Simplicity of in-in Correlators

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and

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Simplicity of Scattering Amplitudes

1. Scattering is one of the simplest process one can study in QFT and QG
2. To do this, we compute Scattering amplitudes
3. These are directly related to physical observables that are measurable in experiments.
4. These are typically evaluated for (time-ordered) correlation functions along with an LSZ prescription.

1. For convenience they are usually evaluated in momentum space to exploit the translational invariance of flat space.

2. Final answer is obtained by summing over Feynman diagrams in perturbation theory. Quickly becomes very complicated, has spurious poles in intermediate steps, etc..

3. Many times, the final answer is much simpler after all the algebra!
   Early Examples include: De Witt (1967), Parke-Taylor Formula (1985).

4. The simplicity of amplitudes has been explored very heavily from early 2000's and led to the Amplitudes program.

Moving Forward

1. How much simplicity for observables extend beyond scattering amplitudes?

2. Today: We focus on correlators without time-translational invariance,

$$\langle\psi|O(\vec{k}_1)\cdots O(\vec{k}_n)|\psi\rangle$$

   where O's are operators at $t=0$. Often known as in-in correlators.

3. While we mostly work with theories in Flat space, they are often related to theories in (A)dS via a Weyl transform.
   – Will not discuss examples with IR divergences in this talk.

$$\langle O_1 \cdots O_n \rangle = \langle \psi | O(\vec{k}_1) \cdots O(\vec{k}_n) | \psi \rangle$$

Interesting for two reasons

1. $|\psi\rangle$ (wave function of the universe) are related to AdS correlators via analytic continuation.
2. The correlator itself are called cosmological (or in-in) correlators.

Summary:

1. We find a novel integral representation for these correlator in terms of the massive flat space S-matrix.
2. Show that (for some examples) Loops for in-in correlator have lower transcendality than ψ.

Going away from Amplitudes

1. Since we have translational invariance along spatial directions we work in momentum space. Therefore $x, y, z \in (-\infty, \infty)$ but $t \in (-\infty, 0)$.

2. Correlation functions can be computed from ψ via

$$\langle \phi_1 \cdots \phi_n \rangle = \int D\phi \, |\Psi[\phi]|^2 \, \phi_1 \cdots \phi_n$$

where $\phi_n = \phi(t = 0, \vec{k}_n)$. At tree level Ψ and $\langle \phi \cdots \phi \rangle$ very similar.

3. The ground state Wave function is obtained via [Hartle-Hawking]

$$\Psi[\varphi(\vec{x})] = \int_{\phi(-\infty)=0}^{\phi(0,\vec{x})=\varphi(\vec{x})} D\phi \; e^{iS[\phi]} = e^{iS_{on-shell}[\varphi]} \; .$$

– By analytical continuation, these are equivalent to computing correlators in AdS [Maldacena]

Harmonic Oscillator

1. These Wave functions also satisfy Schrodinger Equation

$$H\Psi = 0$$

2. For example: Ground state wave function for Harmonic Oscillator

$$H = \frac{d^2}{dx^2} + \omega^2 x^2 \implies \psi(x) = e^{-\frac{1}{2}\omega^2 x^2}$$

3. Similarly for a free scalar field you integrate over all oscillator modes [Hatfiled]

$$H = \int d^3k \frac{\partial^2}{\partial\varphi_{\vec{k}}\,\partial\varphi_{-\vec{k}}} + \omega_k^2 \varphi_{\vec{k}}\varphi_{-\vec{k}} \implies \Psi[\varphi] = e^{-\frac{1}{2}\int d^3k\ \omega_k^2 \varphi_{\vec{k}}\varphi_{-\vec{k}}}$$

1. We perturbatively evaluate the higher order corrections

2. Generic structure is of the form:

$$\Psi[\varphi] \sim \exp\Big[-\int d^3k_1 d^3k_2\, \psi_2(\vec{k}_1,\vec{k}_2)\, \varphi(\vec{k}_1)\varphi(\vec{k}_2) + \int d^3k_1 \cdots d^3k_4\, \psi_4(\vec{k}_1,\cdots,\vec{k}_4)\, \varphi(\vec{k}_1)\cdots\varphi(\vec{k}_4) + \cdots \Big]$$

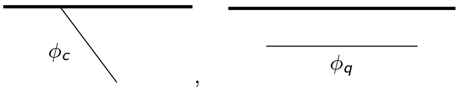
   - $\psi_n(\vec{k}_1,\cdots,\vec{k}_n)$ are called Wave function coefficients.
   - For the free field case: $\psi_2 = \omega_k$; $\psi_4, \psi_6, \cdots = 0$

3. Described as Old Fashioned Perturbation Theory (non-covariant)

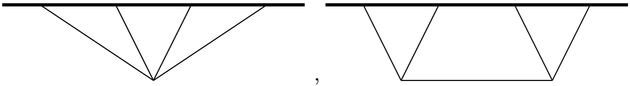
Witten Diagrams

1. Perturbation theory can be expressed in terms of Witten Diagrams

2. No time-translation invariance



Bulk-Boundary & Bulk-Bulk Propagators.
Example: for ψ_4 in ϕ^4 and ϕ^3 theory:



3. Momentum conservation along spatial directions only.

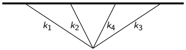
Example of a Witten diagram

1. Propagators are (Notation: $k \equiv |\vec{k}|$)

$$\phi_c(t;k) = e^{ikt},$$
$$G(t,t';k) = \frac{1}{2k}\Big[\underbrace{\theta(t-t')e^{ik(t-t')} + \theta(t'-t)e^{ik(t'-t)}}_{Feynman} - \underbrace{e^{ik(t+t')}}_{B.C}\Big]$$

   Satisfies Dirichlet boundary conditions: $G(0,t') = 0$. Not translational inv.

2. Example: Contribution to ψ_4 for ϕ^4,

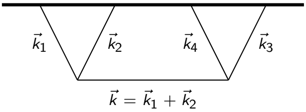


$$= \int_{-\infty}^{0} dt e^{i(k_1+k_2+k_3+k_4)t} \int d^3x e^{i\sum_i \vec{k}_i\cdot\vec{x}} = \frac{\delta(\vec{k}_1 + \cdots \vec{k}_4)}{k_1+k_2+k_3+k_4}$$

– Hence instead of getting $\delta(k_t)$ we get poles!

ϕ^3 theory

Consider an exchange diagram in ϕ^3 theory: $\left(k = |\vec{k}_1 + \vec{k}_2| \text{ and } k_{ij} = |\vec{k}_i| + |\vec{k}_j|\right)$



$$= \int_{-\infty}^{0} dt_1 dt_2 e^{i(k_1+k_2)t_1} e^{i(k_3+k_4)t_2} G(t_1, t_2, \vec{k}_1 + \vec{k}_2)$$

$$= \int_{-\infty}^{0} \frac{dt_1 dt_2 e^{ik_{12}t_1} e^{ik_{34}t_2}}{2k} \left[\Theta(t_1 - t_2)e^{ik(t_1-t_2)} + \Theta(t_2 - t_1)e^{ik(t_2-t_1)} - e^{ik(t_1+t_2)}\right]$$

$$= \frac{1}{2k}\left[\frac{1}{(k + k_{12})(k_{12} + k_{34})} + \frac{1}{(k + k_{34})(k_{12} + k_{34})} - \frac{1}{(k_{12} + k)(k_{34} + k)}\right]$$

$$= \frac{1}{(k_{12} + k_{34})(k + k_{12})(k + k_{34})},$$

Cancellation of spurious poles. Simple answer! Recursive formulas via IBP [Arkani-Hamed, Benincasa, Postnikov] .

Singularities: Flat Space Limit

1. Singularities of Scattering amplitudes typically contain physical information.
   — Do poles of correlator also contain physical info?

2. No poles for physical momenta. Singularities exist after analytically continuing the momenta.

3. One important singularity: Flat Space Limit [Maldacena-Pimentel; Raju]

4. Translational invariance exists in 3-directions, not in 4-th.
   – Flat space limit Restores translational invariance along 4-th direction,

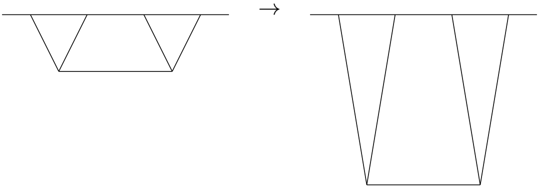
$$\int_{-\infty}^{0} dt e^{iEt} \longrightarrow \int_{-\infty}^{\infty} dt e^{iEt}$$

$$\implies \frac{1}{E} \longrightarrow \delta(E)$$

Flat Space limit

1. For example: Residue at $\frac{1}{E}$ $\rightarrow$ Flat space Scattering amplitude

$\text{Res}_{k_{12}+k_{34}=0}$



$$\implies \text{Res}_{k_{12}+k_{34}=0} \frac{1}{(k_{12}+k_{34})(k_{12}+k)(k_{34}+k)} = \frac{1}{(\underbrace{\vec{k}_1+\vec{k}_2}_{\vec{k}})^2 - (\underbrace{|\vec{k}_1|+|\vec{k}_2|}_{k_{12}})^2} = \frac{1}{s}$$

2. Wave function contains the Scattering amplitude!

Correlators from Wave Function

1. Correlation functions can be computed via

$$\langle \phi_1 \cdots \phi_n \rangle = \int D\phi \, |\Psi[\phi]|^2 \, \phi_1 \cdots \phi_n \qquad (*)$$

2. Therefore the cosmological correlator is one more path integral away from the computation of the wave function. Hence it is expected to be more complicated.

3. But for conformally coupled scalar we find that it is simpler than the wave function at loop level, due to many non-trivial cancellations!
   – Not manifest from (*) .
   – SK formalism and AdS effective actions are better in practice [Weinberg; Sleight, Tarrona; di Pietro, Gorbenko, Komatsu] .

Loops

1. Loop Corrections are perhaps not measurable anytime soon. But they are of theoretical interest
2. For computing higher order corrections to AdS/CFT correlators
3. Of Mathematical interest: What kind of functions appear after integrations?
4. Several works on loop integrals and integrands: [all speakers in the conference! + Banados, Bianchi, Munoz, Skenderis; Heckelbacher, Sachs, Skvortsov, Vanhove; Senatore, Gorbenko; Pajer, Lee, Anninos, Melville, Jazayeri, Anous, Freedman, Konstantinidis, Mahajan, Shaghoulian, Benincasa, Pueyo, Brunello, Mandal, Mastrolia, Vazao; Seery, Starobinsky, Qin, Xianyu, Baumann, Pimentel, Joyce, Arkani-Hamed, etc.]
5. I will review some explicit examples loop integrals for Ψ and for the cosmological correlators.

Why is it difficult?

1. Scattering amplitudes in flat space are squares of momenta

$$\text{Bubble} = \int \frac{d^4 l}{l^2 (l+k)^2}$$

2. Ψ Loops are not squares!

$$\text{Generic Loop Integral for } \Psi = \int \frac{d^3 l}{|\vec{l} + \vec{k}| \times (|\vec{l}| + |\vec{k}| + |\vec{l} + \vec{k}'|) \times \cdots}$$

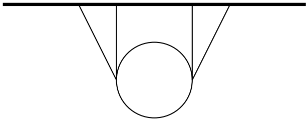
3. Not easy to combine denominators via Schwinger or Feynman parametrizations.

4. All examples explicitly computed till now in momentum space have axi-symmetry.

[See Benincasa, Brunello, Mandal, Mastrolia, Vazao for an analysis of the triangle diagram.]

Loops: Examples

1. Consider the 4-pt bubble diagram for the wave function [Albayrak, CC, Kharel; CC, Singh]



$$= \frac{1}{k}\left[\frac{\pi^2}{3} + \frac{4k\log 2}{k_{12}+k_{34}} + \log^2\left(\frac{k_{34}-k}{k+k_{12}}\right) + \log^2\left(\frac{k_{12}-k}{k+k_{34}}\right) - \log^2\left(\frac{k+k_{12}}{k+k_{34}}\right)\right.$$
$$\left. + 2\mathrm{Li}_2\frac{k+k_{34}}{k-k_{12}} + 2\mathrm{Li}_2\frac{k+k_{12}}{k-k_{34}} + \frac{4k}{k_{12}^2-k_{34}^2}\left(k_{34}\log\frac{k+k_{12}}{\Lambda} - k_{12}\log\frac{k+k_{34}}{\Lambda}\right)\right]$$

– This has higher transcendality (Li_2) than its scattering amplitude counterpart (log)! (independent of choice of regulator)

– Flat Space limit recovers expected structure as UV properties are unaffected.

in-in Correlator

Computing the in-in correlator at same order gives: [Lee; CC, Lipstein, Mei, Sachs, Vanhove]

1. Bubble in hard-cutoff

$$\frac{1}{(k_{12}+k_{34})}\left[\ln\left(\frac{(k_{12}+|\vec{k}_{12}|)(k_{34}+|\vec{k}_{12}|)}{4\Lambda^2}\right)+\frac{k_{12}+k_{34}}{k_{12}-k_{34}}\ln\left(\frac{k_{34}+|\vec{k}_{12}|}{k_{12}+|\vec{k}_{12}|}\right)\right]$$

   No Li_2 or $\log^2$, hence simpler than Ψ! Above answer does not satisfy CWI.
   — Analytic regularization preserves conformal invariances; same structure with $\Lambda \to k_{12}+k_{34}$. Gives ratios of momenta inside Logs.

2. Similar simplicity observed for necklace diagram at 2-loops, tadpoles, etc.

3. The simplicity in this case be traced back to the integrand.

1. In this simple example, the simplicty for the in-in correlator can be explained by the following structure of the integrand:

$$\psi : \int \frac{d^3 l}{(E+2l)(k_{12}+l+|\vec{l}+k|)(k_{34}+l+|\vec{l}+k|)},$$

$$\text{corr} : \int \frac{d^3 l}{l(k_{12}+l+|\vec{l}+k|)(k_{34}+l+|\vec{l}+k|)}$$

In the case of correlator, the pole at $l=0$ does not increase transcendentality because of the measure $d^3 l$. $(E = k_{12}+k_{34})$

2. For every example we have computed, the wave function always has the pole $\frac{1}{E+2l}$ whereas correlator never has this pole and instead has $\frac{1}{l}$.

3. Hints that the poles of a correlator has an overlap with the poles of S-matrix [Lee] .

4. Recent analysis via differential equations analysis might aid in studying these more explicitly [de la Cruz, Vanhove; Benincasa, Brunello, Mandal, Mastrolia, Vazao] .

Correlator[S-matrix]

1. All these hints led us to the following formula, [CC, Lipstein, Mei, Sachs, Vanhove]

$$\langle\phi_1(k_1)\cdots\phi_n(k_n)\rangle = \int_{-\infty}^{\infty}\prod_i dp_i\ R(p_i)\ S(p_1,k_1;\cdots,p_n,k_n)$$

– Suggests an "Inverse LSZ": correlator obtained from the S-matrix ! Similar in spirit to the in-out approach [Pajer, Donath]

2. Auxiliary propagators $R(p)$ are theory dependent. [(in progress)]
– For ϕ^4 theory they are simply [CC, Lipstein, Mei, Sachs, Vanhove]

$$R(p) = \frac{1}{p^2+k^2}$$

3. Example: the in-in bubble in this representation is

$$= \int_{-\infty}^{\infty}\frac{dp}{(p^2+k_{12}^2)(p^2+k_{34}^2)}\int\frac{d^4L}{L^2(L+K)^2}$$

where $L^\mu = (l+p,\vec{l})$, $K^\mu = (p,\vec{k})$.

Conclusion

1. Find a hint of simplification beyond amplitudes in flat space.
2. Many tools used for studying amplitudes can be generalized to Wave functions/Correlators: Differential equations, IBP, etc.
3. How general is the Inverse-LSZ like formula? Can one use it to study discontinuities/cuts?
4. Cosmological correlators have simpler poles than wave functions (at least at conformal coupling).
5. How does all of this generalize to cases with IR divergences?