Slow-roll inflation at N3LO (as deviations from a purely de Sitter spacetime)

Pierre AUCLAIR, Christophe Ringeval arxiv:2205.12608 (PRD)

Looping in the primordial Universe, CERN 2024

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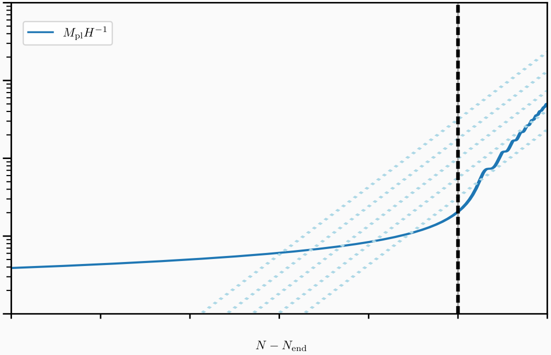
Motivation



Context: Slow-Roll inflation

- Slow-roll inflation: hundreds of scenarios are competing[1]
- To discriminate: we derive the expected scalar and tensor power-spectra perturbatively
  - Not perturbations in terms of h, ζ, these remain linear and independent
  - Not loop corrections
  - Deviations from a purely de Sitter background

Hubble flow functions

$$\epsilon_1(N) \equiv \frac{\mathrm{d}\ln H^{-1}}{\mathrm{d}N} = \mathcal{O}(\epsilon), \qquad \epsilon_{i+1}(N) \equiv \frac{\mathrm{d}\ln|\epsilon_i|}{\mathrm{d}N} = \mathcal{O}(\epsilon)\,.$$

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- Hubble flow function determined from the field's potential $V(\phi)$[2]
- Determined by the equation-of-state parameter in fluid representations of single-field inflation[3]

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History of computation of perturbations

- 2001: Second-order corrections for the scalar spectral index[4]
- 2001: Fully expanded scalar power spectrum at second order (N2LO)[5]
- 2002: Tensor power spectrum at N2LO was derived soon after[6]
- 2004: Third-order corrections to the scalar amplitude for a minimal kinetic term[7], but not fully expanded around a pivot wavenumber
- 2013: Scalar and tensor power spectra with a non-minimal kinetic term at N2LO[8]

In this work, we compute third-order corrections completely, including a non-minimal kinetic term to accommodate the next generation of CMB observations and the incoming large-scale structure surveys.

[4] Jin-Ook Gong and Stewart 2001.
[5] Schwarz, Terrero-Escalante, and Garcia 2001.
[6] Leach et al. 2002.
[7] Choe, Jinn-Ouk Gong, and Stewart 2004.
[8] Jérôme Martin, Ringeval, and Vennin 2013; Beltran Jimenez, Musso, and Ringeval 2013.

Outline

Gravitational Wave power-spectrum from inflation

Challenges when going at N3LO

Curvature perturbations from inflation

Summary

Equation of motion for gravitational waves

- $\mu(\eta, k) \equiv h_\lambda(\eta, k) a(\eta)$ verify, in Fourier space[9] (prime = differentiation wrt conformal time η)

$$\mu''(\eta, k) + \left(k^2 - \frac{a''}{a}\right) \mu(\eta, k) = 0,$$

- Parametric oscillator evolving in a time-dependent effective potential

$$U_{\mathrm{T}}(\eta) \equiv \frac{a''}{a} = \mathcal{H}^2(\eta) \left[2 - \epsilon_1(\eta)\right],$$

where $\mathcal{H}(\eta) = a(\eta) H(\eta)$ is the conformal Hubble parameter

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$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\eta} &= \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{a'}{a}\right) = \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = \frac{a''}{a} - \mathcal{H}^2 \\ &= \frac{\mathrm{d}a}{\mathrm{d}\eta}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}a} = a\mathcal{H}\left(H + a\frac{\mathrm{d}H}{\mathrm{d}a}\right) = \mathcal{H}^2(1 - \epsilon_1)\end{aligned}$$

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Perturbative expansion of the potential U_{T} (1/3)

Hubble flow functions are used to provide a perturbative expression for $\mathcal{H}$

$$-\eta = \int_{t(\eta)}^{0^-} \frac{\mathrm{d}t}{a(t)} = \int_{a(\eta)}^{+\infty} \frac{\mathrm{d}a}{a^2 H(a)}$$

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&= \frac{1}{\mathcal{H}} - \int_a^{+\infty} \frac{\epsilon_1}{a^2 H} \,\mathrm{d}a = \frac{1+\epsilon_1}{\mathcal{H}} + \int_a^{+\infty} \frac{1}{a} \frac{\mathrm{d}}{\mathrm{d}a} \left(\frac{\epsilon_1}{H}\right) \mathrm{d}a
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&= \frac{1}{\mathcal{H}} - \int_a^{+\infty} \frac{\epsilon_1}{a^2 H}\,\mathrm{d}a = \frac{1+\epsilon_1}{\mathcal{H}} + \int_a^{+\infty} \frac{1}{a} \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\epsilon_1}{H}\right)\mathrm{d}a \\
&= \frac{1+\epsilon_1+\epsilon_1^2+\epsilon_1\epsilon_2}{\mathcal{H}} + \int_a^{+\infty} \frac{1}{a} \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\epsilon_1^2+\epsilon_1\epsilon_2}{H}\right)\mathrm{d}a
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\end{aligned}
$$

Hence

$$
\eta\mathcal{H} = -\left(1+\epsilon_1+\epsilon_1^2+\epsilon_1\epsilon_2+\mathcal{O}\left(\epsilon^3\right)\right)
$$

Perturbative expansion of the potential U_{T} (2/3)

$$\mu''(\eta, k) + \left[k^2 - U_{\mathrm{T}}(\eta)\right] \mu(\eta, k) = 0,$$

- U_{T} can be expressed in terms of the Hubble flow functions ϵ_i

$$\eta^2 U_{\mathrm{T}}(\eta) = (\eta\mathcal{H})^2 (2 - \epsilon_1) = 2 + 3\epsilon_1 + 4\epsilon_1^2 + 4\epsilon_1\epsilon_2 + \mathcal{O}\left(\epsilon^3\right)$$

Notice that all $\epsilon_i(\eta)$ are still function of η!

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- derivatives of the ϵ_i

$$\frac{\mathrm{d}\epsilon_i}{\mathrm{d}\ln|\eta|} = \frac{\mathrm{d}N}{\mathrm{d}\ln|\eta|}\frac{\mathrm{d}\epsilon_i}{\mathrm{d}N} = \eta\mathcal{H}\epsilon_i\epsilon_{i+1} = -\epsilon_i\epsilon_{i+1}\left(1 + \epsilon_1 + \epsilon_1^2 + \epsilon_1\epsilon_2\right) + \mathcal{O}\left(\epsilon^5\right)$$

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... and Taylor expand

Perturbative expansion of the potential U_{T} (3/3)

- One can now pick a peculiar time, say $\eta_\flat$, and Taylor expand

$$\epsilon_i(\eta) = \epsilon_{i\flat} - \epsilon_{i\flat}\epsilon_{i+1\flat}\left(1 + \epsilon_{1\flat} + \epsilon_{1\flat}^2 + \epsilon_{1\flat}\epsilon_{2\flat}\right)\ln\left(\frac{\eta}{\eta_\flat}\right)$$
$$+ \frac{\epsilon_{i\flat}\epsilon_{i+1\flat}}{2}\left(\epsilon_{i+1\flat} + \epsilon_{i+2\flat} + \epsilon_{1\flat}\epsilon_{2\flat} + 2\epsilon_{1\flat}\epsilon_{i+1\flat} + 2\epsilon_{1\flat}\epsilon_{i+2\flat}\right)\ln^2\left(\frac{\eta}{\eta_\flat}\right) + \mathcal{O}(\epsilon^4)$$

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- Using the dimensionless positive variable $x \equiv -k\eta$, and a convenient pivot $\eta_\flat(k) = -\dfrac{1}{k}$

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}x^2} + \left[1 - \frac{U_{\mathrm{T}}(x)}{k^2}\right]\mu = 0$$

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$$\frac{\mathrm{d}^2\mu}{\mathrm{d}x^2} + \left(1 - \frac{2}{x^2}\right)\mu = \frac{g[\ln(x)]}{x^2}\mu$$

with

$$g[\ln(x)] = g_{1\flat} + g_{2\flat}\ln(x) + g_{3\flat}\ln^2(x) + \mathcal{O}\left(\epsilon^4\right) = \mathcal{O}(\epsilon),$$

Perturbative solution using Green's function (1/2)

- LHS = Riccati-Bessel equation

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- Advanced Green's function using the Wronskian method

$$G_y(x) = \frac{i}{2}\left[u(x)\overline{u}(y) - u(y)\overline{u}(x)\right]\Theta(y-x)\,,$$

where $u(x)$ is the Riccati-Hankel function of order one

$$u(x) \equiv \left(1 + \frac{i}{x}\right)e^{ix},$$

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- The exact solution therefore reads

$$\mu(x) = \mu_0(x) + \frac{i}{2}\int_x^\infty \frac{g[\ln(y)]}{y^2}\,\mu(y)\left[\overline{u}(y)u(x) - \overline{u}(x)u(y)\right]\mathrm{d}y$$

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Perturbative solution using Green's function (2/2)

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- Defining the rescaled mode function $\hat{\mu}(x) \equiv \sqrt{k}\mu(x)$, we expand it as

$$\hat{\mu}(x) = \hat{\mu}_0(x) + \hat{\mu}_1(x) + \hat{\mu}_2(x) + \hat{\mu}_3(x) + \mathcal{O}\big(\epsilon^4\big), \text{ with } \hat{\mu}_p = \mathcal{O}(\epsilon^p)$$

- The zeroth order term is fixed by imposing a Bunch-Davis vacuum for $x = -k\eta \to \infty$

$$\mu_0(x) = \frac{u(x)}{\sqrt{k}} \implies \hat{\mu}_0(x) = u(x)$$

- One obtains the recursive solutions

$$\hat{\mu}_1(x) = g_{1\flat}\int_x^\infty \frac{G_y(x)}{y^2}\hat{\mu}_0(y)\,\mathrm{d}y$$

$$\hat{\mu}_2(x) = g_{2\flat}\int_x^\infty \frac{G_y(x)\ln(y)}{y^2}\hat{\mu}_0(y) + g_{1\flat}\int_x^\infty \frac{G_y(x)}{y^2}\hat{\mu}_1(y)\,\mathrm{d}y$$

$$\hat{\mu}_3(x) = g_{3\flat}\int_x^\infty \frac{G_y(x)\ln^2(y)}{y^2}\hat{\mu}_0(y)\,\mathrm{d}y + g_{2\flat}\int_x^\infty \frac{G_y(x)\ln(y)}{y^2}\hat{\mu}_1(y)\,\mathrm{d}y + g_{1\flat}\int_x^\infty \frac{G_y(x)}{y^2}\hat{\mu}_2(y)\,\mathrm{d}y$$

Our perturbative expansion depends on the asymptotic expansion $x \to 0^+$

$$F_n(x) \equiv \int_x^\infty \frac{e^{+2iy}}{y} \ln^n(y)\,\mathrm{d}y\,, \text{for } n=0,\ n=1 \text{ and } n=2$$

These then enter into the definition of three other two-dimensional integrals

$$F_{00}(x) = \int_x^\infty \frac{e^{-2iy}}{y} F_0(y)\,\mathrm{d}y\,,$$
$$F_{01}(x) = \int_x^\infty \frac{e^{-2iy}}{y} F_1(y)\,\mathrm{d}y\,,$$
$$F_{10}(x) = \int_x^\infty \frac{e^{-2iy}}{y} \ln(y) F_0(y)\,\mathrm{d}y$$

Finally, the third-order terms involve the three-dimensional integral

$$F_{000}(x) = \int_x^\infty \frac{e^{+2iy}}{y} F_{00}(y)\,\mathrm{d}y$$

Similar to polylogs but with an oscillatory term, convergent at ∞ and divergent at 0^+

The $F_n(x)$ hierarchy (1/2)

- Let us define the generating functional

$$f(\nu, x) \equiv \sum_n \frac{\nu^n}{n!} F_n(x), \text{ hence } F_n(x) = \left.\frac{\partial^n f(\nu, x)}{\partial \nu^n}\right|_{\nu=0}$$

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- It is easy to find the form of the functional $f(\nu, x)$.

$$f(\nu, x) = \int_x^\infty \left(\sum_n \frac{\nu^n \ln^n u}{n!} \right) \frac{e^{2iu}}{u} \,\mathrm{d}u = \int_x^\infty u^{\nu-1} e^{2iu} \,\mathrm{d}u = x^\nu \mathrm{E}_{1-\nu}(-2ix)$$

in which $\mathrm{E}_{1-\nu}$ is the generalized exponential integral.

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in which $\mathrm{E}_{1-\nu}$ is the generalized exponential integral.

- In the limit $x \to 0^+$

$$f(\nu, x) \underset{x \to 0^+}{\sim} -\frac{x^\nu}{\nu} + 2^{-\nu} e^{i\pi\nu/2} \Gamma(\nu).$$

The $F_n(x)$ hierarchy (2/2)

From this expression, we obtain systematically the asymptotic behavior of all the functions F_n

$$F_0(x) = -\ln x - B + \mathcal{O}(x)$$

$$F_1(x) = -\frac{1}{2}\ln^2 x + \frac{B^2}{2} + \frac{\pi^2}{12} + \mathcal{O}(x)$$

$$F_2(x) = -\frac{1}{3}\ln^3 x - \frac{B^3}{3} - \frac{\pi^2}{6}B - \frac{2}{3}\zeta(3) + \mathcal{O}(x)$$

where

$$B \equiv \gamma_E + \ln(2) - \frac{i\pi}{2}$$

and with γ_E the Euler-Mascheroni constant

Gravitational wave power spectrum

The constancy of μ/a after Hubble exit allows us to derive the observable power spectrum of gravitational waves generated during inflation

$$\mathcal{P}_h(k) = \frac{2k^3}{\pi^2} \lim_{x\to 0^+} \left|\frac{\mu}{a}\right|^2,$$

$$\begin{aligned}
\mathcal{P}_h(k) = \frac{2H_*^2}{\pi^2}\Bigg\{ & 1 - 2(C+1)\epsilon_{1*} + \frac{1}{2}(\pi^2 + 4C^2 + 4C - 6)\epsilon_{1*}^2 + \frac{1}{12}(\pi^2 - 12C^2 - 24C - 24)\epsilon_{1*}\epsilon_{2*} \\
& - \frac{1}{3}\left[4C^3 + 3(\pi^2 - 8)C + 14\zeta(3) - 16\right]\epsilon_{1*}^3 + \frac{1}{12}\left[24C^3 + 13\pi^2 + 2(5\pi^2 - 36)C + 36C^2 - 96\right]\epsilon_{1*}^2\epsilon_{2*} \\
& - \frac{1}{12}\left[4C^3 - \pi^2 - (\pi^2 - 24)C + 12C^2 + 8\zeta(3) + 8\right](\epsilon_{1*}\epsilon_{2*}^2 + \epsilon_{1*}\epsilon_{2*}\epsilon_{3*}) \\
& + \Bigg[- 2\epsilon_{1*} + 2(2C+1)\epsilon_{1*}^2 - 2(C+1)\epsilon_{1*}\epsilon_{2*} - (\pi^2 + 4C^2 - 8)\epsilon_{1*}^3 + \frac{1}{6}(5\pi^2 + 36C^2 + 36C - 36)\epsilon_{1*}^2\epsilon_{2*} \\
& + \frac{1}{12}(\pi^2 - 12C^2 - 24C - 24)(\epsilon_{1*}\epsilon_{2*}^2 + \epsilon_{1*}\epsilon_{2*}\epsilon_{3*})\Bigg] \ln\left(\frac{k}{k_*}\right) \\
& + \left[2\epsilon_{1*}^2 - \epsilon_{1*}\epsilon_{2*} - 4C\epsilon_{1*}^3 + 3(2C+1)\epsilon_{1*}^2\epsilon_{2*} - (C+1)(\epsilon_{1*}\epsilon_{2*}^2 + \epsilon_{1*}\epsilon_{2*}\epsilon_{3*})\right] \ln^2\left(\frac{k}{k_*}\right) \\
& + \frac{1}{3}\left(- 4\epsilon_{1*}^3 + 6\epsilon_{1*}^2\epsilon_{2*} - \epsilon_{1*}\epsilon_{2*}^2 - \epsilon_{1*}\epsilon_{2*}\epsilon_{3*}\right) \ln^3\left(\frac{k}{k_*}\right)\Bigg\}.
\end{aligned}$$

Equation of motion and mapping method (1/2)

Mukhanov-Sasaki variable $v(\eta, k) \equiv a(\eta)\sqrt{2\epsilon_1(\eta)}\zeta(\eta, k)$ satisfies[10]

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\eta^2} + \left(k^2 - \frac{1}{z}\frac{\mathrm{d}^2 z}{\mathrm{d}\eta^2}\right) v = 0,$$

where $z(\eta) \equiv a(\eta)\sqrt{\epsilon_1(\eta)}$ is a "generalized scale factor"

Mapping method Beltran Jimenez, Musso, and Ringeval 2013**:**

Introduce a generalized Hubble parameter and e-fold number

$$\tilde{N} \equiv \ln z, \qquad \tilde{\mathcal{H}} \equiv \frac{z'}{z} = \frac{\mathrm{d}\tilde{N}}{\mathrm{d}\eta}, \qquad H_* \to \tilde{H}_*.$$

Define a hierarchy of generalized Hubble-flow functions

$$\alpha_{i+1}(\tilde{N}) \equiv \frac{\mathrm{d}\ln|\alpha_i|}{\mathrm{d}\tilde{N}}, \qquad \alpha_1(\tilde{N}) \equiv -\frac{\mathrm{d}\ln \tilde{H}}{\mathrm{d}\tilde{N}}, \qquad \epsilon_{i*} \to \alpha_{i*}$$

[10] V. F. Mukhanov, Feldman, and Brandenberger 1992.

Equation of motion and mapping method (2/2)

Only work: expressing the generalized parameters in terms of the standard Hubble flow functions.

$$\tilde{H} = \frac{H}{\sqrt{\epsilon_1}}\left(1 + \frac{\epsilon_2}{2}\right), \qquad \tilde{N} = N + \frac{1}{2}\ln\epsilon_1,$$

From which we get

$$\alpha_1 = \frac{4\epsilon_1 + \epsilon_2\left(2 + 2\epsilon_1 + \epsilon_2 - 2\epsilon_3\right)}{\left(\epsilon_2 + 2\right)^2}$$

$$\alpha_{i+1} = \left(1 + \frac{\epsilon_2}{2}\right)\frac{\mathrm{d}\ln|\alpha_i|}{\mathrm{d}N}$$

Scalar power spectrum

$$
\begin{aligned}
\mathcal{P}_\zeta(k) = \frac{H_*^2}{8\pi^2\epsilon_{1*}}\Bigg\{&1 - 2(C+1)\epsilon_{1*} - C\epsilon_{2*} + \left(\frac{\pi^2}{2} + 2C^2 + 2C - 3\right)\epsilon_{1*}^2 + \left(\frac{7\pi^2}{12} + C^2 - C - 6\right)\epsilon_{1*}\epsilon_{2*} \\
&+ \frac{1}{8}\left(\pi^2 + 4C^2 - 8\right)\epsilon_{2*}^2 + \frac{1}{24}(\pi^2 - 12C^2)\epsilon_{2*}\epsilon_{3*} - \frac{1}{24}\left[4C^3 + 3\left(\pi^2 - 8\right)C + 14\zeta(3) - 16\right]\left(8\epsilon_{1*}^3 + \epsilon_{2*}^3\right) \\
&+ \frac{1}{12}\left[13\pi^2 - 8(\pi^2 - 9)C + 36C^2 - 84\zeta(3)\right]\epsilon_{1*}^2\epsilon_{2*} - \frac{1}{24}\left[8C^3 - 15\pi^2 + 6(\pi^2 - 4)C - 12C^2 + 100\zeta(3) + 16\right]\epsilon_{1*}\epsilon_{2*}^2 \\
&+ \frac{1}{24}\left[\pi^2 C - 4C^3 - 8\zeta(3) + 16\right]\left(\epsilon_{2*}\epsilon_{3*}^2 + \epsilon_{2*}\epsilon_{3*}\epsilon_{4*}\right) + \frac{1}{24}\left[12C^3 + \left(5\pi^2 - 48\right)C\right]\epsilon_{2*}^2\epsilon_{3*} \\
&+ \frac{1}{12}\left[8C^3 + \pi^2 + 6\left(\pi^2 - 12\right)C - 12C^2 - 8\zeta(3) - 8\right]\epsilon_{1*}\epsilon_{2*}\epsilon_{3*} \\
&+ \Bigg[-2\epsilon_{1*} - \epsilon_{2*} + 2(2C+1)\epsilon_{1*}^2 + (2C-1)\epsilon_{1*}\epsilon_{2*} + C\epsilon_{2*}^2 - C\epsilon_{2*}\epsilon_{3*} - \frac{1}{8}\left(\pi^2 + 4C^2 - 8\right)(8\epsilon_{1*}^3 + \epsilon_{2*}^3) \\
&- \frac{2}{3}\left(\pi^2 - 9C - 9\right)\epsilon_{1*}^2\epsilon_{2*} - \frac{1}{4}\left(\pi^2 + 4C^2 - 4C - 4\right)\epsilon_{1*}\epsilon_{2*}^2 + \frac{1}{2}\left(\pi^2 + 4C^2 - 4C - 12\right)\epsilon_{1*}\epsilon_{2*}\epsilon_{3*} \\
&+ \frac{1}{24}\left(\pi^2 - 12C^2\right)\left(\epsilon_{2*}\epsilon_{3*}^2 + \epsilon_{2*}\epsilon_{3*}\epsilon_{4*}\right) + \frac{1}{24}\left(5\pi^2 + 36C^2 - 48\right)\epsilon_{2*}^2\epsilon_{3*}\Bigg]\ln\left(\frac{k}{k_*}\right) \\
&+ \frac{1}{2}\Bigg[4\epsilon_{1*}^2 + 2\epsilon_{1*}\epsilon_{2*} + \epsilon_{2*}^2 - \epsilon_{2*}\epsilon_{3*} + 6\epsilon_{1*}^2\epsilon_{2*} - (2C-1)\left(\epsilon_{1*}\epsilon_{2*}^2 - 2\epsilon_{1*}\epsilon_{2*}\epsilon_{3*}\right) \\
&- C\left(8\epsilon_{1*}^3 + \epsilon_{2*}^3 - 3\epsilon_{2*}^2\epsilon_{3*} + \epsilon_{2*}\epsilon_{3*}^2 + \epsilon_{2*}\epsilon_{3*}\epsilon_{4*}\right)\Bigg]\ln^2\left(\frac{k}{k_*}\right) \\
&+ \frac{1}{6}\left(-8\epsilon_{1*}^3 - 2\epsilon_{1*}\epsilon_{2*}^2 + 4\epsilon_{1*}\epsilon_{2*}\epsilon_{3*} - \epsilon_{2*}^3 + 3\epsilon_{2*}^2\epsilon_{3*} - \epsilon_{2*}\epsilon_{3*}^2 - \epsilon_{2*}\epsilon_{3*}\epsilon_{4*}\right)\ln^3\left(\frac{k}{k_*}\right)\Bigg\}.
\end{aligned}
$$

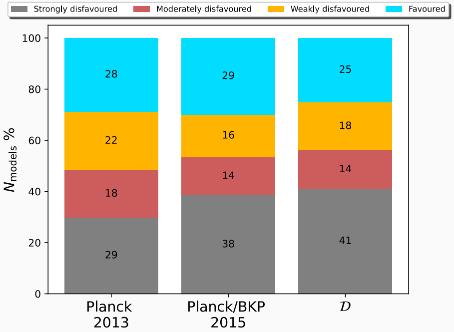
Current status and detection prospects

Datasets

- 2020 post-legacy release Planck
- BICEP/Keck array 2021 data
- South Pole Telescope third gen
- BAO data from Sloan Digital Sky Survey IV

Rule of thumb

- Tensor-to-scalar ratio
  $r \approx 16\epsilon_1 \implies \log(\epsilon_1) < -2.6\,(95\%)$
- Spectral index $n_s - 1 \approx \epsilon_2 \implies \epsilon_2 \approx 0.035$
- Running of the spectral index $\alpha_s \approx -\epsilon_2(2\epsilon_1 + \epsilon_3)$



Jerome Martin, Ringeval, and Vennin 2024a

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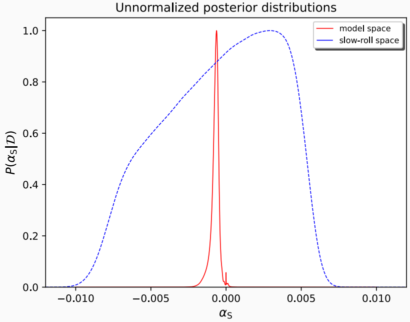

Jerome Martin, Ringeval, and Vennin 2024b

Summary

- Reviewed the Green's function method for the tensor and scalar primordial power spectra...
- extended to N3LO thanks to new asymptotic behaviors of integrals...
- and with non-minimal kinetic term (skipped here).
- Interesting for extrapolations of the power spectra. For instance, if one needs to estimate the amplitude of the curvature perturbations, or gravitational waves, at wavenumbers significantly different than k_*, all higher order terms may play a significant role.
- Highly accurate formulas for the semi-classical slow-roll predictions allow for searching in the data unexpected deviations, such as quantum backreaction

Thank you for your attention

Backup slides

The F_{0^n} hierarchy (1/4)

- Let us introduce the hierarchy of integrals I_n defined by

$$I_{n+1} = \int_x^{+\infty} \frac{e^{+2iy}}{y} \overline{I_n}(y)\,\mathrm{d}y\,, \qquad I_0 = 1.$$

- From this definition, we see that

$$I_{2n}(x) = \overline{F_{0^{2n}}}(x), \qquad I_{2n+1}(x) = F_{0^{2n+1}}(x),$$

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- A generating functional $h(\nu, x)$ can be constructed as

$$h(\nu, x) \equiv \sum_{k=0}^{+\infty} I_k(x)\nu^k, \text{ so that } I_n(x) = \frac{1}{n!}\left.\frac{\partial^n h(\nu, x)}{\partial \nu^n}\right|_{\nu=0}$$

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- $h(\nu, x)$ verifies a complex differential equation

$$\frac{\partial h(\nu, x)}{\partial x} + \frac{\nu e^{2ix}}{x}\overline{h}(\nu, x) = 0$$

The F_{0^n} hierarchy (2/4)

$$\frac{\partial h}{\partial x} + \frac{\nu e^{2ix}}{x}\overline{h}(x) = 0$$

- It can be recast into a matrix equation

$$\frac{\mathrm{d}X}{\mathrm{d}x} = -\frac{\nu}{x}AX, \text{ with } X(x) \equiv \begin{bmatrix} a(x) \\ b(x) \end{bmatrix} \text{ and } A(x) \equiv \begin{bmatrix} \cos(2x) & \sin(2x) \\ \sin(2x) & -\cos(2x) \end{bmatrix}$$

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- If one tries to diagonalize $A = P\Lambda P^{-1}$, with $Z(x) \equiv P^{-1}X$ then

$$\frac{\mathrm{d}Z}{\mathrm{d}x} = -\left(\frac{\nu}{x}\Lambda + P^{-1}\frac{\mathrm{d}P}{\mathrm{d}x}\right)Z = \begin{pmatrix} -\frac{\nu}{x} & 1 \\ -1 & \frac{\nu}{x} \end{pmatrix} Z.$$

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- This system has no longer any oscillatory terms and can be decoupled by differentiation

$$\frac{\mathrm{d}^2 z_1}{\mathrm{d}x^2} = -\left[1 - \frac{\nu(\nu+1)}{x^2}\right] z_1, \quad z_2 = \frac{\mathrm{d}z_1}{\mathrm{d}x} + \frac{\nu}{x} z_1.$$

The F_{0^n} hierarchy (3/4)

- The first of these equations is a Riccati-Bessel differential equation which admits the exact solutions

$$z_1(x) = C_1(\nu)\, x j_\nu(x) + C_2(\nu)\, x y_\nu(x),$$

where j_ν and y_ν are the spherical Bessel functions of first and second kind[11]

[11] Abramowitz and Stegun 1970.

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- One finally obtains the exact expression

$$h(\nu, x) = -x e^{ix} \left\{ \sin\left(\frac{\pi\nu}{2}\right) [j_\nu(x) + i j_{\nu-1}(x)] + \cos\left(\frac{\pi\nu}{2}\right) [y_\nu(x) + i y_{\nu-1}(x)] \right\}$$

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- In the limit $x \to 0^+$

$$h(\nu, x) \underset{x\to 0}{\sim} \frac{2^\nu}{x^\nu \sqrt{\pi}} \cos\left(\frac{\pi\nu}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right) + \frac{x^\nu \sqrt{\pi}}{2^\nu \cos(\pi\nu)} \frac{i \sin\left(\frac{\pi\nu}{2}\right)}{\Gamma\left(\nu + \frac{1}{2}\right)}$$

[11] Abramowitz and Stegun 1970.

The F_{0^n} hierarchy (4/4)

Finally, we can extract our hierarchy of functions $F_{0^n}(x)$ around $x \to 0^+$

$$F_0(x) = -B - \ln(x) + \mathcal{O}(x)$$
$$F_{00}(x) = \frac{\pi^2}{4} + \frac{B^2}{2} + B\ln(x) + \frac{1}{2}\ln^2(x) + \mathcal{O}(x)$$
$$F_{000}(x) = -\frac{7}{3}\zeta(3) - \frac{\pi^2}{4}B - \frac{1}{6}B^3 - \left(\frac{\pi^2}{4} + \frac{B^2}{2}\right)\ln(x) - \frac{B}{2}\ln^2(x) - \frac{1}{6}\ln^3(x) + \mathcal{O}(x)$$