec{x})\,\tilde{\phi}(t_2,\vec{x})\right\rangle \equiv \left\langle \tilde{\phi}(t_1)\,\tilde{\phi}(t_2)\right\rangle = \frac{3H^4}{8\pi^2 m^2}\left(e^{-\frac{m^2}{3H}|t_1-t_2|} - e^{-\frac{m^2}{3H}(t_1+t_2)}\right);$$

where we have used the well-known long-wavelength infrared part of the free massless field

$$\left\langle \phi_0(t_1)\,\phi_0(t_2)\right\rangle = \frac{H^3 t_2}{4\pi^2}, \quad t_2 \leq t_1 \quad \text{(Vilenkin \& Ford, Phys.Rev.D; Linde, Phys.Lett.B, 1982).}$$

The obtained $\left\langle \tilde{\phi}(t_1)\,\tilde{\phi}(t_2)\right\rangle$ exactly coincides with Ornstein-Uhlenbeck stochastic process's one. The notable feature is its tendency to drift towards average: the higher the deviation, the stronger the wandering towards with a mean reversion rate $m^2/3H$.

In our development, the correlation functions have a smooth massless limit, coinciding with the expressions obtained for a massless scalar field. Such a reasoning can be considered as a theory of a massive scalar field with the vacuum «inherited» from the massless one.

Building on this correlator, one reaches the loop series through our Yang-Feldman equation

$$\begin{aligned}\left\langle \phi(t_1)\,\phi(t_2)\right\rangle = \left\langle \tilde{\phi}(t_1)\,\tilde{\phi}(t_2)\right\rangle &+ \left\langle \phi(t_1)\,\phi(t_2)\right\rangle_{1-\text{loop}} + \left\langle \phi(t_1)\,\phi(t_2)\right\rangle_{2-\text{loop}} \\ &+ \left\langle \phi(t_1)\,\phi(t_2)\right\rangle_{3-\text{loop}} + O\left(\lambda^4\right).\end{aligned}$$

Full results up to three loops can be seen in our pre-print on arXiv.

Four-point correlation function

In the case of the four-point correlation function, one can continue in the same calculation manner. Even so, we already have the answer for the tree level

$$\left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{\text{tree}} = \left\langle \tilde{\phi}\left(t_1\right)\tilde{\phi}\left(t_2\right)\right\rangle\left\langle \tilde{\phi}\left(t_3\right)\tilde{\phi}\left(t_4\right)\right\rangle + \text{perm.}$$

Afterwards, for the linear order in λ, we partially also have the answer, since in this case, the complete correlation function is splitted into connected

$$\left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{1-\text{loop}}^{\text{connected}} = -\frac{2\lambda}{H}\mathrm{e}^{-\frac{m^2 t_1}{3H}}\int_0^{t_1} dt'\mathrm{e}^{\frac{m^2 t'}{3H}}\left\langle \tilde{\phi}\left(t'\right)\tilde{\phi}\left(t_2\right)\right\rangle \times$$

$$\times\left\langle \tilde{\phi}\left(t'\right)\tilde{\phi}\left(t_3\right)\right\rangle\left\langle \tilde{\phi}\left(t'\right)\tilde{\phi}\left(t_4\right)\right\rangle \quad + \text{perm.};$$

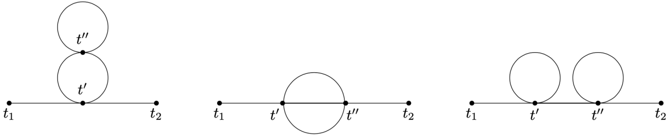
and disconnected (just a combination of two-point functions) diagram' types

$$\left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{1-\text{loop}}^{\text{disconnected}} = \left\langle \tilde{\phi}\left(t_1\right)\tilde{\phi}\left(t_2\right)\right\rangle\left\langle \phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{1-\text{loop}} + \text{perm.}$$

We proceed up to

$$\left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle = \left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{\text{tree}}$$

$$+\left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{1-\text{loop}} + \left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{2-\text{loop}} + O\left(\lambda^3\right).$$



$$\mathcal{I}_{1,a} = \frac{2\lambda^2}{H^2} e^{-\frac{m^2 t_1}{3H}} \int_0^{t_1} dt' \int_0^{t'} dt'' e^{\frac{m^2 t''}{3H}} \langle \tilde{\phi}(t') \tilde{\phi}(t_2) \rangle \langle \tilde{\phi}(t') \tilde{\phi}(t'') \rangle \langle \tilde{\phi}^2(t'') \rangle;$$

$$\mathcal{I}_{2,a} = \frac{2\lambda^2}{H^2} e^{-\frac{m^2 t_1}{3H}} \int_0^{t_1} dt' \int_0^{t'} dt'' e^{\frac{m^2 t''}{3H}} \langle \tilde{\phi}(t'') \tilde{\phi}(t_2) \rangle \left(\langle \tilde{\phi}(t') \tilde{\phi}(t'') \rangle \right)^2;$$

$$\mathcal{I}_{3,a} = \frac{\lambda^2}{H^2} e^{-\frac{m^2 t_1}{3H}} \int_0^{t_1} dt' \int_0^{t'} dt'' e^{\frac{m^2 t''}{3H}} \langle \tilde{\phi}(t'') \tilde{\phi}(t_2) \rangle \langle \tilde{\phi}^2(t') \rangle \langle \tilde{\phi}^2(t'') \rangle;$$

$$\mathcal{I}_{i,b} = \mathcal{I}_{i,a} \text{ with } (t_1 \leftrightarrow t_2)$$

$$\mathcal{I}_4 = \frac{2\lambda^2}{3H^2} e^{-\frac{m^2}{3H}(t_1+t_2)} \int_0^{t_1} dt' e^{\frac{m^2 t'}{3H}} \int_0^{t_2} dt'' e^{\frac{m^2 t''}{3H}} \left(\langle \tilde{\phi}(t') \tilde{\phi}(t'') \rangle \right)^3;$$

$$\mathcal{I}_5 = \frac{\lambda^2}{H^2} e^{-\frac{m^2}{3H}(t_1+t_2)} \int_0^{t_1} dt' e^{\frac{m^2 t'}{3H}} \int_0^{t_2} dt'' e^{\frac{m^2 t''}{3H}} \langle \tilde{\phi}^2(t') \rangle \langle \tilde{\phi}(t') \tilde{\phi}(t'') \rangle \langle \tilde{\phi}^2(t'') \rangle;$$

$$\langle \phi(t_1) \phi(t_2) \rangle^{\text{Snowman}} = \mathcal{I}_{1,a} + \mathcal{I}_{1,b}; \qquad \langle \phi(t_1) \phi(t_2) \rangle^{\text{Sunset}} = \mathcal{I}_{2,a} + \mathcal{I}_{2,b} + \mathcal{I}_4$$

$$\langle \phi(t_1) \phi(t_2) \rangle^{\text{Ind. Loops}} = \mathcal{I}_{3,a} + \mathcal{I}_{3,b} + \mathcal{I}_5$$

Comparison with the stochastic approach

Starobinsky stochastic approach (Starobinsky 1986; Starobinsky & Yokoyama, Phys.Rev.D, 1994) matches the l-w part of the quantum field $\phi(t,\vec{x})$ to the classical stochastic field $\varphi(t,\vec{x})$ with a probability distribution function $\rho\big[\varphi(t,\vec{x})\big]$ that satisfies the Fokker-Planck equation

$$\partial_t\,\rho\big[\varphi(t,\vec{x})\big] = \frac{1}{3H}\,\partial_\varphi\Big(\rho\big[\varphi(t,\vec{x})\big]\,V'_\varphi\big(\varphi(t,\vec{x})\big)\Big) + \frac{H^3}{8\pi^2}\,\partial^2_\varphi\Big(\rho\big[\varphi(t,\vec{x})\big]\Big).$$

Any solution of this Fokker-Planck equation tends to the static solution at late times

$$\rho\big[\varphi(t,\vec{x})\big] \quad \xrightarrow[\text{times}]{\text{late}} \quad \rho_{\mathbf{st}}\,[\varphi] = \frac{1}{\mathcal{N}}\,\mathrm{e}^{-\frac{8\pi^2}{3H^4}V(\varphi)}.$$

Therefore, the expectation values of this stochastic variable at small λ expansion is

$$\langle\varphi^{2n}\rangle = \frac{\int_{-\infty}^{+\infty} d\varphi\,\varphi^{2n}\rho_{\mathbf{st}}\,[\varphi]}{\int_{-\infty}^{+\infty} d\varphi\,\rho_{\mathbf{st}}\,[\varphi]} \to \begin{cases} \dfrac{3H^4}{8\pi^2 m^2} - \dfrac{27\lambda H^8}{64\,\pi^4 m^6} + \dfrac{81\lambda^2 H^{12}}{64\,\pi^6 m^{10}} - \dfrac{24057\lambda^3 H^{16}}{4096\,\pi^8 m^{14}} + O\left(\lambda^4\right); \\[2ex] \dfrac{27H^8}{64\,\pi^4 m^4} - \dfrac{81\lambda H^{12}}{64\,\pi^6 m^8} + \dfrac{24057\lambda^2 H^{16}}{4096\,\pi^8 m^{12}} + O\left(\lambda^3\right). \end{cases}$$

One can reduce the expressions to modified Bessel functions of the 2nd kind $\mathcal{K}_\nu(z)$. It was shown (Kamenshchik, Starobinsky, Vardanyan, Eur.Phys.J.C, 2022), that

$$\int_{-\infty}^{+\infty} d\varphi\,\rho_{\mathbf{st}}\,[\varphi] = 1 \;\to\; \mathcal{N} = \int_{-\infty}^{+\infty} d\varphi\,\mathrm{e}^{-\frac{8\pi^2}{3H^4}V(\varphi)} = \frac{m}{\sqrt{2\lambda}}\exp\left(\frac{\pi^2 m^4}{3\lambda H^4}\right)\mathcal{K}_{1/4}\left(\frac{\pi^2 m^4}{3\lambda H^4}\right).$$

$$\langle\varphi^2\rangle = -\frac{3H^4}{4\pi^2}\frac{1}{\mathcal{N}}\frac{d\mathcal{N}}{dm^2} \quad\Rightarrow\quad \langle\varphi^2\rangle = \frac{m^2}{2\lambda}\left(\frac{\mathcal{K}_{3/4}(z)}{\mathcal{K}_{1/4}(z)} - 1\right), \quad \text{where} \quad z \equiv \frac{\pi^2 m^4}{3\lambda H^4},$$

Comparison with the stochastic approach

One can proceed further and get the expression anew for $\left\langle \varphi^4 \right\rangle$, such as

$$\left\langle \varphi^4 \right\rangle = \left(-\frac{3H^4}{4\pi^2} \right)^2 \frac{1}{\mathcal{N}} \frac{d^{\,2}\mathcal{N}}{d\left(m^2\right)^2} \qquad \Rightarrow \qquad \left\langle \varphi^4 \right\rangle = \frac{3H^4}{8\pi^2\lambda} + \frac{m^4}{2\lambda^2}\left(1 - \frac{K_{3/4}(z)}{K_{1/4}(z)}\right).$$

Let us point out the general structure of any 2n'th expectation value

$$\frac{d^{\,\mathrm{n}}\mathcal{N}}{d\left(m^2\right)^{\mathrm{n}}} = \alpha_{\mathrm{n}}\left(m^2\right) \mathrm{e}^z\, K_{1/4}(z) + \beta_{\mathrm{n}}\left(m^2\right) \mathrm{e}^z\, K_{3/4}(z),$$

$$\text{since} \qquad \frac{d}{dz}\mathcal{K}_\nu(z) = -\frac{\nu}{z}\,\mathcal{K}_\nu(z) - \mathcal{K}_{\nu-1}(z) \quad \text{and} \quad \mathcal{K}_{-\nu}(z) = \mathcal{K}_\nu(z),$$

where from the definition of $\mathcal{N}$, $\alpha_0 = m/\sqrt{2\lambda}$ and $\beta_0 = 0$, and the recurrence relations are

$$\alpha_{\mathrm{n}+1} = \frac{d\alpha_{\mathrm{n}}}{dm^2} - \frac{\alpha_{\mathrm{n}}}{2m^2} + \frac{2\pi^2 m^2}{3\lambda H^4}\left(\alpha_{\mathrm{n}} - \beta_{\mathrm{n}}\right); \quad \beta_{\mathrm{n}+1} = \frac{d\beta_{\mathrm{n}}}{dm^2} - \frac{3\beta_{\mathrm{n}}}{2m^2} - \frac{2\pi^2 m^2}{3\lambda H^4}\left(\alpha_{\mathrm{n}} - \beta_{\mathrm{n}}\right),$$

resulting in

$$\left\langle \varphi^{2\mathrm{n}} \right\rangle = \left(-\frac{3H^4}{4\pi^2} \right)^{\mathrm{n}} \frac{1}{\mathcal{N}} \frac{d^{\,\mathrm{n}}\mathcal{N}}{d\left(m^2\right)^{\mathrm{n}}} \;\; \Rightarrow \;\; \left\langle \varphi^{2\mathrm{n}} \right\rangle = \frac{\sqrt{2\lambda}}{m}\left(-\frac{3H^4}{4\pi^2} \right)^{\mathrm{n}} \left(\alpha_{\mathrm{n}} + \beta_{\mathrm{n}}\,\frac{K_{3/4}(z)}{K_{1/4}(z)}\right).$$

At large $z \gg 1$, this expansion reproduces our outcomes $\left\langle \phi^2(t) \right\rangle$ and $\left\langle \phi^4(t) \right\rangle$ at $t \to \infty$.

To compare our outcomes for $\langle \phi^2(t) \rangle$ to those obtained in the Hartree-Fock (Gaussian) approximation, let us consider the Klein-Gordon equation. After some manipulations with the use of the Hartree-Fock approximation, namely $\langle \phi^4(t) \rangle = 3\langle \phi^2(t) \rangle^2$, one has

$$\frac{d\langle \phi^2 \rangle}{dt} = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H}\langle \phi^2 \rangle - \frac{2\lambda}{H}\langle \phi^2 \rangle^2.$$

The solution to the equation above is

$$\langle \phi^2(t) \rangle_{\mathrm{HF}} = \frac{\frac{3H^4}{4\pi^2 m^2}\left(1 - \exp\left(-\frac{2m^2 t}{3H}\sqrt{1 + \frac{9\lambda H^4}{2\pi^2 m^4}}\right)\right)}{1 + \sqrt{1 + \frac{9\lambda H^4}{2\pi^2 m^4}} - \left(1 - \sqrt{1 + \frac{9\lambda H^4}{2\pi^2 m^4}}\right)\exp\left(-\frac{2m^2 t}{3H}\sqrt{1 + \frac{9\lambda H^4}{2\pi^2 m^4}}\right)};$$

Expanding this solution in a series along a small coupling constant λ, one can see that at the tree and at one-loop levels, this approximation gives the correct results, while at the already two-loop level, it does not.

Through our results for each of the two-loop diagrams, one can conclude that the Hartree-Fock approximation only resummes the «Cactus» and «Double Seagull»-type diagrams, leaving aside the «Sunset» one.

An autonomous equation for the two-point correlation function

One can also make an effort to «correct» the Hartree-Fock approximation. To do so, we construct the autonomous equation (Kamenshchik & Vardanyan, Phys.Rev.D, 2020), which catches the absentee «Sunset» diagrammatic contribution. Our outcomes are

$$\left\langle \phi^2(t) \right\rangle = \frac{3H^4}{8\pi^2 m^2}\left(1 - e^{-\frac{2m^2 t}{3H}}\right) - \frac{27\lambda H^8}{64\pi^4 m^6}\left(1 - \frac{4m^2 t}{3H} e^{-\frac{2m^2 t}{3H}} - e^{-\frac{4m^2 t}{3H}}\right)$$
$$+ \frac{81\lambda^2 H^{12}}{64\pi^6 m^{10}}\left(1 + \left(\frac{21}{8} - \frac{3m^2 t}{2H} - \frac{m^4 t^2}{3H^2}\right) e^{-\frac{2m^2 t}{3H}} - \left(3 + \frac{2m^2 t}{H}\right) e^{-\frac{4m^2 t}{3H}} - \frac{5}{8} e^{-\frac{2m^2 t}{H}}\right).$$

At the zero order in a series $\left\langle \phi^2(t) \right\rangle$ on small λ, we have

$$f(t) = \frac{3H^4}{8\pi^2 m^2}\left(1 - e^{-\frac{2m^2 t}{3H}}\right) \quad \Rightarrow \quad e^{-\frac{2m^2 t}{3H}} = 1 - \frac{8\pi^2 m^2}{3H^4} f(t);$$

To have an autonomous equation, whose solution has the correct expansion up to the terms linear in λ, we substitute it into l.h.s. and r.h.s. and find the correcting term

$$\frac{d}{dt}(f(t)) = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H} f(t) + \Delta f_1(t), \qquad \Delta f_1(t) = -\frac{9\lambda H^7}{32\,\pi^4 m^4}\left(1 - e^{-\frac{2m^2 t}{3H}}\right)^2.$$

and equation appears to be the Hartree-Fock's:

$$\frac{d}{dt}(f(t)) = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H} f(t) - \frac{2\lambda}{H}(f(t))^2,$$

which was able to provide the correct result up to λ.

An autonomous equation for the two-point correlation function

At the two-loop level, we will come to

$$\frac{d}{dt}\left(f(t)\right) = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H}\,f(t) - \frac{2\lambda}{H}\,\left(f(t)\right)^2 - \frac{27\lambda^2 H^{11}}{32\,\pi^6 m^8}\left(\frac{4\,\pi^2 m^2}{H^4}\,f(t) - \frac{16\,\pi^4 m^4}{H^8}\,\left(f(t)\right)^2 \right.$$
$$\left. + \frac{256\,\pi^6 m^6}{27H^{12}}\,\left(f(t)\right)^3 + \frac{3}{2}\left(1 - \frac{8\pi^2 m^2}{3H^4}\,f(t)\right)^2 \ln\left(1 - \frac{8\pi^2 m^2}{3H^4}\,f(t)\right)\right);$$

Taking the function $f(t)$ as $f(t) = \left\langle \phi^2\,(t) \right\rangle_{\mathrm{HF}} + \delta f(t)$, one can find cumbersome though straightforward expression for $\delta f(t)$, and the limiting value at $t \to \infty$ is

$$\delta f(t) \xrightarrow{t\to\infty} \frac{243\lambda^2 H^{12}}{32\pi^6 m^{10}}\left(-\frac{3\mathcal{Z}^2 - 3\mathcal{Z} - 2}{6\mathcal{Z}\left(\mathcal{Z}+1\right)^3} - \frac{\left(\mathcal{Z}-1\right)^2}{4\mathcal{Z}\left(\mathcal{Z}+1\right)^2}\ln\left(\frac{\mathcal{Z}-1}{\mathcal{Z}+1}\right)\right)$$
$$\approx \frac{81\lambda^2 H^{12}}{256\pi^6 m^{10}} + O\left(\lambda^3\right), \quad \text{where} \quad \mathcal{Z} \equiv \sqrt{1 + \frac{3}{2z}} \quad \text{and} \quad z \equiv \frac{\pi^2 m^4}{3\lambda H^4}.$$

It precisely matches with an absent «Sunset» contribution in Hartree-Fock approximation. The full non-analytical on λ result for a two-point correlation function at late times is

$$\left\langle \phi^2\,(t) \right\rangle^{\mathbf{aut}} = \left\langle \phi^2\,(t) \right\rangle_{\mathbf{HF}} + \delta f(t) \xrightarrow{t\to\infty}$$

$$\longrightarrow \frac{H^2}{\pi\sqrt{\lambda}}\left(\frac{\sqrt{12z+18} - \sqrt{12z}}{12} + \frac{\left(3\sqrt{4z^2+6z} - 2z - 9\right)\left(\sqrt{12z+18} - \sqrt{12z}\right)^3}{1728z\sqrt{4z^2+6z}}\right.$$
$$\left. + \frac{3\left(\sqrt{2z+3} - \sqrt{2z}\right)^4}{64z^2\sqrt{12z+18}}\ln\left(\frac{\sqrt{2z+3} + \sqrt{2z}}{\sqrt{2z+3} - \sqrt{2z}}\right)\right),$$

and it almost coincides with the stochastic one in the whole interval of z.

Conclusion

In contrast to the standard theory of a massive scalar field based on the de Sitter-invariant vacuum, we developed vacuum-independent reasoning that may not possess de Sitter invariance but results in a smooth massless limit of the correlation function's infrared part.

Through the Yang-Feldman-type equation, one-, two-, and three-loop quantum corrections for the long-wavelength modes' two-point and four-point correlation functions have been calculated. The main «building block» of elaborated approach is the free massive field's correlation function that coincides with the Ornstein-Uhlenbeck stochastic process's one.

Our outcomes correspond to the Schwinger-Keldysh results at the late-time limit and were also compared to Starobinsky's stochastic approach and the Hartree-Fock approximation.

At last, we constructed an autonomous equation for the two-point function. Integrating its approximate version, one obtains a non-analytic expression with respect to a self-interaction coupling constant λ that reproduces the correct perturbative series up to the two-loop level.

Thank you for your attention!

Backup

$$\phi\left(t,\vec{x}\right)=\phi_{0}\left(t,\vec{x}\right)+\int\limits_{0}^{t}dt'\,F\big(\phi\left(t',\vec{x}\right)\big),$$

where $F\big(\phi\left(t',\vec{x}\right)\big)$ has the form $F\big(\phi\left(t',\vec{x}\right)\big)=\alpha\phi\left(t',\vec{x}\right)+W\big(\phi\left(t',\vec{x}\right)\big)$ and $\phi_{0}\left(t',\vec{x}\right)$ is a given function. Introducing the new scalar field $\tilde{\phi}\left(t,\vec{x}\right)$, which satisfies the equation

$$\tilde{\phi}\left(t,\vec{x}\right)=\phi_{0}\left(t,\vec{x}\right)+\alpha\int\limits_{0}^{t}dt'\,\tilde{\phi}\left(t',\vec{x}\right),$$

and taking the time derivative, we can solve exactly an ordinary inhomogeneous first-order equation

$$\tilde{\phi}\left(t,\vec{x}\right)=\phi_{0}\left(t,\vec{x}\right)+\alpha e^{\alpha t}\int\limits_{0}^{t}dt'\,e^{-\alpha t'}\phi_{0}\left(t',\vec{x}\right),\qquad \phi_{0}\left(t,\vec{x}\right)=\tilde{\phi}\left(t,\vec{x}\right)-\alpha\int\limits_{0}^{t}\tilde{\phi}\left(t',\vec{x}\right).$$

Substituting this expression

$$\phi\left(t,\vec{x}\right)=\tilde{\phi}\left(t,\vec{x}\right)+\alpha\int\limits_{0}^{t}dt'\,\Big(\phi\left(t',\vec{x}\right)-\tilde{\phi}\left(t',\vec{x}\right)\Big)+\int\limits_{0}^{t}dt'\,W\big(\phi\left(t',\vec{x}\right)\big),$$

and hereafter introduce a new equation

$$\phi\left(t,\vec{x}\right)=\tilde{\phi}\left(t,\vec{x}\right)+e^{\alpha t}\int\limits_{0}^{t}dt'\,e^{-\alpha t'}W\big(\phi\left(t',\vec{x}\right)\big),$$

which is equivalent to equation above. To establish this, one can express $\phi\left(t,\vec{x}\right)-\tilde{\phi}\left(t,\vec{x}\right)$, to changing the order of integration and one can show this equation reduces to the previous one.

Backup

$$\left\langle \phi(t_1)\,\phi(t_2)\right\rangle_{1-\mathrm{loop}} = -\frac{27\lambda H^8}{128\,\pi^4 m^6}\Bigg(2\,e^{-\frac{m^2}{3H}|t_1-t_2|} + \frac{4m^2}{3H}\Big(|t_1-t_2| - (t_1+t_2)\Big)\,e^{-\frac{m^2}{3H}(t_1+t_2)}$$

$$- e^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)} + \frac{2m^2}{3H}\,|t_1-t_2|\left(e^{-\frac{m^2}{3H}|t_1-t_2|} - e^{-\frac{m^2}{3H}(t_1+t_2)}\right)$$

$$- e^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)} + e^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)} - e^{-\frac{m^2}{3H}(t_1+t_2)}\Bigg);$$

$$\left\langle \phi(t_1)\,\phi(t_2)\right\rangle_{2-\mathrm{loop}} = \frac{81\lambda^2 H^{12}}{2048\,\pi^6 m^{10}}\Bigg(\left(30 + \frac{12m^2}{H}\,|t_1-t_2| + \frac{2m^4}{3H^2}\,|t_1-t_2|^2\right)e^{-\frac{m^2}{3H}|t_1-t_2|}$$

$$+ 2e^{-\frac{m^2}{H}|t_1-t_2|} - 5\,e^{-\frac{m^2}{H}(t_1+t_2)} + \Bigg(36 + \frac{2m^2}{H}\Big(9\,|t_1-t_2| - 14\,(t_1+t_2)\Big)$$

$$- \frac{2m^4}{3H^2}\Big(|t_1-t_2| - 2\,(t_1+t_2)\Big)^2\Bigg)e^{-\frac{m^2}{3H}(t_1+t_2)} + \frac{15}{2}e^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)}$$

$$+ \left(48 + \frac{2m^2}{H}\Big(7\,|t_1-t_2| + 2\,(t_1+t_2)\Big)\right)e^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)}$$

$$- \left(45 + \frac{2m^2}{H}\Big(|t_1-t_2| + 8\,(t_1+t_2)\Big)\right)e^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)} - \frac{15}{2}e^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)}$$

$$- \left(\frac{117}{2} - \frac{2m^2}{H}\Big(7\,|t_1-t_2| - 8\,(t_1+t_2)\Big)\right)e^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)} - \frac{15}{2}e^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)}\Bigg);$$