IR finite correlation functions in de Sitter space, a smooth massless limit, and an autonomous equation

Polina Petriakova

University of Bologna (DIFA) and INFN, Bologna

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Introduction and motivation

◯ The cosmology of the very early Universe is the unique arena to see how quantum fields and gravity play together, creating the seeds for the observable Universe's structure.
One would like to understand the initial conditions and how to make the precise predictions.

◯ Even in the simplest models of the early Universe, making predictions gets complicated relatively fast. More computation' techniques are needed to test different ideas & scenarios.

◯ The simplest toy model to rely on: scalar fields. They might play a central role at the inflationary (quasi-)de Sitter stage & be responsible for the large-scale structure formation.

◯ For quantum scalar fields living on de Sitter (dS) background, the choice of the vacuum states becomes a non-trivial task. For a massive scalar field, there exists a one-parameter family of dS-invariant vacuum states (Allen, Phys.Rev.D, 1985), while for a massless one does not (Allen & Folacci, Phys.Rev.D, 1987).

◯ Consequently, the abyss between perturbative computation's results for massive and massless scalar fields in de Sitter space appears: there is no regular massless limit.

In this talk

· We consider a particular theory of a massive scalar field living on de Sitter background

$$\mathcal{L}_m = \sqrt{-g}\,\left(\frac{1}{2}\,\phi^{,\mu}\phi_{,\mu} - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\,\phi^4\right), \qquad ds^2 = dt^2 - e^{2Ht}d\vec{x}^2.$$

· In contrast to the standard massive scalar field theory based on the dS-invariant vacuum, we develop some reasoning that may not possess dS invariance but results in a smooth massless limit of the correlation functions in the long-wavelength approximation.

· We employ Yang-Feldman formalism (Yang & Feldman, Phys.Rev., 1950) that recursively defines the interacting field as a coupling constant's formal power series via the free one. Such a formalism in dS appears to be rather convenient for the leading infrared logarithm approximation (Woodard, Nucl. Phys. B, 2005; Tsamis & Woodard, Nucl. Phys. B, 2005, etc.).

· We propose a trick to «hang up» the mass that affords to calculate a correlation function of a free massive scalar field and proceed with quantum corrections relying only on the known correlation function's infrared (IR) part of a free massless one.

· Through our the Yang-Feldman-type equation the quantum corrections for two- & four-point correlation functions have been calculated. We are in agreement at late times with Schwinger-Keldysh technique's results and Starobinsky's stochastic approach.
We compared our results with the Hartree-Fock approximation (leaves aside «Sunset»).

· At last, we have derived an autonomous equation for the two-point function. Integrating its approximate version, one obtains a non-analytic expression with respect to a coupling constant λ that reproduces the correct perturbative series up to the two-loop level.

Yang-Feldman formalism

The Yang-Feldman formalism recursively defines the interacting field as a coupling constant's formal power series through the free field (Yang & Feldman, Phys.Rev., 1950).

In this approach, a solution to the Klein-Gordon equation for a scalar field is placed by

$$\phi\,(t,\vec{x}) = \phi_0\,(t,\vec{x}) - \int d^{\,4}x'\sqrt{-g(x')}\,G_{\mathrm{R}}\,(t,\vec{x}\,;t',\vec{x}\,')\;V'_{\phi}(\phi\,(t',\vec{x}\,')),$$

where $\phi_0\,(t,\vec{x})$ is the solution for the homogeneous equation, $\Box\phi_0\,(t,\vec{x}) = 0$, and the Green's function is any solution to $\Box\, G_{\mathrm{R}}\,(t,\vec{x}\,;t',\vec{x}\,') = \delta(t-t')\,\delta(\vec{x}-\vec{x}\,')/\sqrt{-g(x')}$ with retarded boundary conditions, $G_{\mathrm{R}}\,(t,\vec{x}\,;t',\vec{x}\,') = 0$ for $t \leq t'$. One expresses this solution as

$$G_{\mathrm{R}}\,(t,\vec{x}\,;t',\vec{x}\,') = i\Theta(t-t')\,\langle\left[\phi_0\,(t,\vec{x})\,,\phi_0\,(t',\vec{x}\,')\right]\rangle.$$

The representation of a scalar field in the commutator above upon canonically normalized creation and annihilation operators is the following:

$$\phi_0\,(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\left(u_k(t)\,\mathrm{e}^{i\vec{k}\vec{x}}\,\hat{a}_{\vec{k}} + u_k^*(t)\,\mathrm{e}^{-i\vec{k}\vec{x}}\,\hat{a}^{\dagger}_{\vec{k}}\right),$$

here modes $u_k(t)$ in the de Sitter space, $ds^2 = dt^2 - \mathrm{e}^{2Ht}d\vec{x}^2$, being the solution to

$$\ddot{u}_k + 3H\dot{u}_k + k^2\mathrm{e}^{-2Ht}u_k = 0,$$

must be normalized through the Wronskian

$$W[u_k(t), u_k^*(t)] = \dot{u}_k u_k^* - u_k\dot{u}_k^* = -i\mathrm{e}^{-3Ht}$$

as a consequence of the canonical commutation relations.

Yang-Feldman equation for a massless scalar field in de Sitter space

Then, straightforwardly,

$$G_R\left(t,\vec{x};t',\vec{x}'\right) = i\Theta(t-t')\int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\,(\vec{x}-\vec{x}')}\left(u_k(t)u_k^*(t') - u_k^*(t)\,u_k(t')\right).$$

We are interested in the contribution of the very soft, long-wavelength (l-w) modes, whose wave numbers are small, i.e., $k \leq He^{Ht}$.

Thus, one can neglect the last term $\sim k^2$, leading to the general solution as

$$\ddot{u}_k + 3H\dot{u}_k + k^2 e^{-2Ht}u_k = 0 \quad \Rightarrow \quad u_k^{\text{l-w}}(t) = c_1 + c_2\, e^{-3Ht}.$$

By employing Wronskian $W[u_k(t), u_k^*(t)]$, namely, $c_1^* c_2 + c_1 c_2^* = i/3H$, one gets

$$G_R^{\text{l-w}}\left(t,\vec{x};t',\vec{x}'\right) = \frac{\Theta(t-t')}{3H}\left(e^{-3Ht'} - e^{-3Ht}\right)\delta(\vec{x}-\vec{x}').$$

Therefore, at the leading logarithm, the Yang-Feldman equation takes a simple form

$$\phi\left(t,\vec{x}\right) = \phi_0\left(t,\vec{x}\right) - \frac{1}{3H}\int_0^t dt'\, V'_{\phi}\left(\phi\left(t',\vec{x}\right)\right).$$

Note that we did not use any particular choice of the vacuum.

Nonetheless, one can arrive at the same expressions of the retarded Green's function and equation above (Woodard, Nucl. Phys. B, 2005) owing to the explicit form for the basis functions of the chosen vacuum in the Fock space, the so-called Bunch-Davies vacuum.

Yang-Feldman-type equation for a massive scalar field in de Sitter space

Through the use of Yang-Feldman equation, we define the free massive via massless one as

$$\tilde{\phi}(t,\vec{x}) = \phi_0(t,\vec{x}) - \frac{m^2}{3H} e^{-\frac{m^2 t}{3H}} \int\limits_0^t dt' \, e^{\frac{m^2 t'}{3H}} \phi_0(t',\vec{x}) \, .$$

This introduced relation allows us to calculate the correlation function of a massive scalar field, relying only on the known long-wavelength infrared part of the free massless one.

Furthermore, one can find the corresponding analog to the Yang-Feldman equation:

$$\phi(t,\vec{x}) = \tilde{\phi}(t,\vec{x}) - \frac{\lambda}{3H} e^{-\frac{m^2 t}{3H}} \int\limits_0^t dt' \, e^{\frac{m^2 t'}{3H}} \phi^3(t',\vec{x}) \, ;$$

The iterated Yang-Feldman-type equation can be written out up to a few first terms as

$$\phi(t,\vec{x}) = \tilde{\phi}(t,\vec{x}) - \frac{\lambda}{3H} e^{-\frac{m^2 t}{3H}} \int\limits_0^t dt' \, e^{\frac{m^2 t'}{3H}} \tilde{\phi}^3(t',\vec{x}) + \frac{\lambda^2}{3H^2} e^{-\frac{m^2 t}{3H}} \int\limits_0^t dt' \, \tilde{\phi}^2(t',\vec{x}) \int\limits_0^{t'} dt'' e^{\frac{m^2 t''}{3H}} \tilde{\phi}^3(t'',\vec{x}) + \ldots$$

One can use it to calculate the correlation functions of the massive field $\phi(t,\vec{x})$ through the known correlation function of the free massless one $\phi_0(t,\vec{x})$.

Two-point correlation function for the free massive scalar field and a loop series

Let us calculate the two-point correlation function for the free massive field $\tilde{\phi}(t, \vec{x})$, where the spatial' spacetime points coincide while the time moments are different:

$$\left\langle \tilde{\phi}(t_1, \vec{x})\, \tilde{\phi}(t_2, \vec{x}) \right\rangle \equiv \left\langle \tilde{\phi}(t_1)\, \tilde{\phi}(t_2) \right\rangle = \frac{3H^4}{8\pi^2 m^2} \left(e^{-\frac{m^2}{3H}|t_1 - t_2|} - e^{-\frac{m^2}{3H}(t_1 + t_2)} \right);$$

where we have used the well-known long-wavelength infrared part of the free massless field

$$\left\langle \phi_0(t_1)\, \phi_0(t_2) \right\rangle = \frac{H^3 t_2}{4\pi^2}, \quad t_2 \leq t_1$$

(Vilenkin & Ford, Phys.Rev.D, 1982; Linde, Phys.Lett.B, 1982).

The obtained $\left\langle \tilde{\phi}(t_1)\, \tilde{\phi}(t_2) \right\rangle$ exactly coincides with Ornstein-Uhlenbeck stochastic process's one. The notable feature is its drift towards average with a mean reversion rate of $m^2/3H$.

In our development, the correlation functions have a smooth massless limit, coinciding with the expressions obtained for a massless scalar field. Such a reasoning can be considered as a theory of a massive scalar field with the vacuum «inherited» from the massless one.

Building on this correlator, one reaches the loop series through our Yang-Feldman equation

$$\begin{aligned} \left\langle \phi(t_1)\, \phi(t_2) \right\rangle = \left\langle \tilde{\phi}(t_1)\, \tilde{\phi}(t_2) \right\rangle + \left\langle \phi(t_1)\, \phi(t_2) \right\rangle_{1-\text{loop}} + \left\langle \phi(t_1)\, \phi(t_2) \right\rangle_{2-\text{loop}} \\ + \left\langle \phi(t_1)\, \phi(t_2) \right\rangle_{3-\text{loop}} + O\left(\lambda^4\right). \end{aligned}$$

Full results up to three loops can be seen in our preprint on arXiv:[2410.16226].

Four-point correlation function

In the case of the four-point correlation function, one can continue in the same calculation manner. Even so, we already have the answer for the tree level

$$\left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{\text{tree}} = \left\langle \tilde{\phi}\left(t_1\right)\tilde{\phi}\left(t_2\right)\right\rangle\left\langle \tilde{\phi}\left(t_3\right)\tilde{\phi}\left(t_4\right)\right\rangle + \text{perm.}$$

Afterwards, for the linear order in λ, we partially also have the answer, since in this case, the complete correlation function is splitted into connected

$$\left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{1-\text{loop}}^{\text{connected}} = -\frac{2\lambda}{H}\mathrm{e}^{-\frac{m^2 t_1}{3H}}\int_0^{t_1} dt'\mathrm{e}^{\frac{m^2 t'}{3H}}\left\langle \tilde{\phi}\left(t'\right)\tilde{\phi}\left(t_2\right)\right\rangle\times$$
$$\times\left\langle \tilde{\phi}\left(t'\right)\tilde{\phi}\left(t_3\right)\right\rangle\left\langle \tilde{\phi}\left(t'\right)\tilde{\phi}\left(t_4\right)\right\rangle \quad + \text{perm.};$$

and disconnected (just a combination of two-point functions) diagram' types

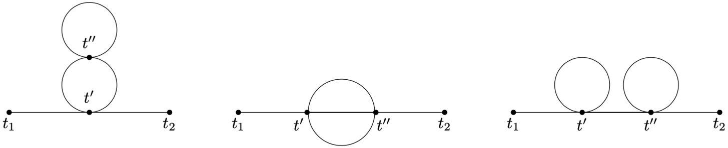
$$\left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{1-\text{loop}}^{\text{disconnected}} = \left\langle \tilde{\phi}\left(t_1\right)\tilde{\phi}\left(t_2\right)\right\rangle\left\langle \phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{1-\text{loop}} + \text{perm.}$$

We proceed up to

$$\left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle = \left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{\text{tree}}$$
$$+\left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{1-\text{loop}} + \left\langle \phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right\rangle_{2-\text{loop}} + O\left(\lambda^3\right).$$

Correspondence between the Yang-Feldman and Schwinger-Keldysh formalisms

All the possible two-loop-level diagrams' structures are the following:



and all integral structures for the two-point function in the Yang-Feldman-type equation:

$$\mathcal{I}_{1,a} = \frac{2\lambda^2}{H^2} e^{-\frac{m^2 t_1}{3H}} \int_0^{t_1} dt' \int_0^{t'} dt'' e^{\frac{m^2 t''}{3H}} \langle \tilde{\phi}(t') \tilde{\phi}(t_2) \rangle \langle \tilde{\phi}(t') \tilde{\phi}(t'') \rangle \langle \tilde{\phi}^2(t'') \rangle;$$

$$\mathcal{I}_{2,a} = \frac{2\lambda^2}{H^2} e^{-\frac{m^2 t_1}{3H}} \int_0^{t_1} dt' \int_0^{t'} dt'' e^{\frac{m^2 t''}{3H}} \langle \tilde{\phi}(t'') \tilde{\phi}(t_2) \rangle \left(\langle \tilde{\phi}(t') \tilde{\phi}(t'') \rangle \right)^2;$$

$$\mathcal{I}_{3,a} = \frac{\lambda^2}{H^2} e^{-\frac{m^2 t_1}{3H}} \int_0^{t_1} dt' \int_0^{t'} dt'' e^{\frac{m^2 t''}{3H}} \langle \tilde{\phi}(t'') \tilde{\phi}(t_2) \rangle \langle \tilde{\phi}^2(t') \rangle \langle \tilde{\phi}^2(t'') \rangle;$$

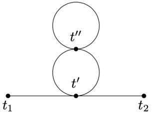
$$\mathcal{I}_{i,\mathbf{b}} = \mathcal{I}_{i,a} \text{ with } (t_1 \leftrightarrow t_2) \quad (i = 1, 2, 3)$$

$$\mathcal{I}_4 = \frac{2\lambda^2}{3H^2} e^{-\frac{m^2}{3H}(t_1+t_2)} \int_0^{t_1} dt' e^{\frac{m^2 t'}{3H}} \int_0^{t_2} dt'' e^{\frac{m^2 t''}{3H}} \left(\langle \tilde{\phi}(t') \tilde{\phi}(t'') \rangle \right)^3;$$

$$\mathcal{I}_5 = \frac{\lambda^2}{H^2} e^{-\frac{m^2}{3H}(t_1+t_2)} \int_0^{t_1} dt' e^{\frac{m^2 t'}{3H}} \int_0^{t_2} dt'' e^{\frac{m^2 t''}{3H}} \langle \tilde{\phi}^2(t') \rangle \langle \tilde{\phi}(t') \tilde{\phi}(t'') \rangle \langle \tilde{\phi}^2(t'') \rangle;$$

Our correspondence hypothesis is based on the following assignment assumption:

In order to obtain one of the diagrams' topologies, the points might be connected either by an explicit correlation function in integral structures or by integration variables' limits.



$$\mathcal{I}_{1,a} = \frac{2\lambda^2}{H^2} e^{-\frac{m^2 t_1}{3H}} \int_0^{t_1} dt' \int_0^{t'} dt'' e^{\frac{m^2 t''}{3H}} \langle \tilde{\phi}(t')\, \tilde{\phi}(t_2) \rangle \langle \tilde{\phi}(t')\, \tilde{\phi}(t'') \rangle \langle \tilde{\phi}^2(t'') \rangle$$

$$\mathcal{I}_{1,b} = \mathcal{I}_{1,a} \quad \text{with} \quad (t_1 \leftrightarrow t_2)$$

$$\langle \phi(t_1)\, \phi(t_2) \rangle^{\text{Snowman}} = \mathcal{I}_{1,a} + \mathcal{I}_{1,b} \xrightarrow[\text{times}]{\text{late}} \frac{243\lambda^2 H^{12}}{512\,\pi^6 m^{10}} \left(1 + \frac{m^2}{3H}|t_1 - t_2|\right) e^{-\frac{m^2}{3H}|t_1 - t_2|};$$

$$\langle \phi(t_1)\, \phi(t_2) \rangle^{\text{Sunset}} = \mathcal{I}_{2,a} + \mathcal{I}_{2,b} + \mathcal{I}_4$$

$$\xrightarrow[\text{times}]{\text{late}} \frac{243\lambda^2 H^{12}}{1024\,\pi^6 m^{10}} \left(\left(1 + \frac{2m^2}{3H}|t_1 - t_2|\right) e^{-\frac{m^2}{3H}|t_1 - t_2|} + \frac{1}{3} e^{-\frac{m^2}{H}|t_1 - t_2|} \right);$$

$$\langle \phi(t_1)\, \phi(t_2) \rangle^{\text{Ind. Loops}} = \mathcal{I}_{3,a} + \mathcal{I}_{3,b} + \mathcal{I}_5$$

$$\xrightarrow[\text{times}]{\text{late}} \frac{243\lambda^2 H^{12}}{1024\,\pi^6 m^{10}} \left(2 + \frac{2m^2}{3H}|t_1 - t_2| + \frac{m^4}{9H^2}|t_1 - t_2|^2\right) e^{-\frac{m^2}{3H}|t_1 - t_2|};$$

These expressions at late times coincide with those obtained directly via Schwinger-Keldysh (Gautier, Serreau, Phys. Lett. B, 2013; Kamenshchik, Starobinsky, Vardanyan, Eur.Phys.J. C, 2022).

Comparison with the stochastic approach

Starobinsky stochastic approach (Starobinsky 1986; Starobinsky & Yokoyama, Phys.Rev.D, 1994) matches the l-w part of the quantum field $\phi(t,\vec{x})$ to the classical stochastic field $\varphi(t,\vec{x})$ with a probability distribution function $\rho\big[\varphi(t,\vec{x})\big]$ that satisfies the Fokker-Planck equation

$$\partial_t\,\rho\big[\varphi(t,\vec{x})\big] = \frac{1}{3H}\,\partial_\varphi\Big(\rho\big[\varphi(t,\vec{x})\big]\,V'_\varphi\big(\varphi(t,\vec{x})\big)\Big) + \frac{H^3}{8\pi^2}\,\partial^2_\varphi\Big(\rho\big[\varphi(t,\vec{x})\big]\Big).$$

Any solution of this Fokker-Planck equation tends to the static solution at late times

$$\rho\big[\varphi(t,\vec{x})\big] \quad \xrightarrow[\text{times}]{\text{late}} \quad \rho_{\mathbf{st}}\,[\varphi] = \frac{1}{\mathcal{N}}\,\mathrm{e}^{-\frac{8\pi^2}{3H^4}V(\varphi)}.$$

Therefore, the expectation values of this stochastic variable at small λ expansion is

$$\big\langle\varphi^{\,2\mathrm{n}}\big\rangle = \frac{\int\limits_{-\infty}^{+\infty} d\varphi\,\varphi^{\,2\mathrm{n}}\rho_{\mathbf{st}}\,[\varphi]}{\int\limits_{-\infty}^{+\infty} d\varphi\,\rho_{\mathbf{st}}\,[\varphi]} \to \begin{cases} \dfrac{3H^4}{8\pi^2 m^2} - \dfrac{27\lambda H^8}{64\,\pi^4 m^6} + \dfrac{81\lambda^2 H^{12}}{64\,\pi^6 m^{10}} - \dfrac{24057\lambda^3 H^{16}}{4096\,\pi^8 m^{14}} + O\left(\lambda^4\right); \\[2ex] \dfrac{27H^8}{64\,\pi^4 m^4} - \dfrac{81\lambda H^{12}}{64\,\pi^6 m^8} + \dfrac{24057\lambda^2 H^{16}}{4096\,\pi^8 m^{12}} + O\left(\lambda^3\right). \end{cases}$$

Since our obtained massive correlation function coincides with Ornstein-Uhlenbeck's mean-reverting stochastic process, our outcomes, as they must be, are in agreement at late times with Starobinsky's stochastic approach, which operates with a near-equilibrium state.

One can also reduce these expressions to modified Bessel functions of the 2nd kind $\mathcal{K}_\nu(z)$. As it was shown (Kamenshchik, Starobinsky, Vardanyan, Eur.Phys.J. C, 2022),

$$\int\limits_{-\infty}^{+\infty} d\varphi\,\rho_{\mathbf{st}}\,[\varphi] = 1 \;\; \to \;\; \mathcal{N} = \int\limits_{-\infty}^{+\infty} d\varphi\,\mathrm{e}^{-\frac{8\pi^2}{3H^4}V(\varphi)} = \frac{m}{\sqrt{2\lambda}}\exp\left(\frac{\pi^2 m^4}{3\lambda H^4}\right)\mathcal{K}_{1/4}\left(\frac{\pi^2 m^4}{3\lambda H^4}\right).$$

Comparison with the stochastic approach

One can notice that

$$\left\langle \varphi^2 \right\rangle = -\frac{3H^4}{4\pi^2}\frac{1}{\mathcal{N}}\frac{d\mathcal{N}}{dm^2} \quad \Rightarrow \quad \left\langle \varphi^2 \right\rangle = \frac{m^2}{2\lambda}\left(\frac{\mathcal{K}_{3/4}(z)}{\mathcal{K}_{1/4}(z)} - 1\right), \quad \text{where} \quad z \equiv \frac{\pi^2 m^4}{3\lambda H^4}.$$

One can also proceed further and get the expression anew for $\left\langle \varphi^4 \right\rangle$, such as

$$\left\langle \varphi^4 \right\rangle = \left(-\frac{3H^4}{4\pi^2}\right)^2 \frac{1}{\mathcal{N}}\frac{d^2\mathcal{N}}{d\left(m^2\right)^2} \quad \Rightarrow \quad \left\langle \varphi^4 \right\rangle = \frac{3H^4}{8\pi^2\lambda} + \frac{m^4}{2\lambda^2}\left(1 - \frac{K_{3/4}(z)}{K_{1/4}(z)}\right).$$

Let us point out the general structure of any 2n'th expectation value

$$\frac{d^{\mathrm{n}}\mathcal{N}}{d\left(m^2\right)^{\mathrm{n}}} = \alpha_{\mathrm{n}}\left(m^2\right) \mathrm{e}^z K_{1/4}(z) + \beta_{\mathrm{n}}\left(m^2\right) \mathrm{e}^z K_{3/4}(z),$$

$$\text{since} \qquad \frac{d}{dz}\mathcal{K}_\nu(z) = -\frac{\nu}{z}\mathcal{K}_\nu(z) - \mathcal{K}_{\nu-1}(z) \quad \text{and} \quad \mathcal{K}_{-\nu}(z) = \mathcal{K}_\nu(z),$$

where from the definition of $\mathcal{N}$, $\alpha_0 = m/\sqrt{2\lambda}$ and $\beta_0 = 0$, and the recurrence relations are

$$\alpha_{\mathrm{n}+1} = \frac{d\alpha_{\mathrm{n}}}{dm^2} - \frac{\alpha_{\mathrm{n}}}{2m^2} + \frac{2\pi^2 m^2}{3\lambda H^4}\left(\alpha_{\mathrm{n}} - \beta_{\mathrm{n}}\right); \quad \beta_{\mathrm{n}+1} = \frac{d\beta_{\mathrm{n}}}{dm^2} - \frac{3\beta_{\mathrm{n}}}{2m^2} - \frac{2\pi^2 m^2}{3\lambda H^4}\left(\alpha_{\mathrm{n}} - \beta_{\mathrm{n}}\right),$$

resulting in

$$\left\langle \varphi^{2\mathrm{n}} \right\rangle = \left(-\frac{3H^4}{4\pi^2}\right)^{\mathrm{n}} \frac{1}{\mathcal{N}}\frac{d^{\mathrm{n}}\mathcal{N}}{d\left(m^2\right)^{\mathrm{n}}} \quad \Rightarrow \quad \left\langle \varphi^{2\mathrm{n}} \right\rangle = \frac{\sqrt{2\lambda}}{m}\left(-\frac{3H^4}{4\pi^2}\right)^{\mathrm{n}}\left(\alpha_{\mathrm{n}} + \beta_{\mathrm{n}}\frac{K_{3/4}(z)}{K_{1/4}(z)}\right).$$

At large $z \gg 1$, this expansion reproduces our results for $\left\langle \phi^2(t) \right\rangle$ and $\left\langle \phi^4(t) \right\rangle$ at $t \to \infty$.

Comparison with the Hartree-Fock approximation

To compare our outcomes for $\left\langle \phi^2 (t) \right\rangle$ to those obtained in the Hartree-Fock (Gaussian) approximation (HF), let us consider the Klein-Gordon equation.
After some manipulations with the use of HF, namely $\left\langle \phi^4 (t) \right\rangle = 3\left\langle \phi^2 (t) \right\rangle^2$, one has

$$\frac{d\left\langle \phi^2 \right\rangle}{dt} = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H}\left\langle \phi^2 \right\rangle - \frac{2\lambda}{H}\left\langle \phi^2 \right\rangle^2.$$

The solution to the equation above is

$$\left\langle \phi^2 (t) \right\rangle_{\mathrm{HF}} = \frac{\frac{3H^4}{4\pi^2 m^2}\left(1 - \exp\left(-\frac{2m^2 t}{3H}\sqrt{1 + \frac{9\lambda H^4}{2\pi^2 m^4}}\right)\right)}{1 + \sqrt{1 + \frac{9\lambda H^4}{2\pi^2 m^4}} - \left(1 - \sqrt{1 + \frac{9\lambda H^4}{2\pi^2 m^4}}\right)\exp\left(-\frac{2m^2 t}{3H}\sqrt{1 + \frac{9\lambda H^4}{2\pi^2 m^4}}\right)}.$$

Expanding this solution in a series along a small self-interaction coupling constant λ, one can find out that at the tree and at one-loop levels, HF gives the correct results, while at the already two-loop level, it does not.

Through our results for each of the two-loop diagrams, we conclude that the Hartree-Fock approximation only resummes the «Cactus» and «Double Seagull»-type diagrams, leaving aside the «Sunset» one.

To «correct» the Hartree-Fock approximation, we further construct an autonomous equation, which catches the absentee «Sunset» diagrammatic contribution.

An autonomous equation for the two-point correlation function

The main idea is to get through the known perturbative series an autonomous first-order differential equation (Kamenshchik & Vardanyan, Phys.Rev.D, 2020). The solution of such an equation is the non-analytic on λ, while providing the correct series up to the two-loop level.

Our obtained perturbative series is

$$\left\langle \phi^2(t) \right\rangle = \frac{3H^4}{8\pi^2 m^2}\left(1 - e^{-\frac{2m^2 t}{3H}}\right) - \frac{27\lambda H^8}{64\pi^4 m^6}\left(1 - \frac{4m^2 t}{3H} e^{-\frac{2m^2 t}{3H}} - e^{-\frac{4m^2 t}{3H}}\right)$$
$$+ \frac{81\lambda^2 H^{12}}{64\pi^6 m^{10}}\left(1 + \left(\frac{21}{8} - \frac{3m^2 t}{2H} - \frac{m^4 t^2}{3H^2}\right) e^{-\frac{2m^2 t}{3H}} - \left(3 + \frac{2m^2 t}{H}\right) e^{-\frac{4m^2 t}{3H}} - \frac{5}{8} e^{-\frac{2m^2 t}{H}}\right).$$

At the zero order in that series $\left\langle \phi^2(t) \right\rangle$ on small coupling constant λ, we have

$$f(t) = \frac{3H^4}{8\pi^2 m^2}\left(1 - e^{-\frac{2m^2 t}{3H}}\right) \quad \Rightarrow \quad e^{-\frac{2m^2 t}{3H}} = 1 - \frac{8\pi^2 m^2}{3H^4} f(t);$$

This tree-level expression above is a solution to the following autonomous equation:

$$\frac{d}{dt}\left(f(t)\right) = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H} f(t).$$

One can take the time derivatives from both its l.h.s. and r.h.s. and extract the exponent through $f(t)$ to establish that.

An autonomous equation for the two-point correlation function

To have an autonomous equation, whose solution has the correct expansion up to the terms linear in λ, we substitute $f(t)$ into l.h.s. and r.h.s. and find the correcting term

$$\frac{d}{dt}(f(t)) = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H} f(t) + \Delta f_1(t), \qquad \Delta f_1(t) = -\frac{9\lambda H^7}{32\,\pi^4 m^4}\left(1 - e^{-\frac{2m^2 t}{3H}}\right)^2.$$

and equation appears to be the Hartree-Fock's:

$$\frac{d}{dt}(f(t)) = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H} f(t) - \frac{2\lambda}{H}(f(t))^2,$$

which was able to provide the correct result up to λ.

At the two-loop level, we will come to

$$\frac{d}{dt}(f(t)) = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H} f(t) - \frac{2\lambda}{H}(f(t))^2 - \frac{27\lambda^2 H^{11}}{32\,\pi^6 m^8}\left(\frac{4\,\pi^2 m^2}{H^4} f(t) - \frac{16\,\pi^4 m^4}{H^8}(f(t))^2 + \frac{256\,\pi^6 m^6}{27 H^{12}}(f(t))^3 + \frac{3}{2}\left(1 - \frac{8\pi^2 m^2}{3H^4} f(t)\right)^2 \ln\left(1 - \frac{8\pi^2 m^2}{3H^4} f(t)\right)\right);$$

Taking the function $f(t)$ as

$$f(t) = \langle \phi^2(t) \rangle_{\text{HF}} + \delta f(t), \quad \delta f(t) \sim O(\lambda^2),$$

one can find a cumbersome though straightforward expression for $\delta f(t)$ through the linearized version of the equation above; see our arXiv: [2410.16226] for the full one.

An autonomous equation for the two-point correlation function

The limiting value of the obtained solution at $t \to \infty$ is

$$\delta f(t) \xrightarrow{t\to\infty} \frac{243\lambda^2 H^{12}}{32\pi^6 m^{10}} \left(-\frac{3\mathcal{Z}^2 - 3\mathcal{Z} - 2}{6\mathcal{Z}\left(\mathcal{Z}+1\right)^3} - \frac{\left(\mathcal{Z}-1\right)^2}{4\mathcal{Z}\left(\mathcal{Z}+1\right)^2} \ln\left(\frac{\mathcal{Z}-1}{\mathcal{Z}+1}\right)\right)$$

$$\approx \frac{81\lambda^2 H^{12}}{256\pi^6 m^{10}} + O\left(\lambda^3\right), \quad \text{where} \quad \mathcal{Z} \equiv \sqrt{1+\frac{3}{2z}} \quad \text{and} \quad z \equiv \frac{\pi^2 m^4}{3\lambda H^4}.$$

It precisely matches with an absent «Sunset» contribution in Hartree-Fock approximation.

The full non-analytical on λ result for a two-point correlation function at late times is

$$\left\langle \phi^2(t)\right\rangle^{\text{aut}} = \left\langle \phi^2(t)\right\rangle_{\text{HF}} + \delta f(t) \xrightarrow{t\to\infty}$$

$$\longrightarrow \frac{H^2}{\pi\sqrt{\lambda}} \left(\frac{\sqrt{12z+18}-\sqrt{12z}}{12} + \frac{\left(3\sqrt{4z^2+6z}-2z-9\right)\left(\sqrt{12z+18}-\sqrt{12z}\right)^3}{1728z\sqrt{4z^2+6z}} \right.$$

$$\left. + \frac{3\left(\sqrt{2z+3}-\sqrt{2z}\right)^4}{64z^2\sqrt{12z+18}} \ln\left(\frac{\sqrt{2z+3}+\sqrt{2z}}{\sqrt{2z+3}-\sqrt{2z}}\right)\right) \xrightarrow{m\to 0} \frac{7\sqrt{2}\,H^2}{24\,\pi\sqrt{\lambda}}.$$

Besides, it almost coincides with the Starobinsky's stochastic approach' result:

$$\frac{\left\langle\phi^2\right\rangle^{\text{aut}}_{t\to\infty}}{\left\langle\varphi^2\right\rangle_{\text{Stoch}}} \xrightarrow{z\to 0} \frac{7\pi}{6\sqrt{6}\,\Gamma^2\left(\frac{3}{4}\right)} \approx 0.9964 \quad \text{and} \quad \frac{\left\langle\phi^2\right\rangle^{\text{aut}}_{t\to\infty}}{\left\langle\varphi^2\right\rangle_{\text{Stoch}}} \xrightarrow{z\to\infty} 1$$

in the whole interval of a new dimensionless parameter $0 \le z \equiv \frac{\pi^2 m^4}{3\lambda H^4} < \infty$.

Conclusion

In contrast to the standard theory of a massive scalar field based on the de Sitter-invariant vacuum, we developed some reasoning that may not possess de Sitter invariance but results in a smooth massless limit of correlation functions in the long-wavelength approximation.

Through the Yang-Feldman-type equation, loop quantum corrections for the two-point and four-point correlation functions's infrared part have been calculated. The main «building block» of ourelaborated approach is the free massive field's two-point correlation function that coincides with the Ornstein-Uhlenbeck stochastic process's one.

Our outcomes are in agreement with the Schwinger-Keldysh results at late times and were also compared to Starobinsky's stochastic approach and the Hartree-Fock approximation.

At last, we constructed an autonomous equation for the two-point function. Integrating its approximate version, one obtains a non-analytic expression with respect to a self-interaction coupling constant λ that reproduces the correct perturbative series up to the two-loop level.

Outlook

Physically, the more interesting case is when space points do not coincide, i.e., the case of non-zero modes. Is our approach convenient to catch the leading infrared logarithm there?

In principle, one can use our approach to go beyond de Sitter spacetime. For the power-law inflation, $a(t) \sim t^s$, in the Friedmann background (Lucchin & Matarrese, Phys. Rev. D, 1985), one finds the analog to our Yang-Feldman-type equation:

$$\phi(t, \vec{x}) = \tilde{\phi}(t, \vec{x}) - \frac{\lambda}{3s-1} e^{-\frac{m^2 t}{3s-1}} \int_0^t dt' \, e^{\frac{m^2 t'}{3s-1}} \tilde{\phi}^3(t', \vec{x}) + \ldots$$

It would be also inquiring to bridge direct calculations and more novel techniques and ideas, such as the wave-functional approach, cosmological polytopes, intersection theory, etc.

Thank you for your attention!

Backup

$$\phi(t,\vec{x}) = \phi_0(t,\vec{x}) + \int_0^t dt'\, F\big(\phi(t',\vec{x})\big),$$

where $F\big(\phi(t',\vec{x})\big)$ has the form $F\big(\phi(t',\vec{x})\big) = \alpha\phi(t',\vec{x}) + W\big(\phi(t',\vec{x})\big)$ and $\phi_0(t',\vec{x})$ is a given function. Introducing the new scalar field $\tilde{\phi}(t,\vec{x})$, which satisfies the equation

$$\tilde{\phi}(t,\vec{x}) = \phi_0(t,\vec{x}) + \alpha\int_0^t dt'\, \tilde{\phi}(t',\vec{x}),$$

and taking the time derivative, we can solve exactly an ordinary inhomogeneous first-order equation

$$\tilde{\phi}(t,\vec{x}) = \phi_0(t,\vec{x}) + \alpha e^{\alpha t}\int_0^t dt'\, e^{-\alpha t'}\phi_0(t',\vec{x}), \qquad \phi_0(t,\vec{x}) = \tilde{\phi}(t,\vec{x}) - \alpha\int_0^t \tilde{\phi}(t',\vec{x}).$$

Substituting this expression

$$\phi(t,\vec{x}) = \tilde{\phi}(t,\vec{x}) + \alpha\int_0^t dt'\, \Big(\phi(t',\vec{x}) - \tilde{\phi}(t',\vec{x})\Big) + \int_0^t dt'\, W\big(\phi(t',\vec{x})\big),$$

and hereafter introduce a new equation

$$\phi(t,\vec{x}) = \tilde{\phi}(t,\vec{x}) + e^{\alpha t}\int_0^t dt'\, e^{-\alpha t'} W\big(\phi(t',\vec{x})\big),$$

which is equivalent to equation above. To establish this, one can express $\phi(t,\vec{x}) - \tilde{\phi}(t,\vec{x})$, to changing the order of integration and one can show this equation reduces to the previous one.

$$\left\langle \phi(t_1)\,\phi(t_2)\right\rangle_{1-\text{loop}} = -\frac{27\lambda H^8}{128\,\pi^4 m^6}\Bigg(2\,e^{-\frac{m^2}{3H}|t_1-t_2|} + \frac{4m^2}{3H}\Big(|t_1-t_2| - (t_1+t_2)\Big)\,e^{-\frac{m^2}{3H}(t_1+t_2)}$$
$$- e^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)} + \frac{2m^2}{3H}\,|t_1-t_2|\left(e^{-\frac{m^2}{3H}|t_1-t_2|} - e^{-\frac{m^2}{3H}(t_1+t_2)}\right)$$
$$- e^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)} + e^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)} - e^{-\frac{m^2}{3H}(t_1+t_2)}\Bigg);$$

$$\left\langle \phi(t_1)\,\phi(t_2)\right\rangle_{2-\text{loop}} = \frac{81\lambda^2 H^{12}}{2048\,\pi^6 m^{10}}\Bigg(\left(30 + \frac{12m^2}{H}\,|t_1-t_2| + \frac{2m^4}{3H^2}\,|t_1-t_2|^2\right)e^{-\frac{m^2}{3H}|t_1-t_2|}$$
$$+ 2e^{-\frac{m^2}{H}|t_1-t_2|} - 5\,e^{-\frac{m^2}{H}(t_1+t_2)} + \left(36 + \frac{2m^2}{H}\Big(9\,|t_1-t_2| - 14\,(t_1+t_2)\Big)\right.$$
$$\left. - \frac{2m^4}{3H^2}\Big(|t_1-t_2| - 2\,(t_1+t_2)\Big)^2\right)e^{-\frac{m^2}{3H}(t_1+t_2)} + \frac{15}{2}e^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)}$$
$$+ \left(48 + \frac{2m^2}{H}\Big(7\,|t_1-t_2| + 2\,(t_1+t_2)\Big)\right)e^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)}$$
$$- \left(45 + \frac{2m^2}{H}\Big(|t_1-t_2| + 8\,(t_1+t_2)\Big)\right)e^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)} - \frac{15}{2}e^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)}$$
$$- \left(\frac{117}{2} - \frac{2m^2}{H}\Big(7\,|t_1-t_2| - 8\,(t_1+t_2)\Big)\right)e^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)} - \frac{15}{2}e^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)}\Bigg);$$