On adiabatic renormalization in U(1)-axion inflation

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Introduction to the adiabatic Renormalization

- **UV divergences:** As in the case of flat space time, observable are characterized by divergences in the deep UV.
- **New divergences:** The presence of gravity led to new divergences that are not matched by the Minkowski ones.
- **Vacuum choice:** There is not a preferred choice of the vacuum.

the "vacuum"-state is not anymore empty due to time dependent background

$$\mathbf{A}_k |0\rangle = 0 \xrightarrow{t-\text{evolution}} \langle 0| \mathbf{A}_k^\dagger \mathbf{A}_k |0\rangle \neq 0 = N_k = |\beta_k|^2$$

Physical request:
particles should not be created when the energy of a single particle is larger w.r.t. the energy scale of the spacetime.

$$\frac{k^2}{a^2(t)} + m^2 > \left(\frac{\dot{a}}{a}\right)^2, \ \frac{\ddot{a}}{a} \qquad \Rightarrow \quad N_k \sim \text{const.}$$

the particle content should not change if the change rate of $a(t)$ is adiabatic.

Adiabatic vacuum:
the vacuum that minimizes the creation of particle due to the presence of a time-dependent metric.

Scalar Field

$$\mathcal{L} = \frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 - \xi R\phi^2\right)$$

Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}\mathbf{x}^2$$

Equation of motion: $$\left(\Box + m^2 + \xi R\right)\phi = 0$$

standard quantization: $$\phi(x) = \sum_{\mathbf{k}}\{A_{\mathbf{k}} f_{\mathbf{k}}(x) + A^\dagger_{\mathbf{k}} f^*_{\mathbf{k}}(x)\}$$

$A^\dagger_{\mathbf{k}}$ and $A_{\mathbf{k}}$ creation and annihilation operators

mode function: $$f_{\mathbf{k}} = (2V)^{-1/2} a(t)^{-3/2} h_k(t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

the rescaled mode function $h_k(t)$ satisfies the equation

$$\ddot{h}_k + \Omega_k^2(t)\, h_k = 0$$

formally solved by the Wentzel-Kramer-Brillouin (WKB) approximation

$$h_k(t) = \frac{1}{\sqrt{2W_k(t)}} e^{-i\int W_k(t')\mathrm{d}t'}$$

inserting the WKB ansatz into the equation of motion

$$W_k(t)^2 = \Omega_k(t)^2 - \left(\frac{\ddot{W}_k(t)}{2W_k(t)} - \frac{3\dot{W}_k(t)^2}{4W_k(t)^2}\right)$$

Adiabatic condition: slowly changes in time

$$\left|\frac{\dot{W}}{W^2}\right| \ll 1$$

introducing an adiabatic parameter $\epsilon \ll 1$

$$\partial_t \to \epsilon \partial_t$$

solution for $W_k(t)$ as a power series in time derivatives

$$W_k(t) = W_k^{(0)}(t) + \epsilon\, W_k^{(1)}(t) + \cdots + \epsilon^n\, W_k^{(n)}(t)$$

Adiabatic renormalization prescription:

- evaluate expectation values w.r.t. the adiabatic vacuum
- mode functions are given in terms of WKB ansatz
- expand up to the adiabatic order that matches energy dimension of the operator
- subtract the adiabatic term from the bare quantity

A problematic example: Axion-gauge fields

Pseudo-scalar inflaton field ϕ coupled to $U(1)$ gauge field A_μ

$$\mathcal{L} = -\frac{1}{2}(\nabla\phi)^2 - V(\phi) - \frac{1}{4}(F^{\mu\nu})^2 - \frac{g\phi}{4}F^{\mu\nu}\tilde{F}_{\mu\nu}$$

- Due to the coupling with the inflaton field ϕ, quantum fluctuations of the gauge field A_μ are amplified.
- Backreaction:

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = g\langle\mathbf{E}\cdot\mathbf{B}\rangle$$

$$H^2 = \frac{1}{3M_p^2}\left[\frac{\dot{\phi}^2}{2} + V(\phi) + \frac{\langle\mathbf{E}^2+\mathbf{B}^2\rangle}{2}\right]$$

$$\dot{H} = -\frac{1}{2M_p^2}\left[\dot{\phi}^2 + \frac{2}{3}\langle\mathbf{E}^2+\mathbf{B}^2\rangle\right]$$

Energy Density

$$\frac{\langle\mathbf{E}^2+\mathbf{B}^2\rangle}{2} = \int\frac{\mathrm{d}kk^2}{(2\pi)^2a(\tau)^4}\left[|A'_+|^2 + |A'_-|^2 + k^2\left(|A_+|^2+|A_-|^2\right)\right]$$

Helicity Integral

$$\langle\mathbf{E}\cdot\mathbf{B}\rangle = -\int\frac{\mathrm{d}kk^3}{(2\pi)^2a(\tau)^4}\frac{\partial}{\partial\tau}\left(|A_+|^2-|A_-|^2\right)$$

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Energy Density

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Helicity Integral

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The fourier mode functions $A_\pm$ satisfy the EOM:

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} A_\pm(\tau, k) + \left(k^2 \mp kg\phi'\right) A_\pm(\tau, k) = 0$$

assuming de Sitter: $a(\tau) = -1/(H\tau),\ H = \text{const.}$

$$\xi \equiv g\phi'/(2a(\tau)H) = g\dot{\phi}/(2H)$$

$$\boxed{A_\pm(\tau, k) = \frac{1}{\sqrt{2k}} e^{\pm\pi\xi/2} W_{\pm i\xi, \frac{1}{2}}(-2ik\tau)}$$

- **Divergences:** $\frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} \supset \Lambda^4, \Lambda^2, \log\Lambda,$
  $\langle \mathbf{E} \cdot \mathbf{B} \rangle \supset \Lambda^2, \ \log\Lambda$
  - $\Lambda^4,\ \Lambda^2$ and $\log[\Lambda]$ UV divergences for the energy density.
  - Λ^2 and $\log[\Lambda]$ UV divergences for the helicity integral.
  - well-behaved in the infrared

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- **Divergences:**

$$\frac{\langle \mathbf{E}^2+\mathbf{B}^2\rangle}{2} \supset \Lambda^4,\ \Lambda^2, \log\Lambda,$$
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Renormalization

For each polarization $\lambda = \pm$:

$$A_\lambda^{\mathsf{WKB}}(k,\tau) = \frac{1}{\sqrt{2\Omega_\lambda(k,\tau)}} e^{-i\int \Omega_\lambda(k,\tau')\mathrm{d}\tau'}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} A_\pm^{\mathsf{WKB}}(\tau,k) + \left(k^2 \mp gk\phi' + \frac{m^2}{H^2\tau^2}\right) A_\pm^{\mathsf{WKB}}(\tau,k) = 0$$

$$\Downarrow$$

$$\Omega_\lambda^2(k,\tau) = \bar{\Omega}_\lambda^2(k,\tau) + \frac{3}{4}\left(\frac{\Omega_\lambda'(k,\tau)}{\Omega_\lambda(k,\tau)}\right)^2 - \frac{1}{2}\frac{\Omega_\lambda''(k,\tau)}{\Omega_\lambda(k,\tau)}$$

- **Adiabatic condition**: slowly changes in time: $\left|\frac{\dot{\Omega}}{\Omega^2}\right| \ll 1$

$$\epsilon \ll 1 : \partial_t \to \epsilon\partial_t$$

$$\Omega_k(t) = \Omega_k^{(0)}(t) + \epsilon\,\Omega_k^{(1)}(t) + \cdots + \epsilon^n\,\Omega_k^{(n)}(t)$$

Standard adiabatic regularization: ill defined for $m \to 0$

$$\frac{1}{2}\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle_{\text{ad}} = \int_0^{a(\tau)\Lambda} \frac{\mathrm{d}k k^2}{(2\pi)^2 a(\tau)^4} (\cdots)_{\text{ad}}^{n=4} \supset \Lambda^4, \Lambda^2, \log\Lambda, \log m$$

$$\langle \mathbf{E} \cdot \mathbf{B} \rangle_{\text{ad}} = -\int_0^{a(\tau)\Lambda} \frac{\mathrm{d}k k^3}{(2\pi)^2 a(\tau)^4} \frac{\partial}{\partial\tau} (\cdots)_{\text{ad}}^{n=4} \supset \Lambda^2, \log\Lambda, \log m$$

- The standard adiabatic renormalization correctly removes the divergences in the UV, introducing **unphysical IR divergences**.

Issues and Motivations: the need of a IR cut off

- Adiabatic renormalization concerns the UV divergences
- WKB is well defined for modes that feel small curvature
- Good approximation for sub-horizon modes

$\langle T_{\mu\nu}\rangle_{\mathsf{ad}}$ will have the following general structure:

$$\langle T\rangle^{(n>4)}_{\mathsf{ad}} = H^4 \sum_{n>4} \left(c_n \left(\frac{H}{m}\right)^{n-4} + c'_n \left(\frac{H}{\Lambda}\right)^{n-4} \right)$$

- **deep UV**:when $\Lambda \to \infty$, the higher order terms go to zero and we can truncate the series at the fourth adiabatic order, which is indeed the order needed to remove the UV divergences.

- **IR regime** the IR regime produces higher order terms involving m which are increasingly relevant for $m \to 0$.

New adiabatic regularization

We suggest that the procedure of adiabatic regularization should be always performed on a proper domain which excludes the IR tail of the spectrum.

- the adiabatic subtraction should be considered only up to an IR cut-off $c = \beta a(t)H(t)$.
- the coefficient β, should be determined by a proper physical prescription.

$$\frac{1}{2}\langle \mathbf{E}^2+\mathbf{B}^2\rangle_{\mathsf{ad}} = \int_{\beta a(\tau)H}^{a(\tau)\Lambda} \frac{\mathrm{d}k k^2}{(2\pi)^2 a(\tau)^4}\left(\cdots\right)_{\mathsf{ad}}^{n=4}$$

$$\langle \mathbf{E}\cdot\mathbf{B}\rangle_{\mathsf{ad}} = -\int_{\beta a(\tau)H}^{a(\tau)\Lambda} \frac{\mathrm{d}k k^3}{(2\pi)^2 a(\tau)^4}\frac{\partial}{\partial\tau}\left(\cdots\right)_{\mathsf{ad}}^{n=4}$$

$$\frac{1}{2}\langle \mathbf{E}^2+\mathbf{B}^2\rangle_{\mathsf{ad}}^{c=\beta H a(\tau)} = \frac{\Lambda^4}{8\pi^2}+\frac{H^2\Lambda^2\xi^2}{8\pi^2}+\frac{3H^4\xi^2(5\xi^2-1)\log(2\Lambda/H)}{16\pi^2} - \frac{\beta^4H^4}{8\pi^2}-\frac{\beta^2H^4\xi^2}{8\pi^2}-\frac{3H^4\xi^2(5\xi^2-1)\log(2\beta)}{16\pi^2}$$

$$\langle \mathbf{E}\cdot\mathbf{B}\rangle_{\mathsf{ad}}^{c=\beta H a(\tau)} = -\frac{H^2\Lambda^2\xi}{8\pi^2}-\frac{3H^4\xi(5\xi^2-1)\log(2\Lambda/H)}{8\pi^2} + \frac{\beta^2H^4\xi}{8\pi^2}+\frac{3H^4\xi(5\xi^2-1)\log(2\beta)}{8\pi^2}$$

How to fix the scheme
Conformal anomaly

- In the conformal limit, a proper renormalization scheme should provide the conformal anomaly induced by quantum effects.

- When at the classical level $T^{\mu}{}_{\mu} = 0$

$$\langle T^{\mu}{}_{\mu} \rangle_{\mathsf{phys}} = -\langle T^{\mu}{}_{\mu} \rangle_{\mathsf{reg}}$$

  $\langle T^{\mu}{}_{\mu} \rangle_{\mathsf{reg}}$ is given by the particular renormalization method applied.

- The two helicities of the mode functions $A_\pm$ are equivalent to two conformally coupled massless scalar fields for $\xi = 0$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} A_\pm + \left(k^2 \pm \frac{2k\xi}{\tau} + \frac{m^2}{H^2\tau^2} \right) A_\pm = 0 \;\rightarrow\; \boxed{\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + k^2 \right) A_\pm = 0}$$

$$\lim_{\xi\to 0,\, m\to 0} \langle T^0{}_0 \rangle_{\mathsf{ad}} = \lim_{\xi\to 0,\ m\to 0} \frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle_{\mathsf{ad}}^{c=\beta H a(\tau)}}{2} = -\frac{\beta^4 H^4}{8\pi^2}$$

this term should reproduce the expected value of the anomaly

$$\frac{\beta^4 H^4}{8\pi^2} = \frac{H^4}{480\pi^2} \implies \boxed{\beta = \frac{1}{\sqrt{2} \times 15^{1/4}} \approx 0.359}$$

Summary

- Adiabatic renormalization is a powerful renormalization scheme to regularize UV divergences.
- Should be truncated up to an IR cut-off proportional to the horizon size.
- This cut-off should be fixed by a proper physical prescription.

Thank you for the attention