RENORMALIZATION OF THE PRIMORDIAL INFLATIONARY POWER SPECTRA

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October 30, 2024

Based on arXiv:2402.14910
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Overview

In this talk, I will explore the *effects of renormalization* in the amplitude of the inflationary spectra at scales measurable in the cosmic microwave background.

1. **Motivation and background.**
   Do we need to renormalise the PS?
   Previous results.
2. **Main tools.**
   Adiabatic expansion/adiabatic regularization.
3. **The power spectrum.**
   Cosmological perturbations.
   Instant transition.

Part I

Motivation and background

Inflationary paradigm

It provides an elegant solution to the *horizon and flatness problems*, as well as an explanation for the *homogeneity and isotropy* of the universe.

Inflation + Quantum Field Theory

$\downarrow$

predict the generation of a nearly scale-free spectrum of primordial **scalar and tensor fluctuations**.

- **Scalar fluctuations** $\rightarrow$ observed as temperature anisotropies in the CMB. Act as the seeds for structure in the universe.
- **Tensor fluctuations** $\rightarrow$ prediction of a spectrum of relic gravitational waves carrying information from the earliest moments of the universe.

It is very important to have have firm theoretical predictions for the tensor and scalar spectra. **Can renormalization change the standard predictions?**

Do we need renormalization?

$$\langle\zeta(\tau,\mathbf{x})\zeta(\tau,\mathbf{x}')\rangle = \int \frac{\mathrm{d}k}{k}\frac{\sin k|\mathbf{x}-\mathbf{x}'|}{k|\mathbf{x}-\mathbf{x}'|}\mathcal{P}_\zeta(k,\tau),$$

$\langle\zeta(\tau,\mathbf{x})\zeta(\tau,\mathbf{x}')\rangle$ it's finite in the distributional sense.

$\mathcal{P}_\zeta(k,\tau)$ goes as k^{-1} in the UV limit.

The correlator is *divergent* in the coincident limit $x \to x'$.

It was suggested that renormalization results in a significant reduction of the amplitude of the spectra at CMB scales.

(Parker, 2007; Agullo, Navarro-Salas, Olmo, Parker, 2008/2009/2010)

- They apply the *adiabatic renormalization method*.
- The final result was obtained by evaluating $\mathcal{P}_{\zeta}(k, \tau)$ at the moment of horizon crossing during inflation. *Quantum-to-classical transition.*

This result sparked a vigorous debate

- "Renormalization is not needed."

  (F. Finelli, G. Marozzi, G. P. Vacca, and G. Venturi, 2007)
- "We need to introduce an arbitrary scale in the adiabatic method."

  (A. Ferreiro and F. Torrenti, 2023)
- "The adiabatic subtractions shouldn't be applied at super-horizon scales."

  (R. Durrer, G. Marozzi, and M. Rinaldi, 2009/2011)

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- They apply the *adiabatic renormalization method*.
- The final result was obtained by evaluating $\mathcal{P}(k, \tau)$ at the moment of horizon crossing during inflation. *Quantum-to-classical transition.*

Our claim:

- The adiabatic method works as it is, and should be applied at all scales.
- We shouldn't evaluate the subtractions at horizon crossing. We should let them evolve. The *quantum-to-classical transition will occur dynamically.*

Part II

Framework and main tools

The adiabatic expansion.

It is an asymptotic expansion that captures the large-k behaviour of the modes $\to$ it can be used for renormalization.

1. Starting point: the mode equation

$$\varphi_k'' + (\omega_k^2 + \sigma)\varphi_k = 0\,.$$

2. Assume the WKB ansatz for the field modes

$$\varphi_k \sim \frac{1}{\sqrt{2\,\Omega_k(\tau)}}\, e^{-i\int^{\tau}\Omega_k(\tau')\,d\tau'}\,,$$

The function Ω_k admits the following adiabatic expansion

$$\Omega_k = \sum_{n=0}^{\infty} \omega_k^{(n)}\,.$$

The adiabatic expansion

3. Fix the adiabatic order of the background fields: ω is of adiabatic order zero. σ is of adiabatic order two.

4. Insert the WKB ansatz and the adiabatic expansion in the eom and group terms with the same adiabatic order.

5. Solve iteratively. First orders:

$$\begin{aligned}\omega^{(0)} &= \omega\,,\\ \omega^{(1)} &= \omega^{(3)} = 0\,,\\ \omega^{(2)} &= \frac{\sigma}{2\omega} + \frac{3}{8}\frac{(\omega')^2}{\omega^3} - \frac{1}{4}\frac{\omega''}{\omega^2}\,,\end{aligned}$$

From the adiabatic expansion of the field modes, we can obtain the adiabatic expansion of composite quantities

$$2|\varphi_k|^2_{\text{Ad}} \sim (\Omega_k^{-1})^{(0)} + (\Omega_k^{-1})^{(2)} + (\Omega_k^{-1})^{(4)} + \ldots$$

The adiabatic expansion captures the UV behavior of modes.

↓

it removes the UV divergences by simply subtracting the adiabatic counterterms mode-by-mode.

The number of subtractions is determined by the scaling dimension of the observable. Two-point function:

$$\langle\hat{\phi}^2\rangle_{\text{phys}} = \int \frac{\mathrm{d}k}{k}\frac{k^3}{4\pi a^2}\left(2|\varphi_k|^2 - (\Omega_k^{-1})^{(0)} - (\Omega_k^{-1})^{(2)}\right) .$$

Adiabatic regularization is known to be equivalent to other renormalization methods, e.g., *dimensional regularization or point-splitting* (up to the well-known renormalization ambiguities).

$$\langle\hat{\phi}^2\rangle_1 - \langle\hat{\phi}^2\rangle_2 = c_1 m^2 + c_2 R$$

It is compatible with *locality* and *general covariance*.

It is a powerful tool for practical cosmological applications in FRW spacetime. Specially when a numerical analysis is required.

Note: despite it's name, it works in any FRW background, not only when the expansion rate of the universe is adiabatic.

Part III

The power spectrum of primordial perturbations

Cosmological perturbations.

We start with the perturbed FRW metric in the longitudinal gauge

$$\mathrm{d}s^2 = a^2\Big[-(1+2\Phi)\mathrm{d}\tau^2 + [(1-2\Psi)\delta_{ij} + h_{ij}]\mathrm{d}x^i\mathrm{d}x^j\Big]\,,$$

- The tensor fluctuations h_{ij} are chosen to satisfy the transverse-traceless condition $\partial_i h_{ij} = h^i_{\ i} = 0$, which yields two physical polarizations $h = h_{+,\times}$.
- For the scalars, (we assume $\Phi = \Psi$) one can define the gauge-invariant quantity ζ

$$\zeta = \Psi + H\frac{\delta\phi}{\dot{\phi}}\,,$$

Our goal is to compute the coincident two-point functions of ζ and h and renormalize them using the *adiabatic regularization technique.*

It is very convenient to work with Mukhanov variables

$$v_s = z_s \zeta \,, \qquad z_s = a M_P \sqrt{2\varepsilon} \,,$$
$$v_t = z_t h \,, \qquad z_t = a M_P / 2 \,,$$

$s \equiv$ scalar, $t \equiv$tensor, $\varepsilon = -\dot{H}/H^2$, and M_P is the reduced Planck scale.

These variables are very advantageous since they allow the scalar and tensor perturbations to be described by the same action

$$\mathcal{S} = \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}^3\mathbf{x} \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z} v^2 \right] \,,$$

$z = z(\tau)$ contains all the information about the gravitational background.

v is suitable for quantization by via the canonical procedure.

1. Mode expansion:

$$\hat{v}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \left[v_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}} + v_k^*(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}}^\dagger \right] .$$

2. Vacuum state:

$$\hat{a}_{\mathbf{k}}|0\rangle = 0 .$$

3. Mode equation:

$$v_k'' + \left(k^2 - \frac{z''}{z} \right) v_k = 0 .$$

4. Wronskian condition:

$$v_k v_k'^* - v_k^* v_k' = i .$$

We can compute now the (coincident) two-point function

$$\langle v^2 \rangle = \int_0^\infty \frac{\mathrm{d}k}{k} \mathcal{P}_v(k, \tau) , \qquad \mathcal{P}_v \equiv \frac{k^3}{2\pi^2} |v_k(\tau)|^2 .$$

$\mathcal{P}_v$ is the unregularized power spectrum of the Mukhanov variable.

$$\langle v^2 \rangle = \int_0^\infty \frac{\mathrm{d}k}{k} \mathcal{P}_v(k,\tau)\,, \qquad \mathcal{P}_v \equiv \frac{k^3}{2\pi^2} |v_k(\tau)|^2\,.$$

The coincident two-point function is UV divergent.

It can be regularised using the adiabatic method. The regularized spectrum is defined as

$$\mathcal{P}_v^{\mathrm{reg}} \equiv \mathcal{P}_v(k,\tau) - \mathcal{P}_v^{\mathrm{ct}}(k,\tau)\,,$$

where $\mathcal{P}_v^{\mathrm{ct}}$ contains the adiabatic counterterms

$$\mathcal{P}_v^{\mathrm{ct}}(k,\tau) = \frac{k^2}{4\pi^2} + \frac{1}{8\pi^2}\frac{z''}{z}\,.$$

Subtracting $\mathcal{P}_v^{\mathrm{ct}}(k,\tau)$ leads to an exact cancellation of both divergent terms.

For simplicity we assumed $\varepsilon = -\dot{H}/H^2 \approx const.$

In this limit

$$\frac{z''}{z} = \frac{a''}{a}\,, \qquad a \propto \tau^{1/2-\nu}\,,$$

Note: ν is the "bessel index" that appears in the mode equations. It is related to ε (and to the equation of state w).

We analysed two cases:

- The power spectrum in an **inflating universe** $(0 \leq \varepsilon < 1;\ w < -1/3)$.
- The power spectrum in a universe that:
  1. Starts in an **inflationary phase**, described by a constant equation of state
  $$w_1 < -1/3\,.$$
  2. At $\tau = \tau_0$ experiences an *instant transition* to a FRW universe with a **growing horizon**, described by another constant
  $$w_2 > -1/3\,.$$

Instant transition.

The solutions for the "*in*" and "*out*" regions are given in terms of Hankel functions with indices ν and μ, respectively. These indices are related with w_1 and w_2, as follows

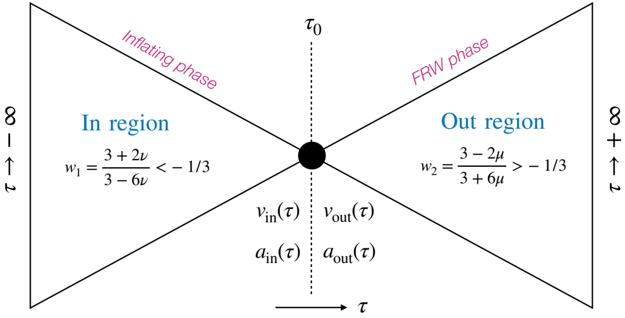


Figure: Credits to B. Stefanek

UV limit

The counterterms in the *out* region can be readily computed

$$\mathcal{P}_v^{\mathrm{ct}}(k,\tau) = \frac{k^2}{4\pi^2} + \frac{(\mu^2 - 1/4)}{8\pi^2(\tau-\bar{\tau})^2}\,.$$

Let us focus first on the **UV behaviour** of $\mathcal{P}_v(k,\tau)$. It is not difficult to find the following asymptotic form

$$\mathcal{P}_v(k,\tau) \;\to\; \frac{k^2}{4\pi^2} + \frac{(\mu^2 - 1/4)}{8\pi^2(\tau-\bar{\tau})^2} - \frac{(\mu+\nu)\cos(2k(\tau-\tau_0))}{8\pi^2\gamma\tau_0^2} + \mathcal{O}(k^{-2})\,.$$

The first two UV divergent terms in $\mathcal{P}_v$ are exactly canceled by $\mathcal{P}_v^{\mathrm{ct}}$, while the oscillatory term is UV finite.

IR limit

After some algebra, it can be shown that the unrenormalised power spectrum in the IR is *independent* of μ. It reads

$$\mathcal{P}_v(k,\tau) \to \frac{a^2 H_0^2}{4\pi^2} \frac{2^{2\nu-3}\Gamma(\nu)^2}{\Gamma(3/2)^2} \left(\nu - \frac{1}{2}\right)^{1-2\nu} \left(\frac{k}{k_0}\right)^{3-2\nu} ,$$

On the other hand, the counterterms only depend on μ

$$\mathcal{P}_v^{\rm ct}(k,\tau) \to \frac{a^2 H_0^2}{8\pi^2} \left(\frac{\mu - 1/2}{\mu + 1/2}\right) e^{-\frac{(3+2\mu)}{(\mu+1/2)}(N-N_0)} ,$$

where $N - N_0$ measures the number of e-folds after the end of inflation.

Conclusion

- The IR counterterm spectrum rapidly decays after the end of inflation, while $\mathcal{P}_v$ is independent of time.
- This means that $\mathcal{P}_v^{\mathrm{reg}} \to \mathcal{P}_v$ is an attractor solution in the IR that is reached a few e-folds after the end of inflation.
- We can safely take the limit $\tau \to \infty$ to obtain our final answer for the IR spectra of ζ and h that would be measured at late times

$$\mathcal{P}_\zeta^{\mathrm{reg}}(k_*, \infty) \approx \frac{1}{8\pi^2 \varepsilon_{\mathrm{in}}} \frac{H_*^2}{M_P^2} \equiv A_s \,,$$
$$\mathcal{P}_h^{\mathrm{reg}}(k_*, \infty) \approx \frac{2}{\pi^2} \frac{H_*^2}{M_P^2} \equiv A_t \,.$$

It gives the standard prediction for the tensor-to-scalar ratio $r = A_t/A_s = 16\varepsilon_{\mathrm{in}}$.

THANKS FOR YOUR ATTENTION

Instant transition.

1. We start with an **inflationary universe**, described by a constant equation of state

$$w_1 < -1/3\,.$$

2. At $\tau = \tau_0$ ($a = a_0$, $H = H_0$) it experiences an *instant transition* to a FRW universe with a **growing horizon**, described by another constant

$$w_2 > -1/3\,.$$

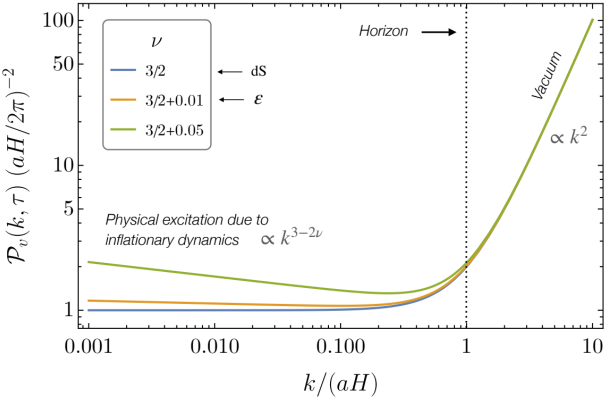
Scale factor:

$$\frac{a(\tau)}{a_0} = \begin{cases} (\tau/\tau_0)^{\frac{2}{(1+3w_1)}} & \tau < \tau_0 \\ \left[\frac{1}{2}a_0 H_0(\tau - \bar{\tau})(1+3w_2)\right]^{\frac{2}{(1+3w_2)}} & \tau > \tau_0\,, \end{cases}$$

where

$$\frac{\bar{\tau}}{\tau_0} = \frac{(w_2 - w_1)}{(w_2 + 1/3)}\,, \qquad a_0 H_0 \tau_0 = \frac{2}{(1+3w_1)}$$

"Bare" $\mathcal{P}$ in an inflating universe.



The solutions for the "*in*" and "*out*" regions are given in terms of Hankel functions with indices ν and μ, respectively. These indices are related with w_1 and w_2, as follows

$$w_1 = \frac{3+2\nu}{3-6\nu}\,, \qquad\qquad w_2 = \frac{3-2\mu}{3+6\mu}\,.$$

The solution for the modes in the inflating *in* phase is $(q = -k\tau)$

$$v_k^{\rm in}(\tau) = \sqrt{\frac{\pi}{4k}}\, e^{i\frac{\pi}{4}(1+2\nu)}\sqrt{q}\, H_\nu^{(1)}(q) \equiv \frac{f_\nu(q)}{\sqrt{2k}}\,.$$

In the growing horizon *out* phase we have

$$v_k^{\rm out}(\tau) = \frac{1}{\sqrt{2k}}\Big(\alpha_k\, f_\mu(q-\bar{q}) + \beta_k\, f_\mu^*(q-\bar{q})\Big)\,,$$

where $|\alpha_k|^2 - |\beta_k|^2 = 1$.

The coefficients α_k and β_k are determined by requiring the mode function and its derivative to be continuous at τ_0, namely

$$v_k^{\text{in}}(\tau_0) = v_k^{\text{out}}(\tau_0)\,, \qquad v_k'^{\text{in}}(\tau_0) = v_k'^{\text{out}}(\tau_0)$$

We find

$$\alpha_k = \frac{1}{2}\Big[f_\nu(q_0)f^*_{1+\mu}(\gamma q_0) + f_{\nu-1}(q_0)f^*_\mu(\gamma q_0)\Big]\,,$$
$$\beta_k = \frac{1}{2}\Big[f_\nu(q_0)f_{1+\mu}(\gamma q_0) - f_{\nu-1}(q_0)f_\mu(\gamma q_0)\Big]\,.$$

where $\gamma = (1+2\mu)/(1-2\nu)$ and $q_0 = -k\tau_0$.

Let's study $\mathcal{P}_v^{\text{reg}}$ in the *out* region!

The counterterms in the *out* region can be readily computed

$$\mathcal{P}_v^{\rm ct}(k,\tau) = \frac{k^2}{4\pi^2} + \frac{(\mu^2 - 1/4)}{8\pi^2(\tau-\bar{\tau})^2}\,.$$

Let us focus first on the **UV behaviour** of $\mathcal{P}_v(k,\tau)$. It is not difficult to find the following asymptotic form

$$\mathcal{P}_v(k,\tau) \;\to\; \frac{k^2}{4\pi^2} + \frac{(\mu^2 - 1/4)}{8\pi^2(\tau-\bar{\tau})^2} - \frac{(\mu+\nu)\cos(2k\,(\tau-\tau_0))}{8\pi^2\gamma\tau_0^2} + \mathcal{O}(k^{-2})\,.$$

The first two UV divergent terms in $\mathcal{P}_v$ are exactly canceled by $\mathcal{P}_v^{\rm ct}$, while the oscillatory term is UV finite.

In the **IR limit** we find $\alpha_k \approx \beta_k e^{-i\pi(\mu+\frac{1}{2})}$ at leading order, and

$$|\beta_k|^2 \to \frac{4^{\nu+\mu}}{4\pi^2} \frac{q_0^{-2(\nu+\mu)}}{\gamma^{1+2\mu}} \Gamma(\nu)^2 \Gamma(1+\mu)^2 \,.$$

After some algebra, it can be shown that the unrenormalised power spectrum in the IR is *independent* of μ. It reads

$$\mathcal{P}_v(k,\tau) \to \frac{a^2 H_0^2}{4\pi^2} \frac{2^{2\nu-3}\Gamma(\nu)^2}{\Gamma(3/2)^2} \left(\nu - \frac{1}{2}\right)^{1-2\nu} \left(\frac{k}{k_0}\right)^{3-2\nu} \,,$$

On the other hand, the counterterms only depend on μ

$$\mathcal{P}_v^{\rm ct}(k,\tau) \to \frac{a^2 H_0^2}{8\pi^2} \left(\frac{\mu - 1/2}{\mu + 1/2}\right) e^{-\frac{(3+2\mu)}{(\mu+1/2)}(N-N_0)} \,,$$

where $N - N_0$ measures the number of e-folds after the end of inflation.

We have also solved the mode equation numerically for a universe that transitions out of inflation in a *finite time* to a matter domination universe.

- The **instant transition** leads to an over-excitation of UV modes.
- The scale-invariant UV oscillations appearing in the **instant transition** case are due to the finite term that appears in the (UV expansion).
- The **instant transition** provides an excellent approximation to the full numerical solution in the IR a few e-folds after inflation ends, which is the region of interest for cosmology.

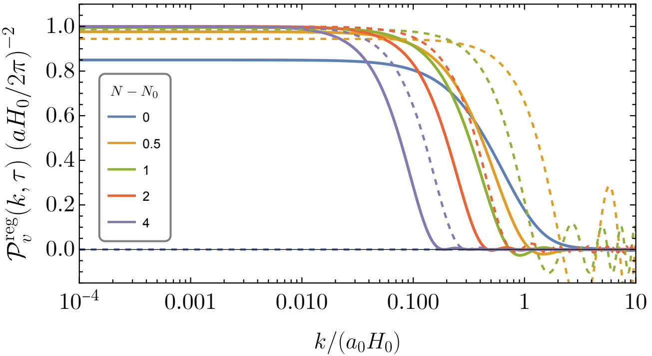


Figure: Regularized power spectrum for a universe that makes a transition from an inflating phase with $w_1 = -1$ (de-Sitter) to a growing horizon phase with $w_2 = 0$ (matter). Solid lines are obtained numerically (for a transition on a timescale H_0^{-1}), while dashed lines give the instant transition approximation

We have computed the renormalized spectra of scalar and tensor perturbations from inflation using **adiabatic regularization.**

The adiabatic counterterms must be subtracted at *all times and for all scales*. As a consequence $\mathcal{P}_v^{\rm reg}(k,\tau) = 0$ for all (k,τ) in de Sitter.

We followed the evolution of the renormalized spectra through the inflationary transition using an *instant transition model* (supported by a full numerical solution).

The standard result for the IR spectrum is recovered just a few e-folds after inflation ends, while the counterterms ensure that UV divergences are canceled at all times. **Standard predictions for inflationary observables are recovered.**

Quantum-to-classical transition?

We can compute the "purity" of the vacuum state

$$\begin{aligned}\gamma_k &= 4 \times \det \begin{pmatrix} |v_k|^2 & \frac{1}{2}(v_k v_k'^* + v_k' v_k^*) \\ \frac{1}{2}(v_k v_k'^* + v_k' v_k^*) & |v_k'| \end{pmatrix} \\ &= -(v_k^* v_k' - v_k v_k'^*)^2 = 1\end{aligned}$$

The determinant of the purity matrix is proportional to the Wronskian

$\downarrow$

the purity is a *conserved quantity*.

An infrared-safe definition for the counterterm spectrum

$\mathcal{P}_\phi^{ct}$ for a free field in FLRW read:

$$\mathcal{P}_v^{\rm ct}(k,\tau) = \frac{k^3}{2\pi^2}\left(\frac{1}{2\omega} - \frac{\omega^{(2)}}{2\omega^2}\right)$$

where

$$\omega^{(2)} = \sigma + f(\omega', \omega'')$$

with $\omega^2 = k^2 + m^2$ and $\sigma = (\xi - 1/6)R$. This result gives rise to a logarithmic IR divergence in $\langle\phi^2\rangle$ when the massless limit $m \to 0$ is taken.

These IR divergences can be removed by an alternative (resummed), IR-safe definition for the counterterm spectrum

$$\overline{\mathcal{P}}^{\rm ct}(k,\tau) = \frac{k^3}{2\pi^2}\frac{1}{2\bar{\omega}},$$

where $\bar{\omega}^2 = k^2 + m^2 + \sigma$.