The one-loop power spectrum of curvature fluctuations in ultra slow-roll inflation

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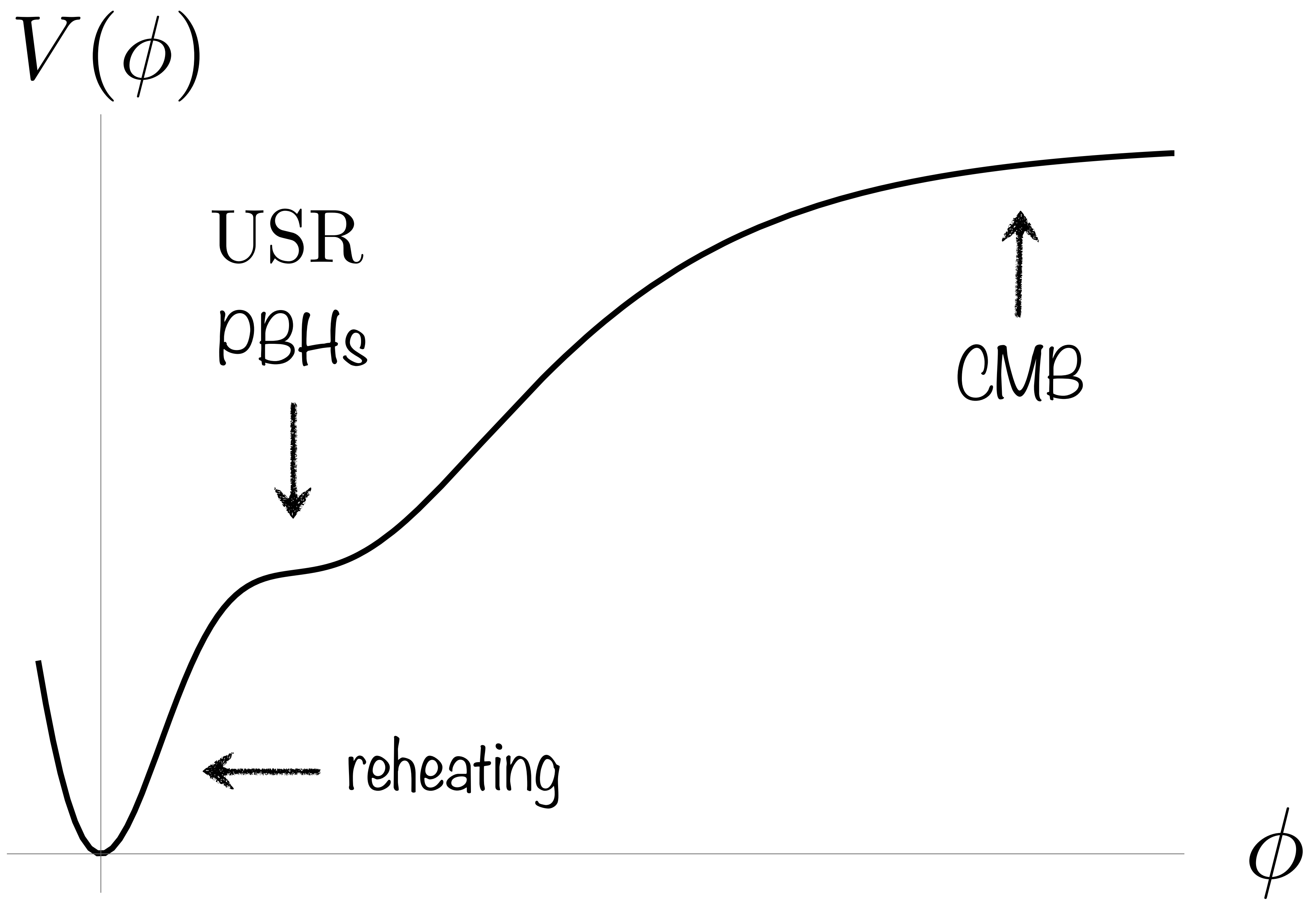
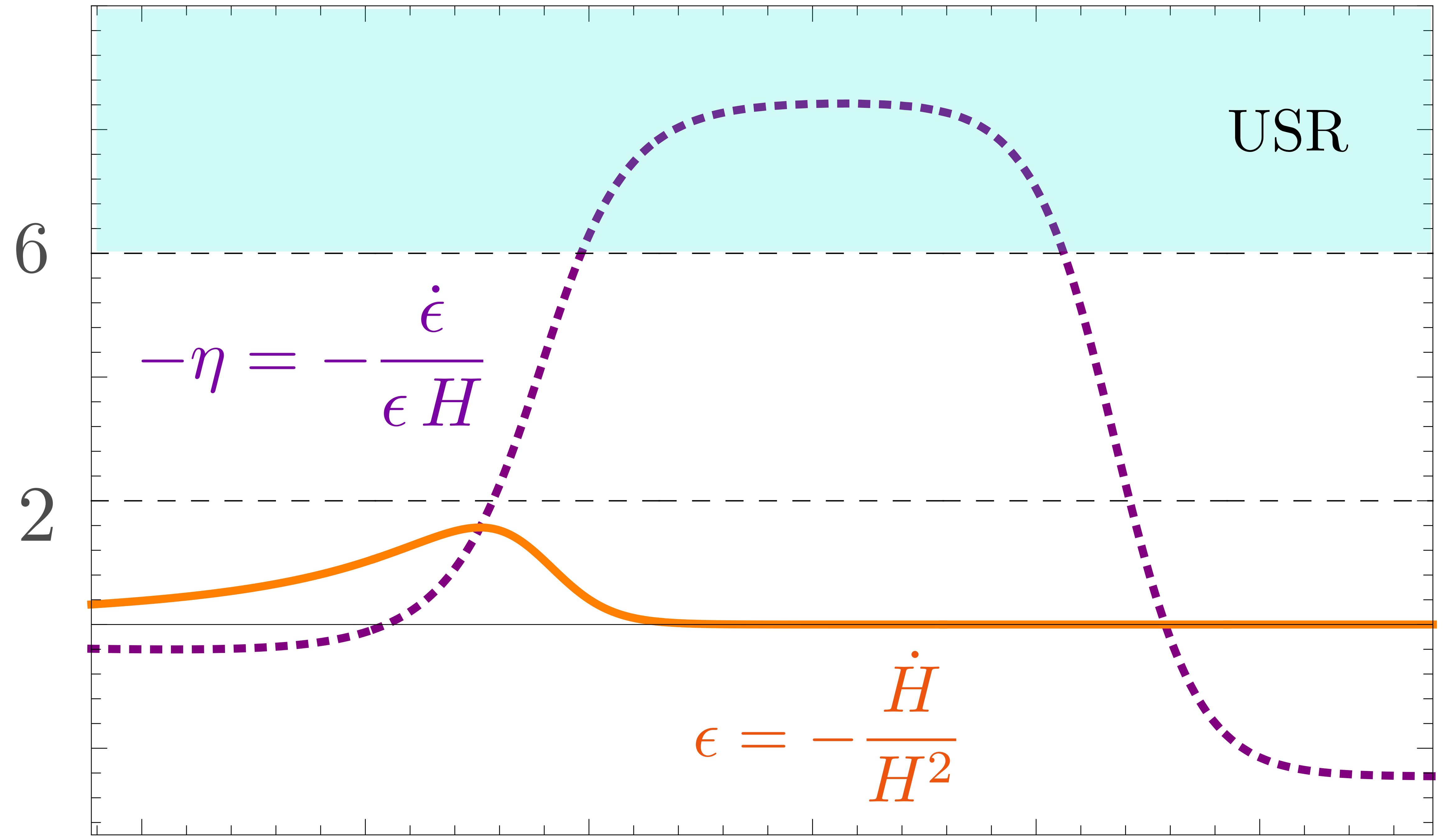
CERN TH Institute. Looping in the primordial universe

Does perturbation theory break in ultra slow roll (USR) **inflation** primordial black holes (PBH) dark matter (DM) models?

Ultra slow roll (USR)





$$\frac{d\zeta}{dN} \propto \exp\left[-\int (3+\epsilon+\eta)\, dN\right]$$

Examples of potentials for USR PBH DM

1. $$V = \sum_{n=2}^{4+\epsilon\,\mathcal{O}(5)} a_n \phi^n$$

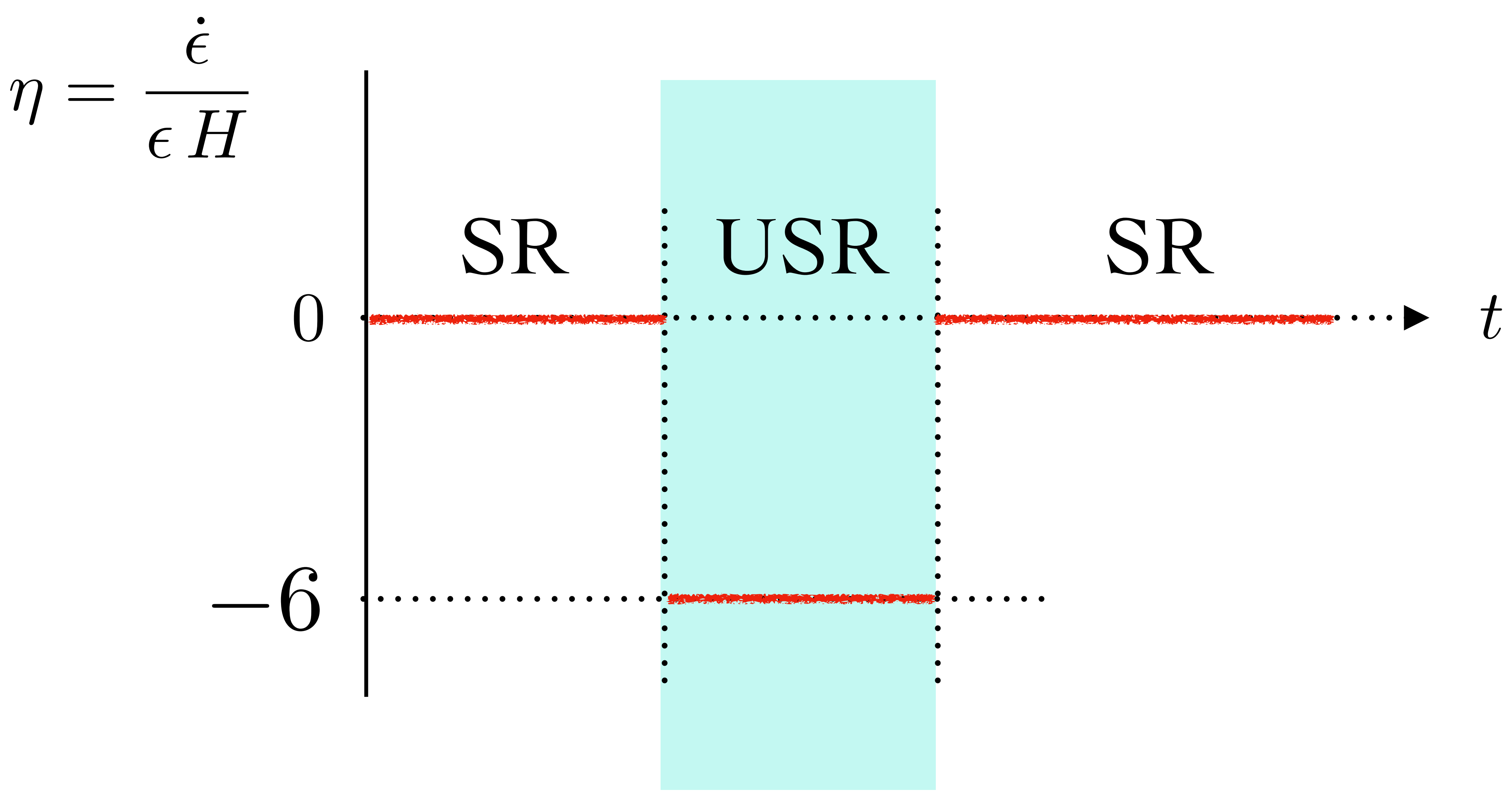
2. $$\frac{V(\phi)}{\phi^4} = \lambda(\phi_0) + \frac{1}{2}\beta_\lambda(\phi_0)\log\frac{\phi^2}{\phi_0^2} + \frac{1}{8}\beta'_\lambda(\phi_0)\left(\log\frac{\phi^2}{\phi_0^2}\right)^2 + \cdots$$

GB, Taoso 2017

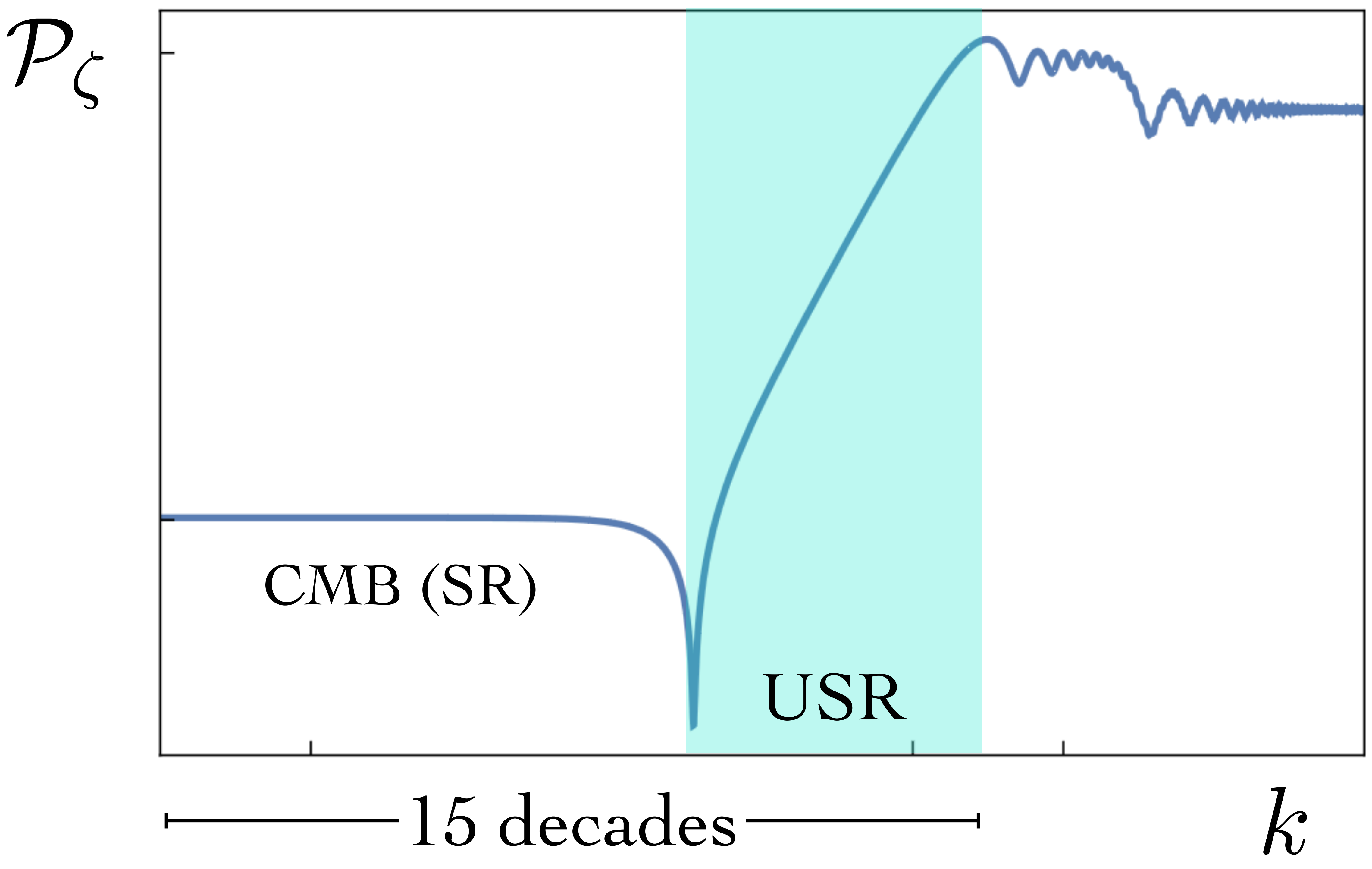
Breakdown of perturbation theory in USR inflation?

Claim: *a large enough tree-level primordial spectrum for PBH DM implies perturbation theory breaks at CMB scales.*

Kristiano and Yokoyama, 2022 & 2023



(with sharp transitions)



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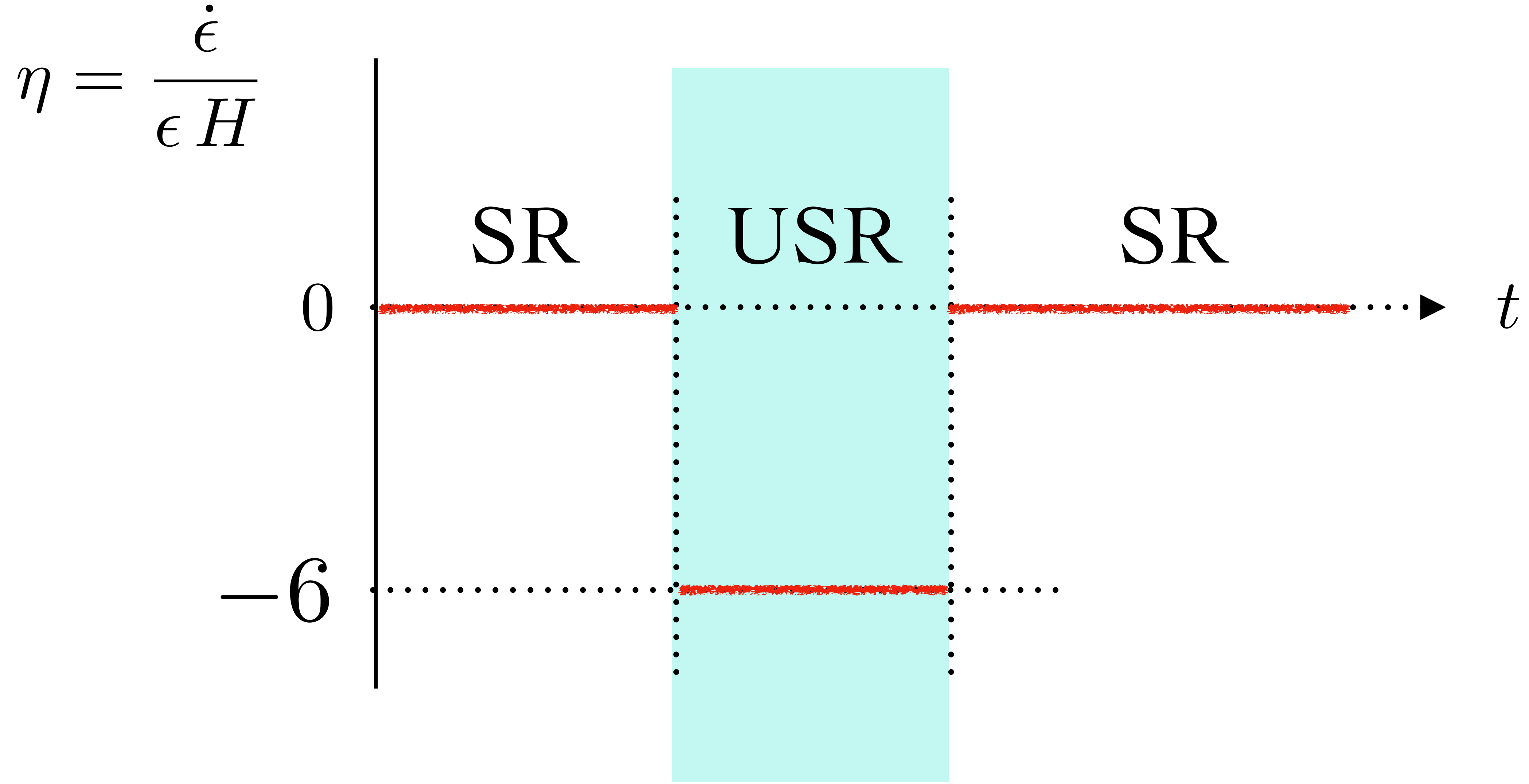


(with sharp transitions)

$$\mathcal{P}_\zeta \ll \frac{1}{(\Delta\eta)^2} \simeq 0.03$$

(for perturbation theory to hold)

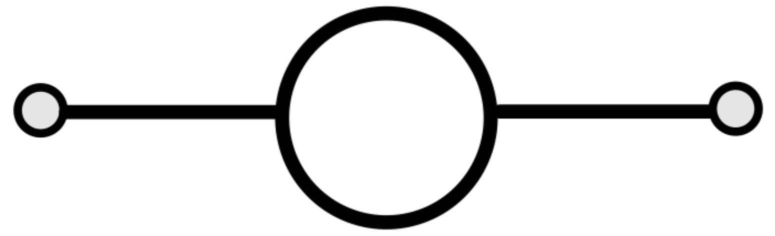
Breakdown of perturbation theory in USR inflation?

Claim: *a large enough tree-level primordial spectrum for PBH DM implies perturbation theory breaks at CMB scales*

Kristiano and Yokoyama, 2022 & 2023

- Method: Primordial spectrum at one-loop with the in-in formalism.

- A single cubic interaction: $H_{\rm int} = -\frac{M_P^2}{2}\int d^3x\, \epsilon\, \eta'\, a^2\, \zeta'\, \zeta^2$



- UV divergence: Cut-off given by location of spectral peak

One-loop power spectrum in ultra slow-roll inflation and implications for primordial black hole dark matter

April 2024

Guillermo Ballesteros and Jesús Gambín Egea

Summary of the analysis we do

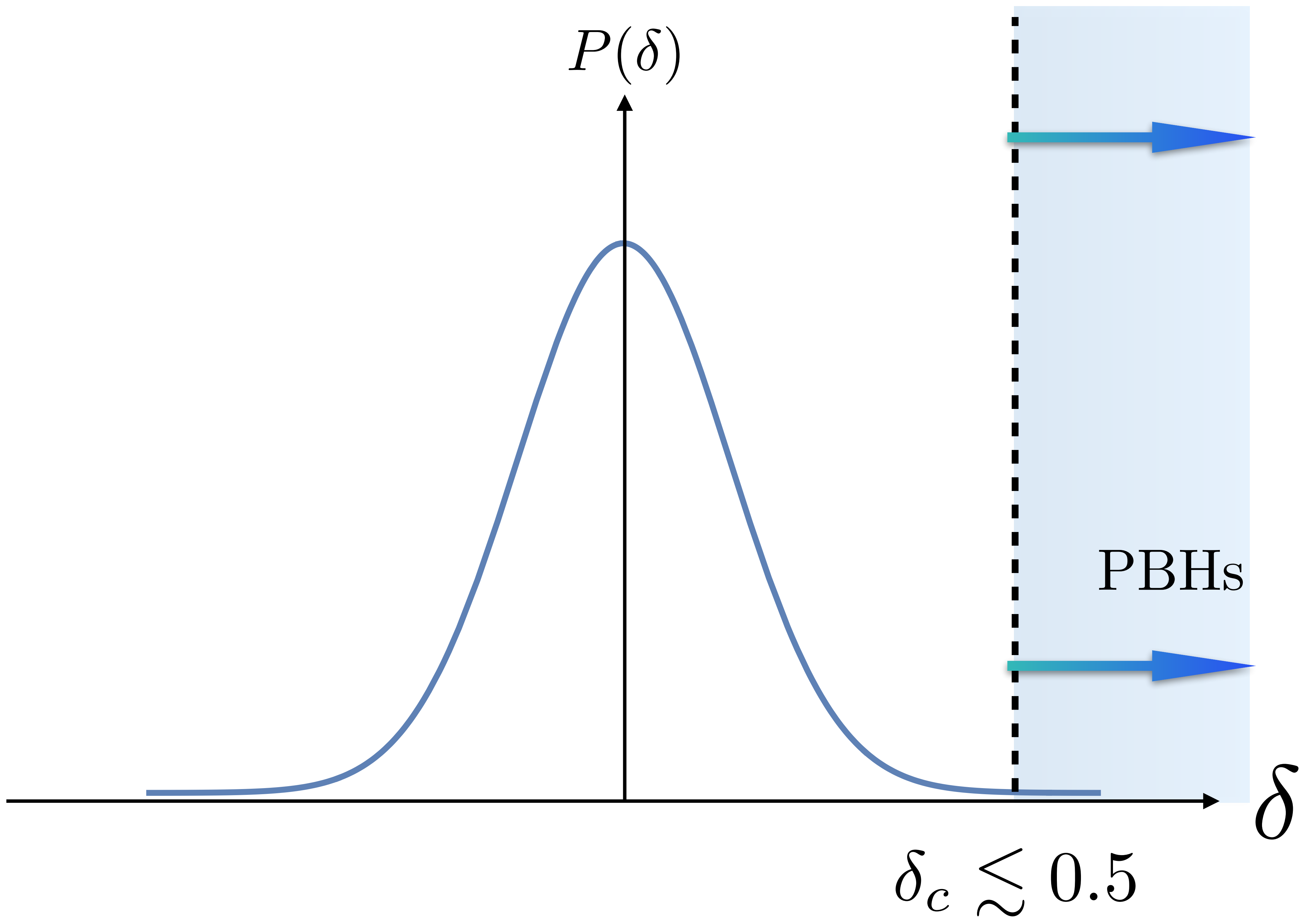
- Two-parameter piece-wise model of inflation.
    1. Duration of USR phase: ΔN
    2. Duration of transitions between SR and USR: δN

- We study $\mathcal{P}_\zeta(k)$ at all k using the in-in formalism
- $\delta\phi$-gauge $(\zeta = 0)$
- Include ll relevant cubic and quartic interactions, plus counterterms
- We use a cut-off to regularize the UV divergences
- **Result**: perturbation theory does not necessarily break in USR PBH DM models

Some key differences from previous works

- No complications with boundary terms thanks to the $\delta\phi$-gauge $(\zeta = 0)$
- We find that $\mathcal{P}_\zeta(k)$ explodes at one-loop in the limit $\delta N = 0$
- Previous analysis using a cutoff gave the cutoff a numerical value. Instead, we regularize in the UV with appropriate counterterms

PBH dark matter

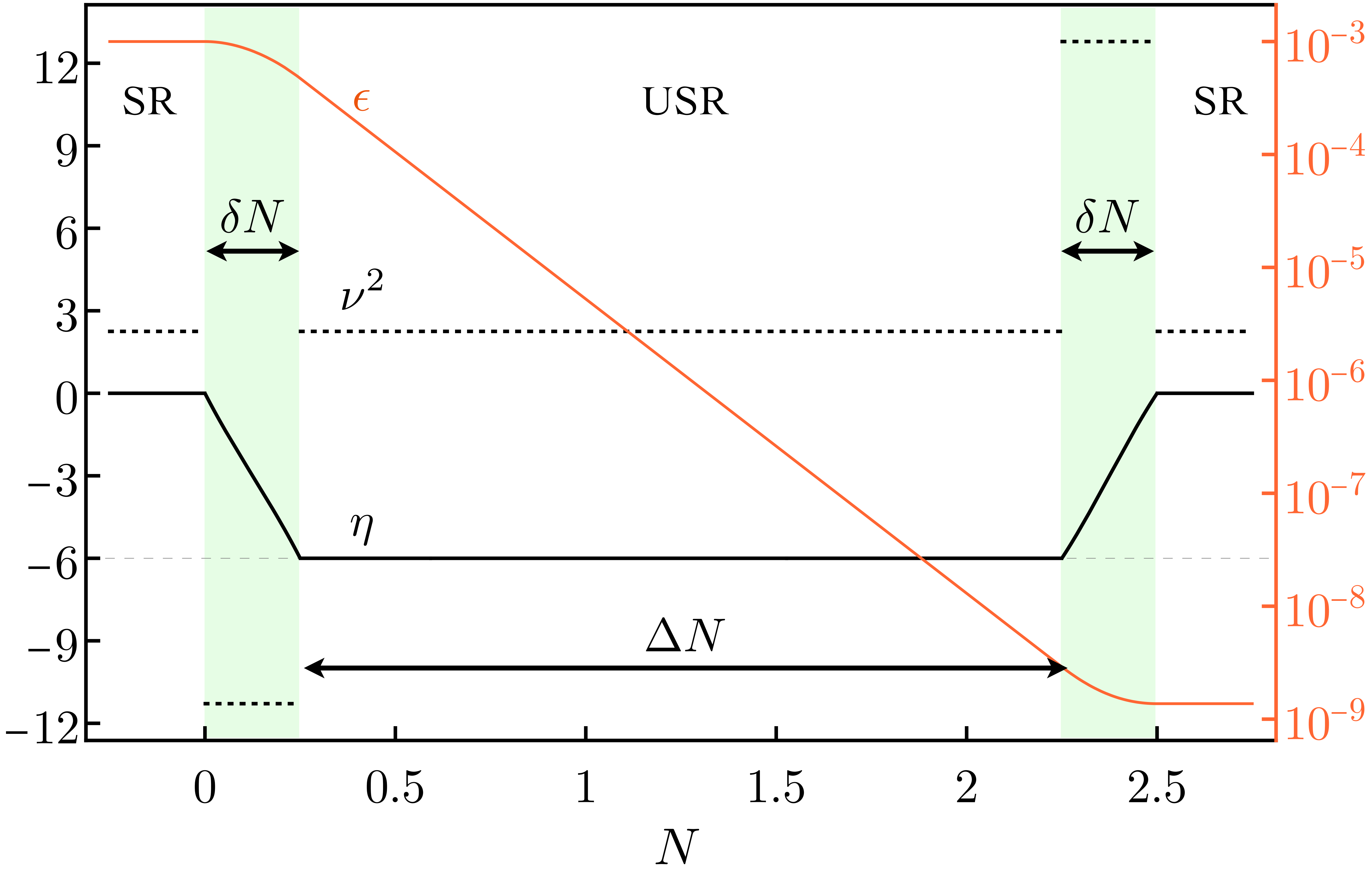


Abundance (Gaussian estimate)

$$f_{\mathrm{PBH}} = \frac{\Omega_{\mathrm{PBH}}}{\Omega_{\mathrm{DM}}} \propto \int_{\delta_c}^{\infty} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) d\delta$$

$$\sigma \sim \mathcal{P}_\zeta \sim 10^{-2} \quad \Longrightarrow \quad \frac{\Omega_{\mathrm{PBH}}}{\Omega_{\mathrm{DM}}} \sim 1$$

Two-parameter model



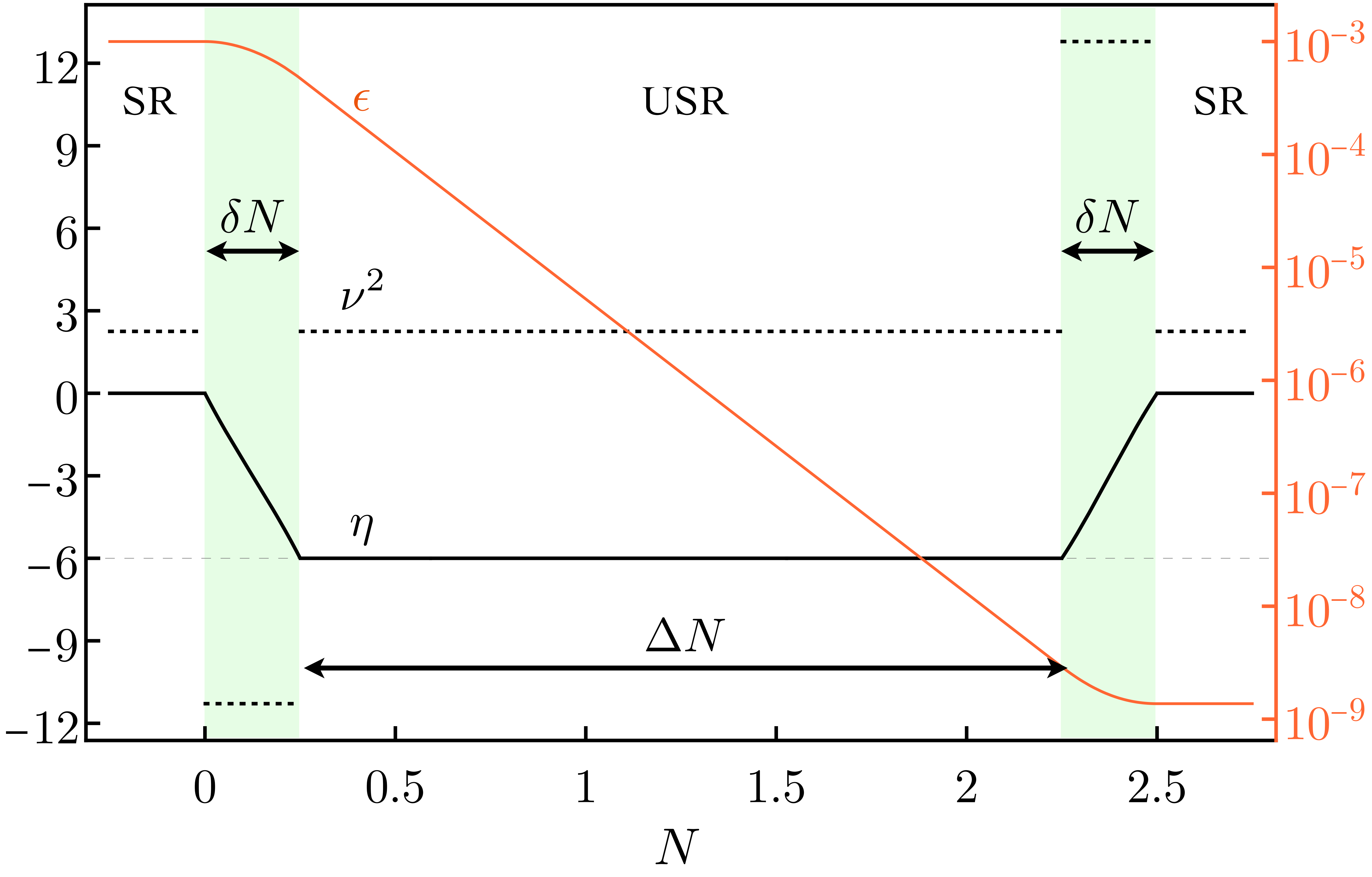
$$\eta = \begin{cases} 0 & \text{SR} \\ -6 & \text{USR} \end{cases}$$

$$\nu^2 = \frac{9}{4} + \frac{1}{2}\left(3\eta + \frac{\eta^2}{2} + \frac{\eta'}{aH}\right)$$

Piece-wise constant

Equal to 9/4 in SR & USR

Two-parameter model



$$V_{3,4} \sim \Delta\nu^2\, \delta(\tau - \tau_*)$$

$$|\nu^2| \to \frac{3}{\delta N} \quad \text{when} \quad \delta N \to 0$$

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Piece-wise constant
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Quantization of $\delta\phi$

We work in the interaction picture:

- $\delta\phi_k'' + 2aH\delta\phi_k' + (k^2 + a^2 V_2)\delta\phi_k = 0$ (free field)

- $\delta\phi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{kx}} \left(\delta\phi_k(\tau) a_{\mathbf{k}} + \delta\phi_k^*(\tau) a_{-\mathbf{k}}^\dagger\right)$ (with canonical comm. rel.)

$$\delta\phi_k(\tau \to -\infty) = e^{-ik\tau}/\sqrt{2ka^2} \qquad \text{(Bunch-Davies b.c.)}$$

$\delta\phi_k(\tau) = (-k\tau)^{3/2} \left(\alpha_k J_\nu(-k\tau) + \beta_k Y_\nu(-k\tau)\right)$ in each phase

$\delta\phi_k \in C^1$ across phase boundaries

Action for fluctuations in the $\delta\phi$-gauge $(\zeta = 0)$

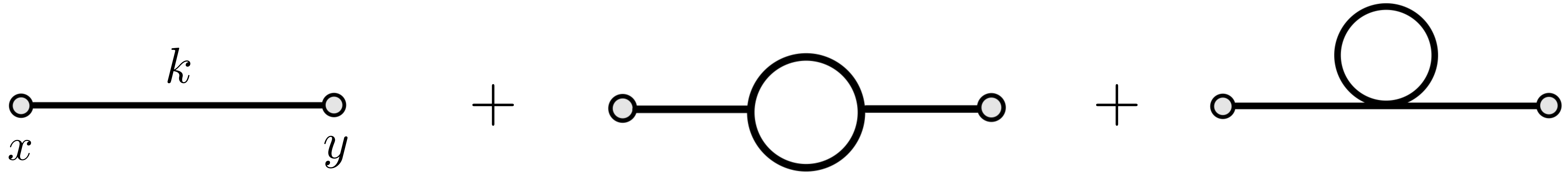
$$S = \int \mathrm{d}\tau\, \mathrm{d}^3\mathbf{x}\, \left[\frac{a^2}{2}\left((\partial_\tau \delta\phi)^2 - (\partial_i \delta\phi)^2\right) - a^4 \sum_{n\geq 2} \frac{V_n\, \delta\phi^{\,n}}{n!}\right]$$

- The interactions coming from the metric are suppressed by ϵ

$$\nu^2 = \frac{9}{4} + \frac{1}{2}\left(3\eta + \frac{\eta^2}{2} + \frac{\eta'}{aH}\right)$$

($'$ = Conformal time derivative)

$$a^2 V_2 = -(aH)^2(\nu^2 - 9/4)\,, \quad a^2 V_3 = -\frac{aH(\nu^2)'}{\sqrt{2\epsilon} M_P}\,, \quad a^2 V_4 = -\frac{1}{2\epsilon M_P^2}\left((\nu^2)'' - aH(\nu^2)'\left(1 + \frac{\eta}{2}\right)\right)$$



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$$V_{3,4} \sim \Delta\nu^2\, \delta(\tau - \tau_*) \qquad\qquad |\nu^2| \to \frac{3}{\delta N} \quad \text{when} \quad \delta N \to 0$$

Two-point correlation of ζ at late times

$\delta\phi$ – gauge

$$ds^2 = -N^2\, dt^2 + a^2(t)\left(N^i\, dt + dx^i\right)\left(N^j\, dt + dx^j\right) \exp\left(h_{ij}\right), \quad \phi = \phi_0 + \delta\phi$$

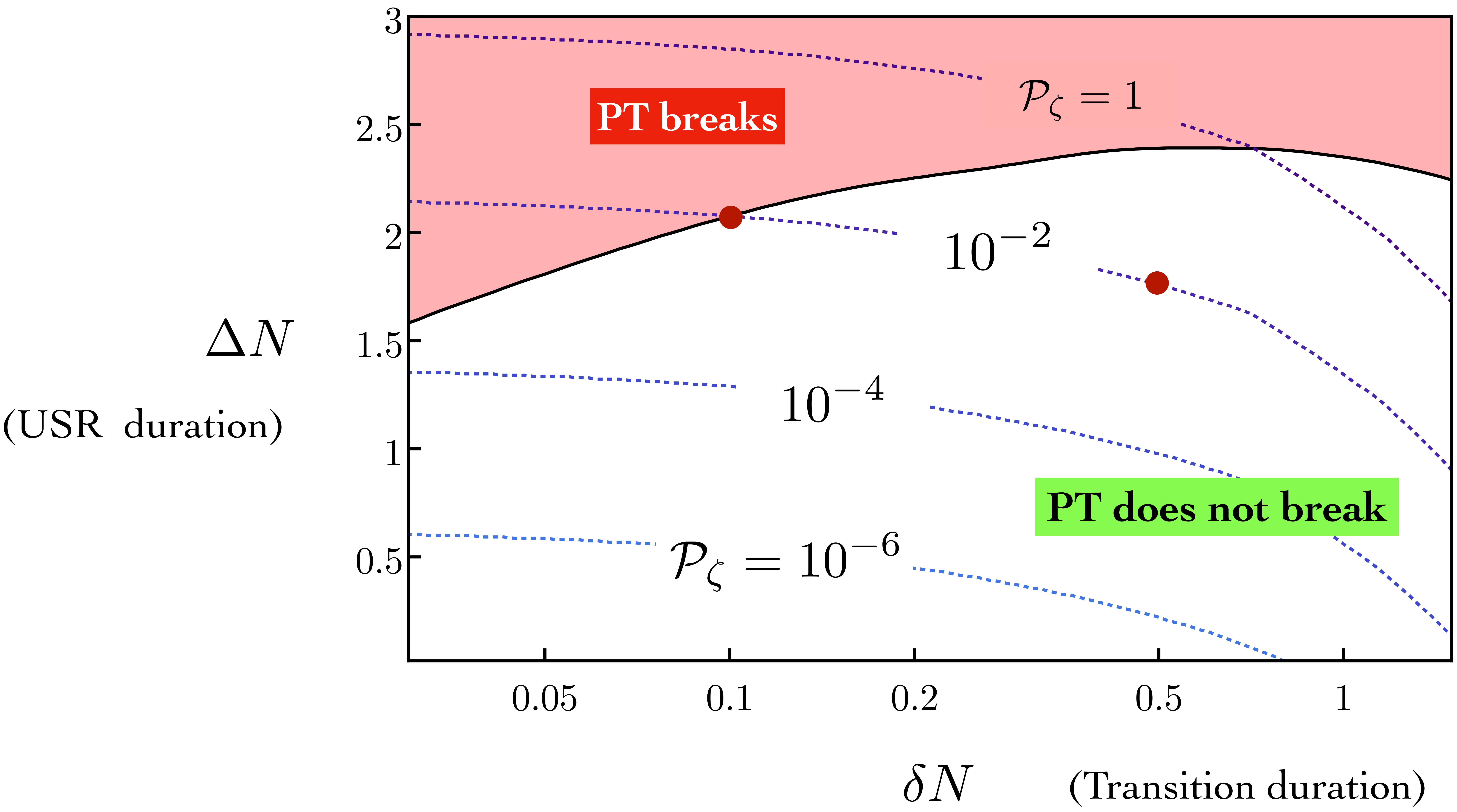
ζ – gauge

$$ds^2 = -N^2\, dt^2 + a^2(t)\left(N^i\, dt + dx^i\right)\left(N^j\, dt + dx^j\right) \exp\left(2\,\zeta\,\delta_{ij} + h_{ij}\right), \quad \phi = \phi_0$$

$$\zeta = -\frac{\delta\phi}{\sqrt{2\epsilon}M_P} + \frac{\eta}{4}\left(\frac{\delta\phi}{\sqrt{2\epsilon}M_P}\right)^2 - \frac{\eta^2 - \eta\epsilon_3}{12}\left(\frac{\delta\phi}{\sqrt{2\epsilon}M_P}\right)^3 + \mathcal{O}\left(\frac{\delta\phi}{M_P}\right)^4$$

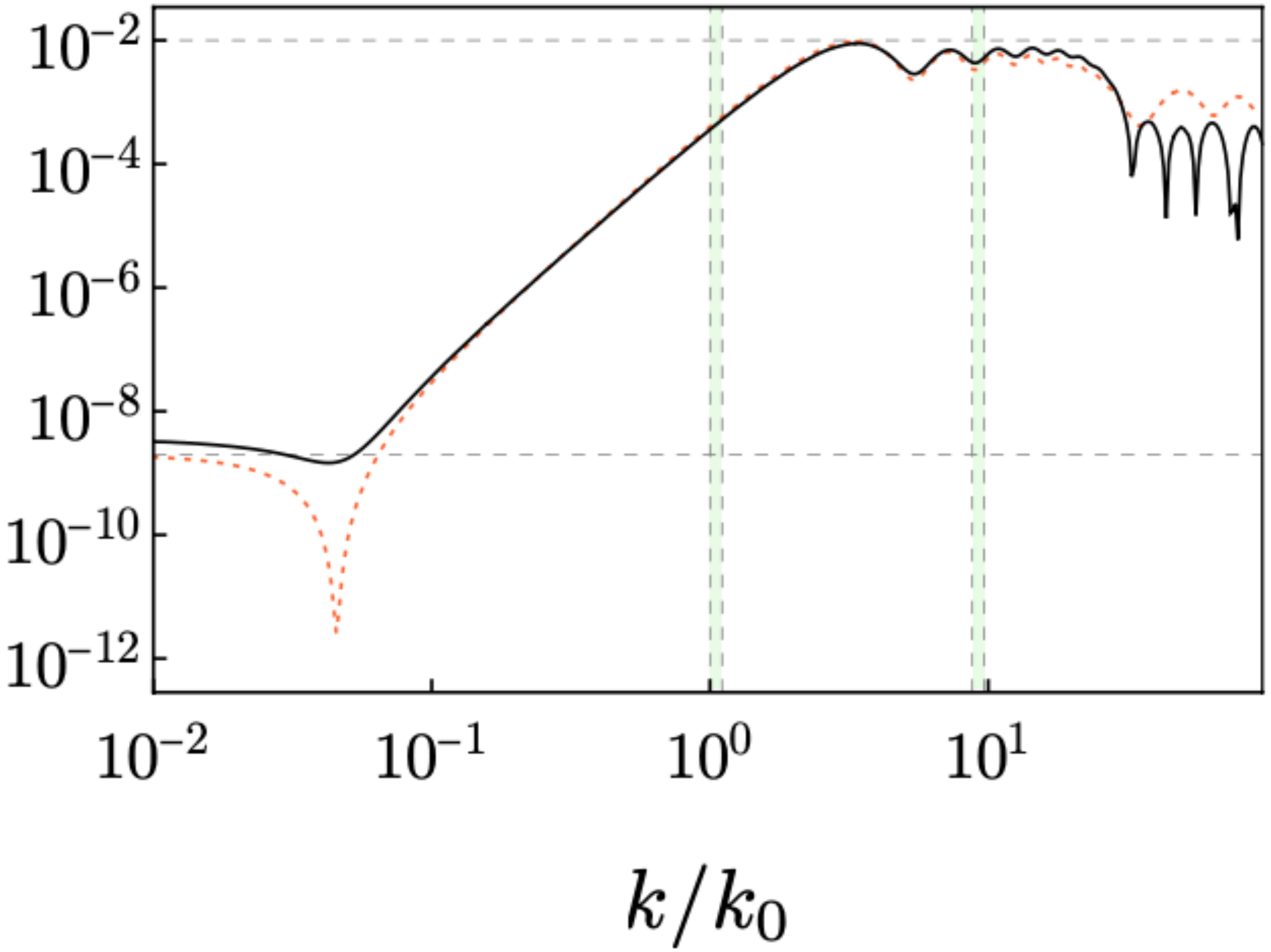
For $\eta = 0$: $\quad \mathcal{P}_\zeta = \frac{1}{2\epsilon\, M_P^2}\mathcal{P}_{\delta\phi} = \frac{k^3}{4\pi^2 M_P^2 \epsilon(\tau)}|\delta\phi_k(\tau)|^2$

Does perturbation theory (PT) break?

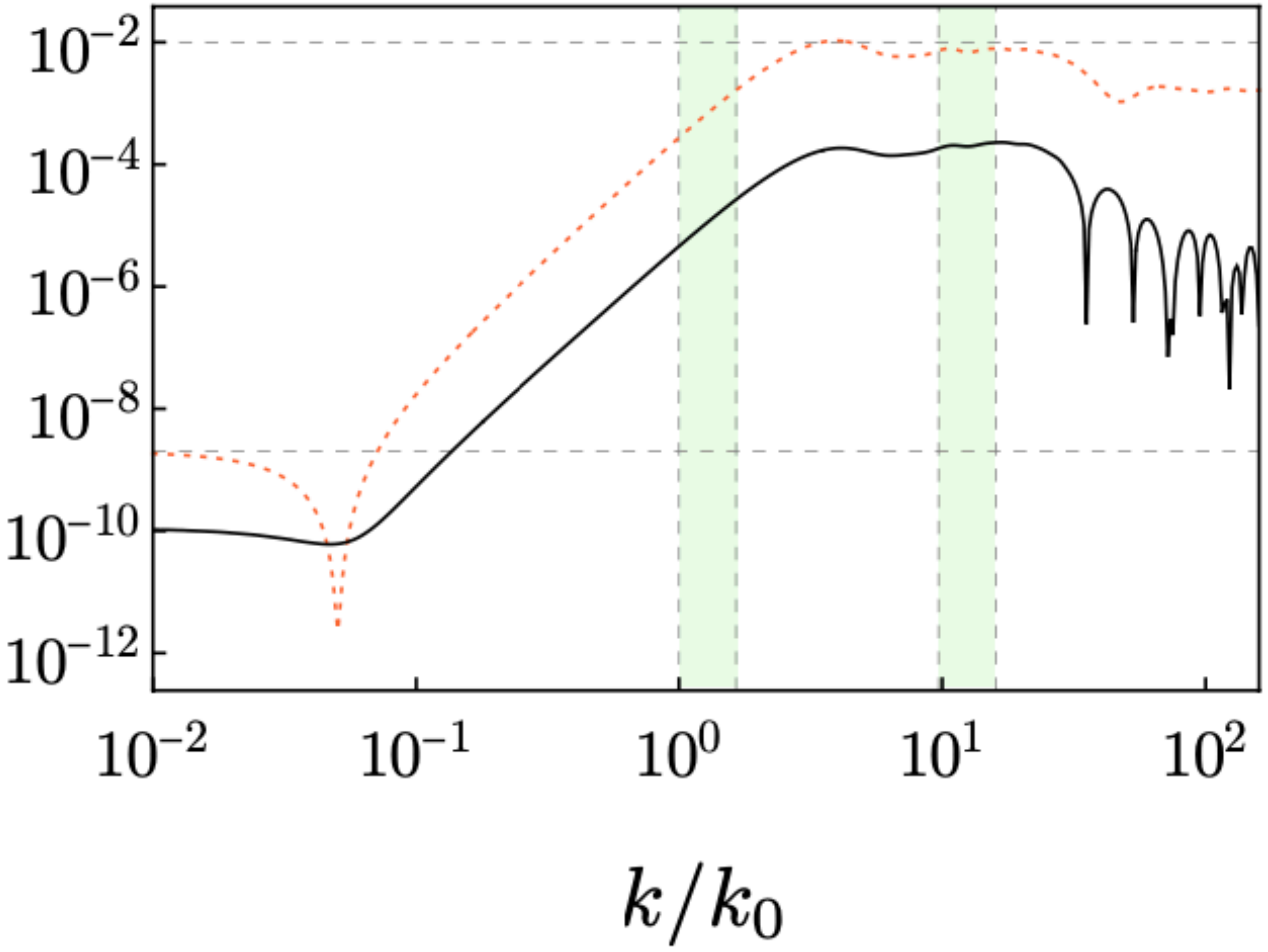


Two examples of $\mathcal{P}_\zeta$

$\delta N = 0.1\,, \quad \Delta N = 2$



$\delta N = 0.5\,, \quad \Delta N = 1.8$



In-in formalism

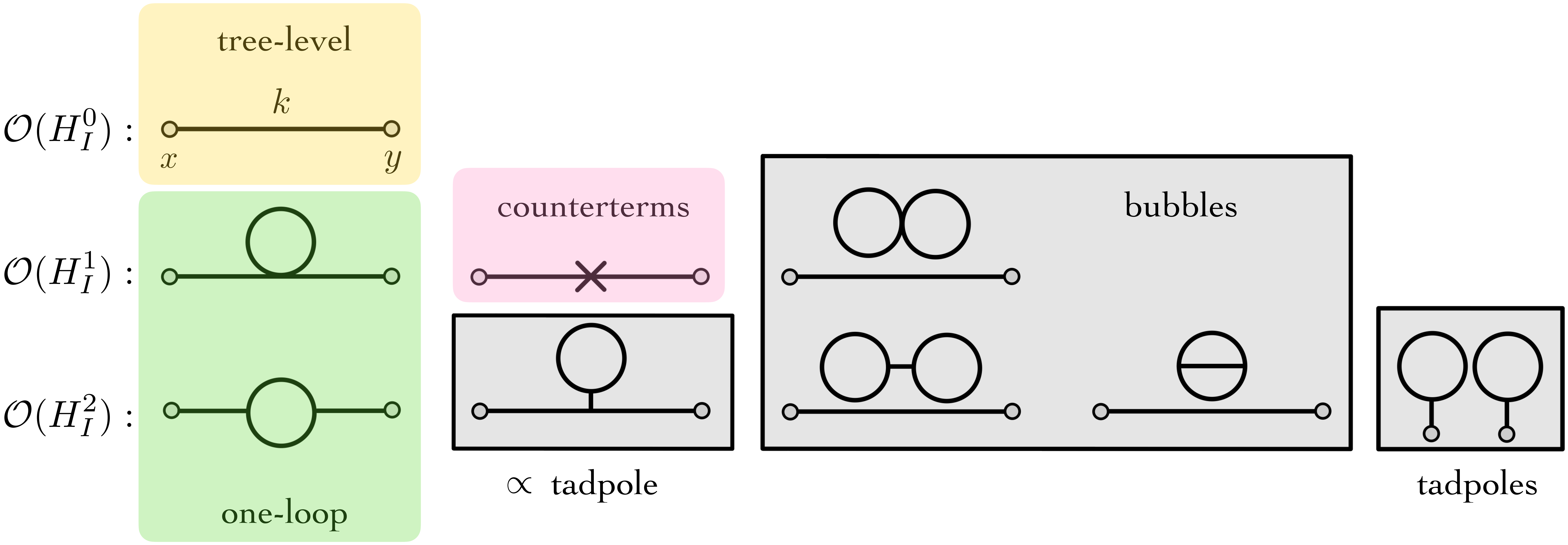
$$\langle Q(t)\rangle = \frac{{}_I\langle 0|\, F^{-1}(t,-\infty^+) Q_I(t) F(t,-\infty^-)\, |0\rangle_I}{{}_I\langle 0|\, F^{-1}(t,-\infty^+) F(t,-\infty^-)\, |0\rangle_I} \;=\; {}_I\langle 0|\, F^{-1}(t,-\infty^+) Q_I(t) F(t,-\infty^-)\, |0\rangle_I \,\Big|_{\text{no bubbles}}$$

$$F(t,-\infty^-) = 1 - i\int_{-\infty}^{t} \mathrm{d}t' H_I(t'_-) - \int_{-\infty}^{t}\mathrm{d}t' \int_{t'}^{t} \mathrm{d}t'' H_I(t''_-) H_I(t'_-)$$

$$t_\pm \equiv t(1 \pm i\omega)\,, \quad \omega > 0$$

$$\begin{aligned}\langle Q(t)\rangle = \langle 0|\, Q_I(t)\, |0\rangle + 2\,\mathrm{Im}\Big\{\int_{-\infty}^{t} \mathrm{d}t' \,\langle 0|\, Q_I(t) H_I(t'_-)\, |0\rangle\Big\} \\ + 2\,\mathrm{Re}\Big\{\int_{-\infty}^{t}\mathrm{d}t' \int_{t'}^{t} \mathrm{d}t'' \,\langle 0|\, \big(H_I(t''_+) Q_I(t) - Q_I(t) H_I(t''_-)\big)\, H_I(t'_-)\, |0\rangle\Big\}\end{aligned}$$

Diagrams



tree-level
k
x
y
$\mathcal{O}(H_I^0)$:
$\mathcal{O}(H_I^1)$:
$\mathcal{O}(H_I^2)$:
one-loop
counterterms
$\propto$ tadpole
bubbles
tadpoles

UV regularization

UV Cutoff

$$\int_0^\infty \mathrm{d}^3p \to \int_0^{a(t)\Lambda_{\mathrm{UV}}} \mathrm{d}^3p\,, \qquad \int_{-\infty}^{\tau} \mathrm{d}\tau' \int_{\tau'}^{\tau} \mathrm{d}\tau'' \to \int_{-\infty}^{\tau+1/(a(\tau)\Lambda_{\mathrm{UV}})} \mathrm{d}\tau' \int_{\tau'+1/(a(\tau')\Lambda_{\mathrm{UV}})}^{\tau} \mathrm{d}\tau''$$

Counterterms for computing $\mathcal{P}_\zeta$

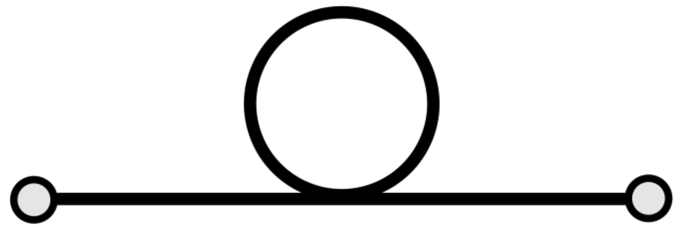
1. $\phi \to (1+\delta_\phi)\,\phi_R$
2. $V_2 \to V_{R,2} + \delta V$

δ_ϕ and δV are functions of time

$$H_I(\tau) = \int \mathrm{d}^3\mathbf{x} \left[a^4 \left(\frac{V_3 \delta\phi^3}{3!} + \frac{V_4 \delta\phi^4}{4!}\right) + (a^2 \delta_\phi\, \delta\phi\, \delta\phi')' + 2a^2\, \delta\phi\, \left(a^2 \tilde{\delta}_V - \delta_\phi \nabla^2\right) \delta\phi\right]$$

$$\tilde{\delta}_V \;=\; \delta_\phi V_2 + \delta_V/4$$

Quartic

 can be completely absorbed by 

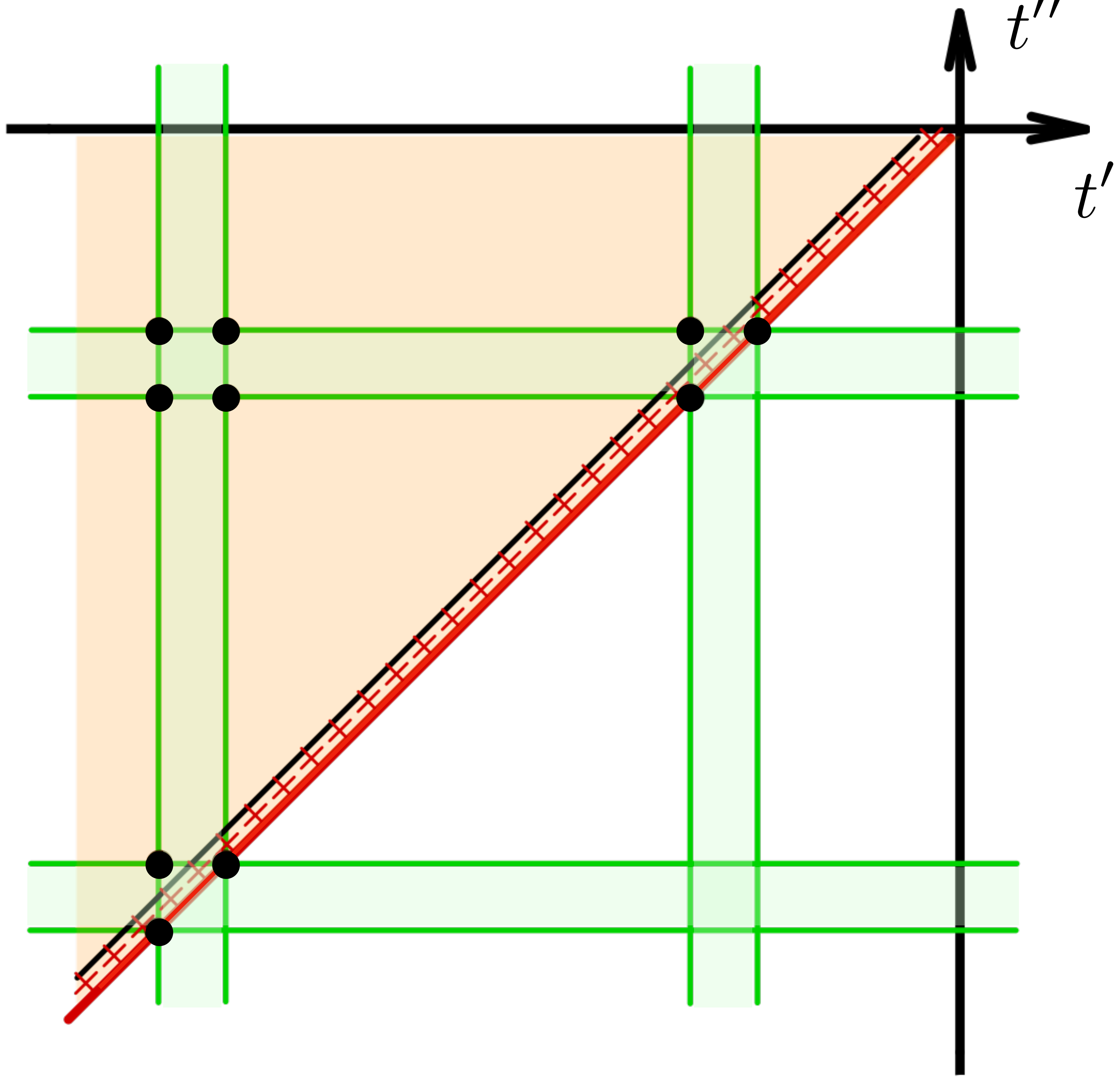
$$\mathcal{P}_\zeta^{1\mathrm{l},\mathrm{V}_4}(\tau,k) = \frac{k^3}{4\pi^2 M_P^2 \epsilon(\tau)}\,\mathrm{Im}\left\{\delta\phi_k^2(\tau)\int_{-\infty}^{\tau}\mathrm{d}\tau' a^4(\tau')V_4(\tau')\delta\phi_k^{*2}(\tau')\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left|\delta\phi_p(\tau')\right|^2\right\}$$

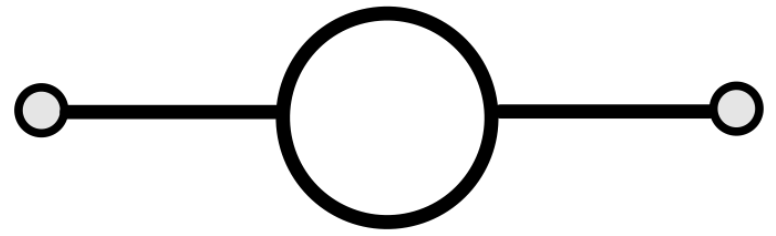
$$\mathcal{P}_\zeta^{\mathrm{ct}}(\tau,k) = \frac{k^3}{4\pi^2 M_P^2 \epsilon(\tau)}\left(2\,\delta_\phi\left|\delta\phi_k(\tau)\right|^2 + 8\,\mathrm{Im}\left\{\delta\phi_k^2(\tau)\int_{-\infty}^{\tau}\mathrm{d}\tau' a^2(\delta_\phi k^2+\tilde{\delta}_V)\delta\phi_k^{*2}\Big|_{\tau'}\right\}\right)$$

$$\tilde{\delta}_V \to \tilde{\delta}_V - \frac{1}{8}a^2 V_4\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left|\delta\phi_p\right|^2$$

Cubic

Two time integrals:





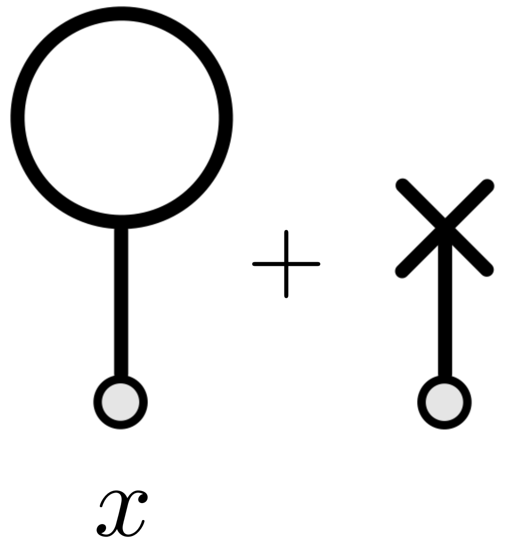
Becomes UV finite after removing the points along the diagonal with the cutoff.

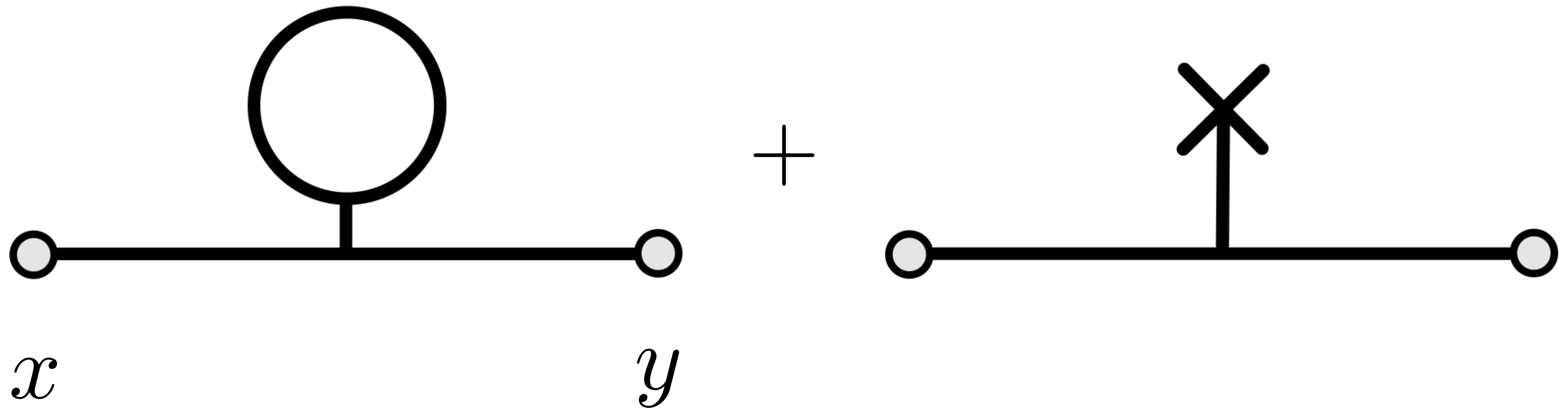
$\tau_\pm \equiv \tau(1 \pm i\omega)$ prescription is key

- If we had kept those points, there would be a UV div. that cannot be absorbed by the counterterms (different k dependence).
- We expect that in dim. reg. the difference is compensated by the finite part of the counterterms.

Tadpoles

In general, they can contribute to the power spectrum, but we can remove them using the counterterms, imposing $\langle \delta\phi(x) \rangle = 0$.

$\langle \delta\phi(x) \rangle =$ 

$\langle \delta\phi(x)\, \delta\phi(y) \rangle =$ 

$+ \cdots$

$$H_I(\tau) = \int \mathrm{d}^3\mathbf{x} \left[a^4 \frac{V_3 \delta\phi^3}{3!} + \delta_{\mathrm{tp}} \delta\phi \right]$$

$$2\delta_{\mathrm{tp}}(\tau) = -a^4(\tau) V_3(\tau) \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} |\delta\phi_p(\tau)|^2$$

Instantaneous transition limit $(\delta N \to 0)$ in both gauges

$$S = \int \mathrm{d}\tau \mathrm{d}^3\mathbf{x}\, M_P^2 a^2 \epsilon \left((\zeta')^2 - (\partial\zeta)^2 + \frac{\eta'}{2}\zeta'\zeta^2 \right)$$

Interaction Hamiltonian in the interaction picture (the conjugate momentum only sees the free action):

$$H_I(\tau) = \int \mathrm{d}^3\mathbf{x}\, M_P^2 a^2 \epsilon \left(-\frac{\eta'}{2}\zeta'\zeta^2 + \frac{(\eta')^2}{16}\zeta^4 \right)$$

See also Firouzjahi, 2023

- Exact agreement between both gauges in the limit $\delta N \to 0$ for the two-point function of ζ
- We also checked that the bispectrum of ζ agrees in both gauges in the same limit.

$\mathcal{P}_\zeta^{1l} \propto 1/\delta N^2$ removing the diagonal points

$\mathcal{P}_\zeta^{1l} \propto 1/\delta N$ not removing them (a different kind of regulator)

Limit $k \to 0$

$$\mathcal{P}_\zeta = \mathcal{P}_\zeta^{\text{tree}} + \mathcal{P}_\zeta^{1\text{l}} + \mathcal{P}_\zeta^{\text{ct}}$$

Tree-level, one-loop and counterterm contributions to $\mathcal{P}_\zeta(\tau, k)$ are all scale invariant in this limit

Do we find a difference between tree-level and one-loop in this limit?

Yes, but the freedom in the time-dependence of the counterterms can keep ζ constant

Summary and outlook

- Perturbation theory does not necessarily break in USR PBH DM models:
    1. Duration of USR phase:
    2. Duration of transitions between SR and USR:

- Modes that leave well before USR cannot be used to answer the question:
    1. Scale invariance
    2. Separation of scales

Summary and outlook

- The freedom in the time dependence of the "couplings" of the model (EFT coeffs.) is both:
    1. A blessing: simplifies the calculation greatly
    2. A curse: it prevents us from doing a full renormalization
        1. Finite parts from counterterms
        2. Show conservation of ζ explicitly.

- What would be better: a complete calculation for a potential $V(\phi)$ with a symmetry preserving regularization scheme.

Lacking that, we have used $\mathcal{P}_\zeta^{\rm ct} \sim \mathcal{P}_\zeta^{\rm 1l} \ll \mathcal{P}_\zeta^{\rm tl}$ as a proxy for the validity of perturbation theory.