LPENS
LABORATOIRE DE PHYSIQUE
DE L'ÉCOLE NORMALE SUPÉRIEURE

Stochastic inflation: key insights, latest advances and future directions

Chiara Animali

Thursday, 31 October 2024

Looping in the Primordial Universe, CERN

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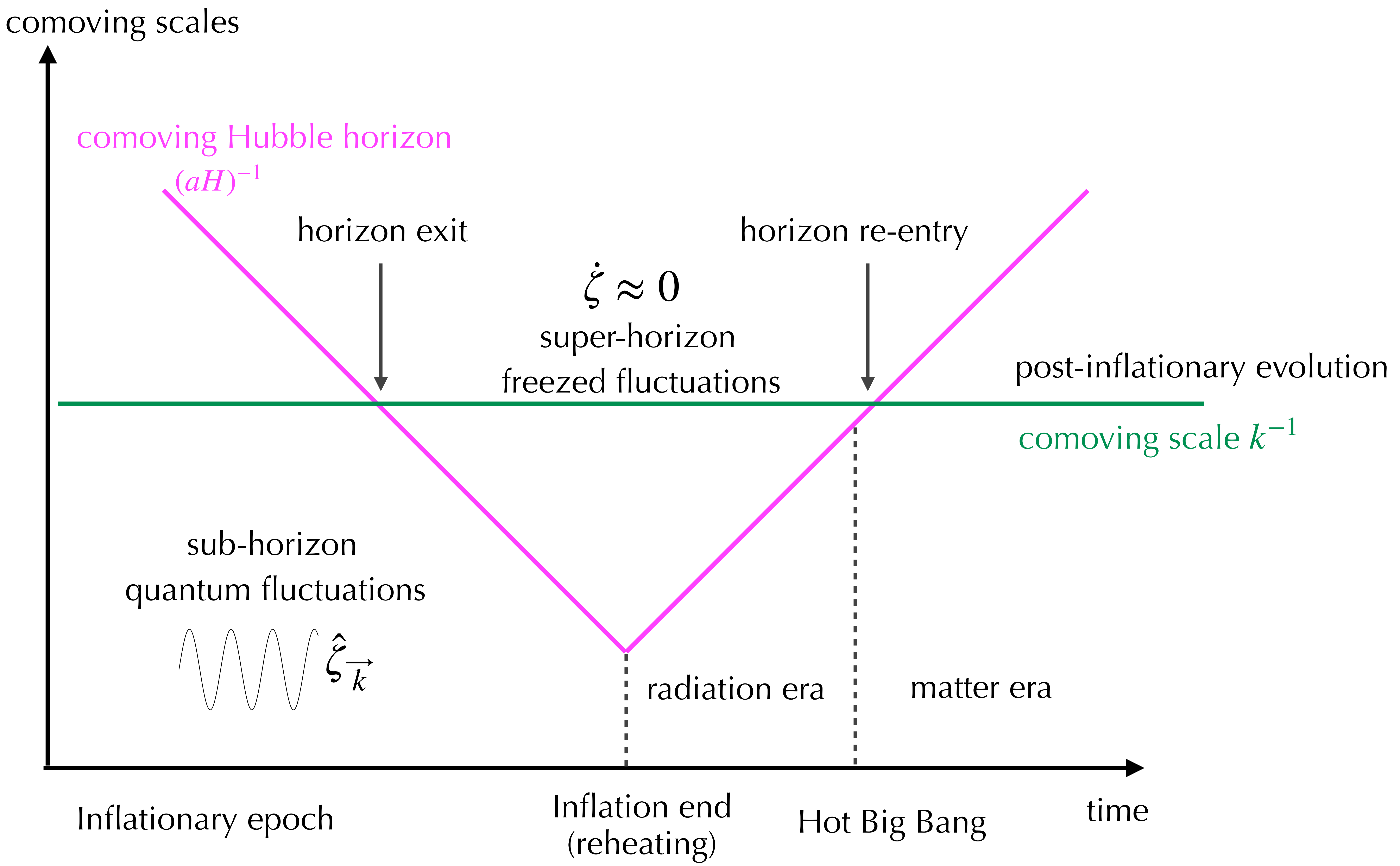
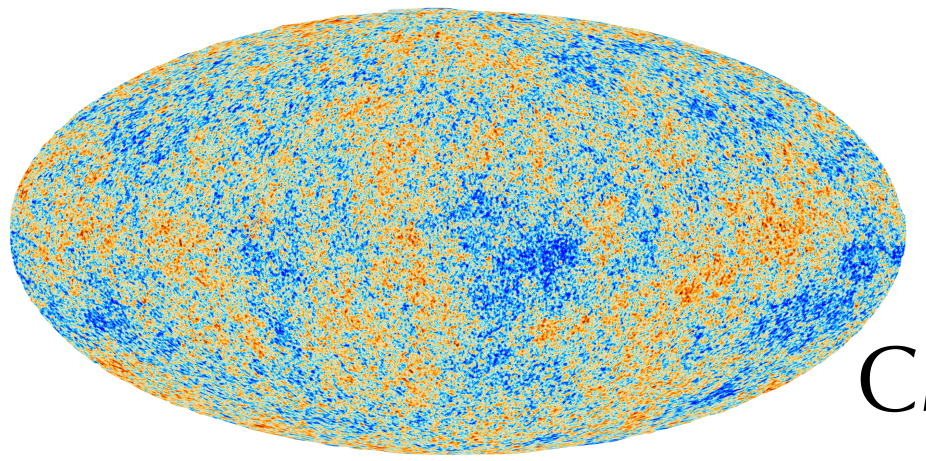
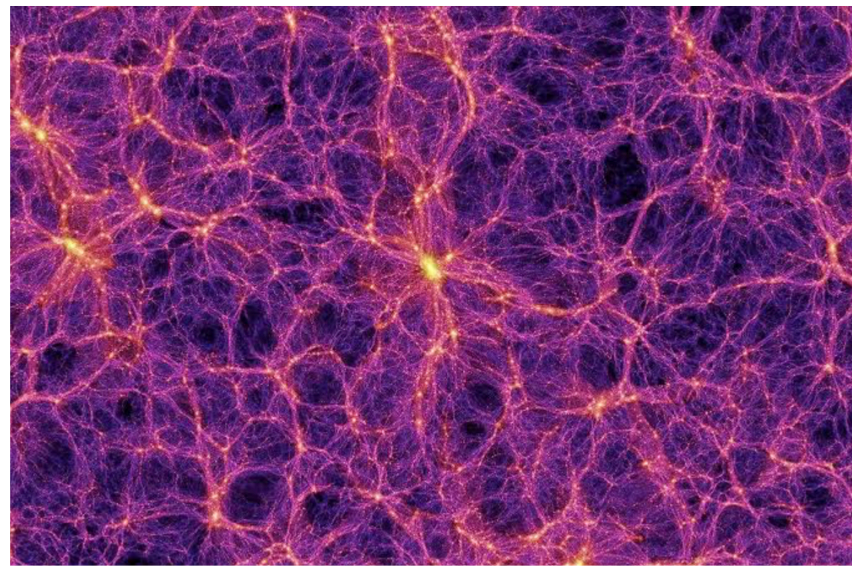
Generation of cosmological perturbation

Inflation explains causally the small fluctuations we observe in the universe and the structures at large scales





CMB



LSS

Cosmological Perturbation Theory

Splitting of the full metric and of the matter fields (inflaton) into a background part and small perturbations:

$$g_{\mu\nu}(\vec{x}, t) = g_{\mu\nu}(t) + \hat{\delta} g_{\mu\nu}(\vec{x}, t)$$

homogeneous and isotropic background classical solution

$$\phi(\vec{x}, t) = \phi(t) + \hat{\delta}\phi(\vec{x}, t)$$

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Small scales: density fluctuations could be large ✗
(PBHS, SIGWs)

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Dynamics of eternal inflation

Stochastic inflation

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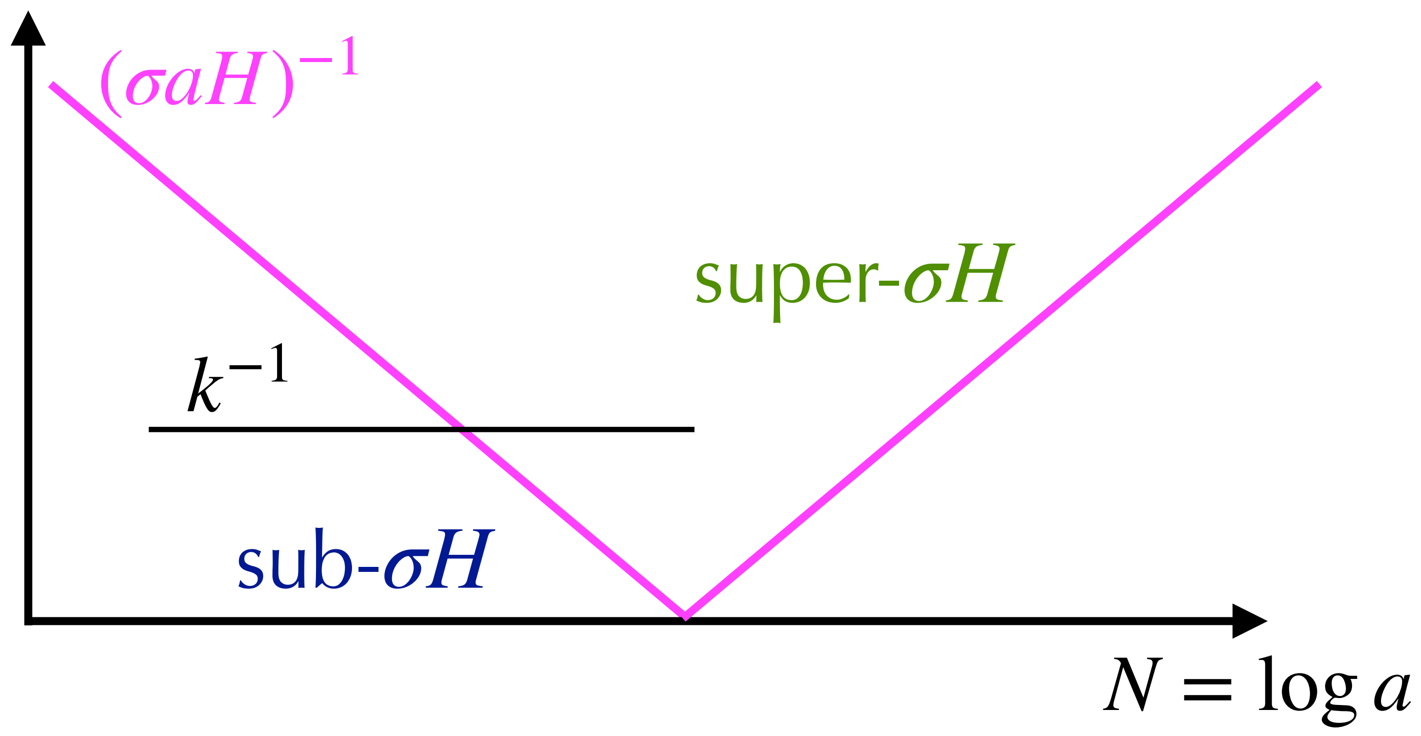
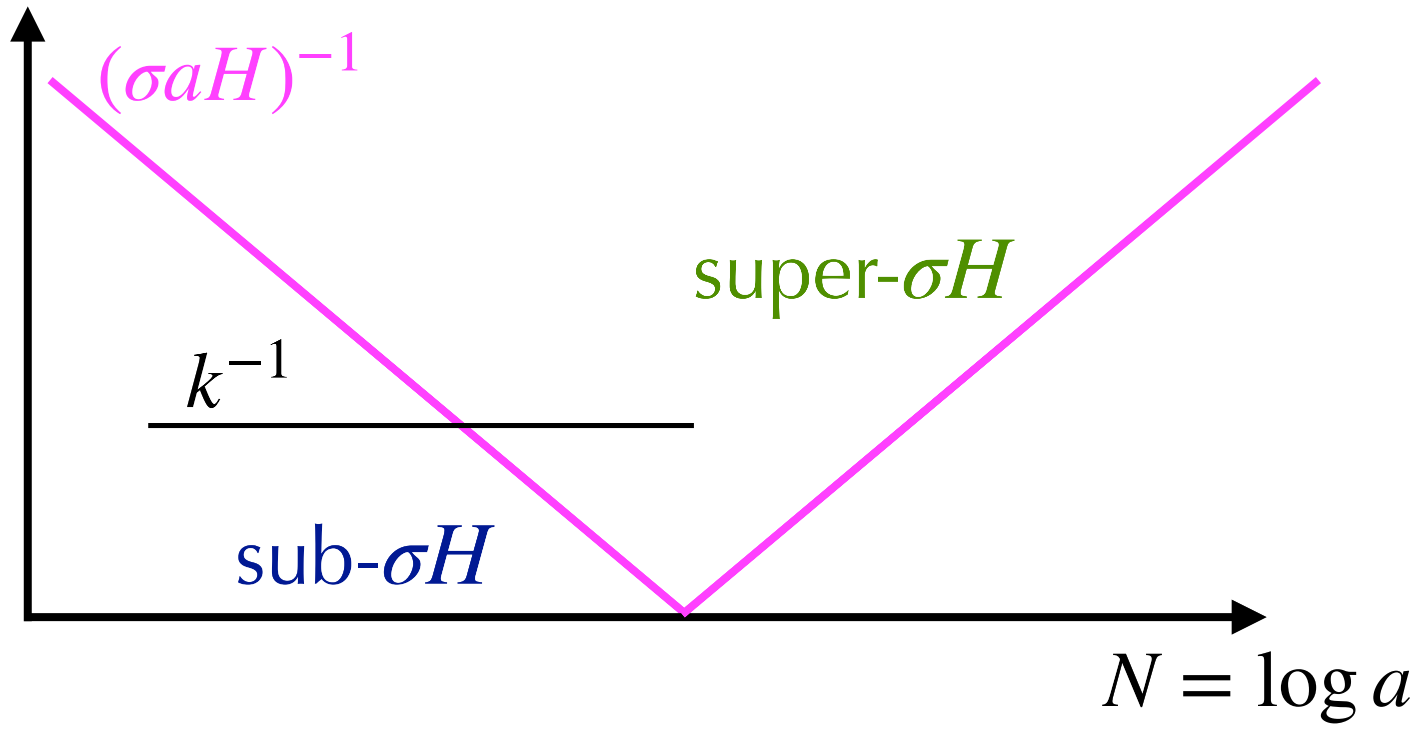


Sub-Hubble scales: small amplitude, cosmological perturbation theory

Super-Hubble scales: large perturbations, stochastic inflation

Stochastic inflation



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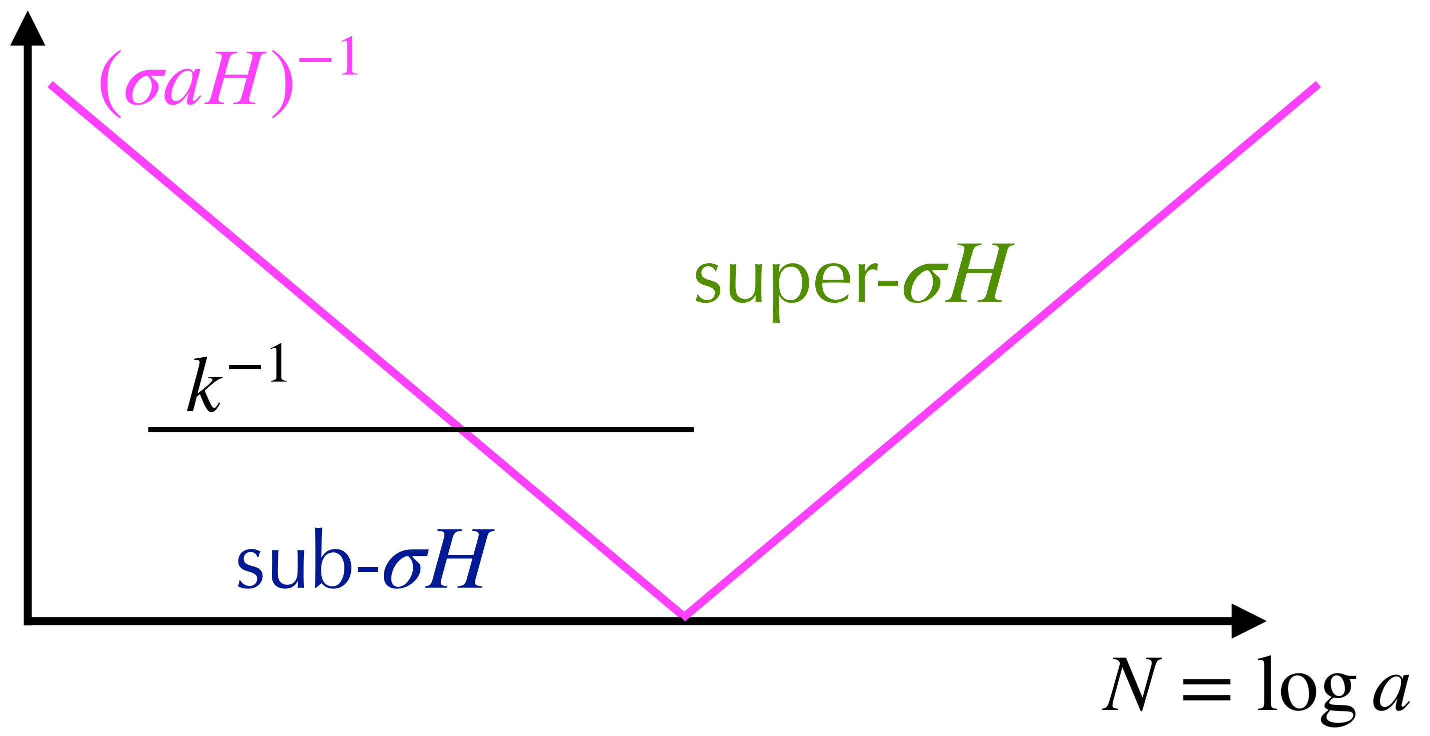
Super-Hubble scales: large perturbations, stochastic inflation

A. Starobinsky [1986]

$$\hat{\Phi}(x)_{\rm cg}(N, \vec{x}) = \int \frac{d\vec{k}}{(2\pi)^{3/2}} \widetilde{W}\left(\frac{k}{\sigma a H}\right) \left[\Phi_k(N)\, e^{-i\vec{k}\cdot\vec{x}}\, \hat{a}_{\vec{k}} + \mathrm{h.c.}\right]$$

$$\Phi = (\phi_1, \pi_1, \ldots \phi_n, \pi_n) \qquad \pi_i = \mathrm{d}\phi_i/\mathrm{d}N$$

Stochastic inflation



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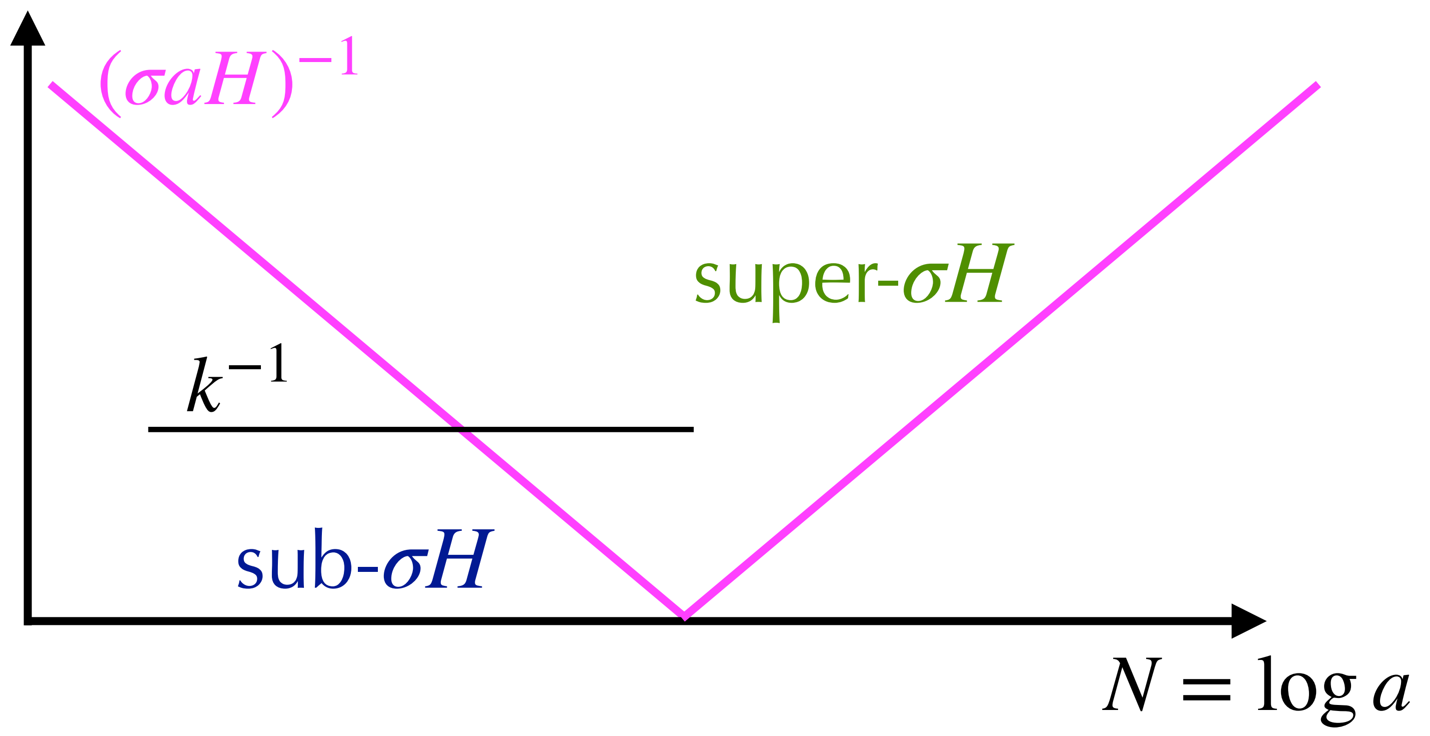

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Stochastic classical theory for $\Phi_{\rm cg}$: $\dfrac{{\rm d}\Phi_{\rm cg}}{{\rm d}N} = F_{\rm cl}(\Phi_{\rm cg}) + {\rm G}(\Phi_{\rm cg}) \cdot \xi$

- $F_{\rm cl}(\Phi_{\rm cg})$: classical background eom
- ξ : white Gaussian noise $\quad \langle \xi(\vec{x}, N)\rangle = 0, \quad \langle \xi(\vec{x}, N)\, \xi(\vec{x}', N')\rangle = \delta(N - N')\, \sin(\sigma a H |\vec{x} - \vec{x}'|)$
- $({\rm G}^2)_{ij} = \dfrac{{\rm d}\ln(\sigma a H)}{{\rm d}N} \mathscr{P}_{\Phi_i,\Phi_j}\left[\sigma a H(N_i), N_i\right]$

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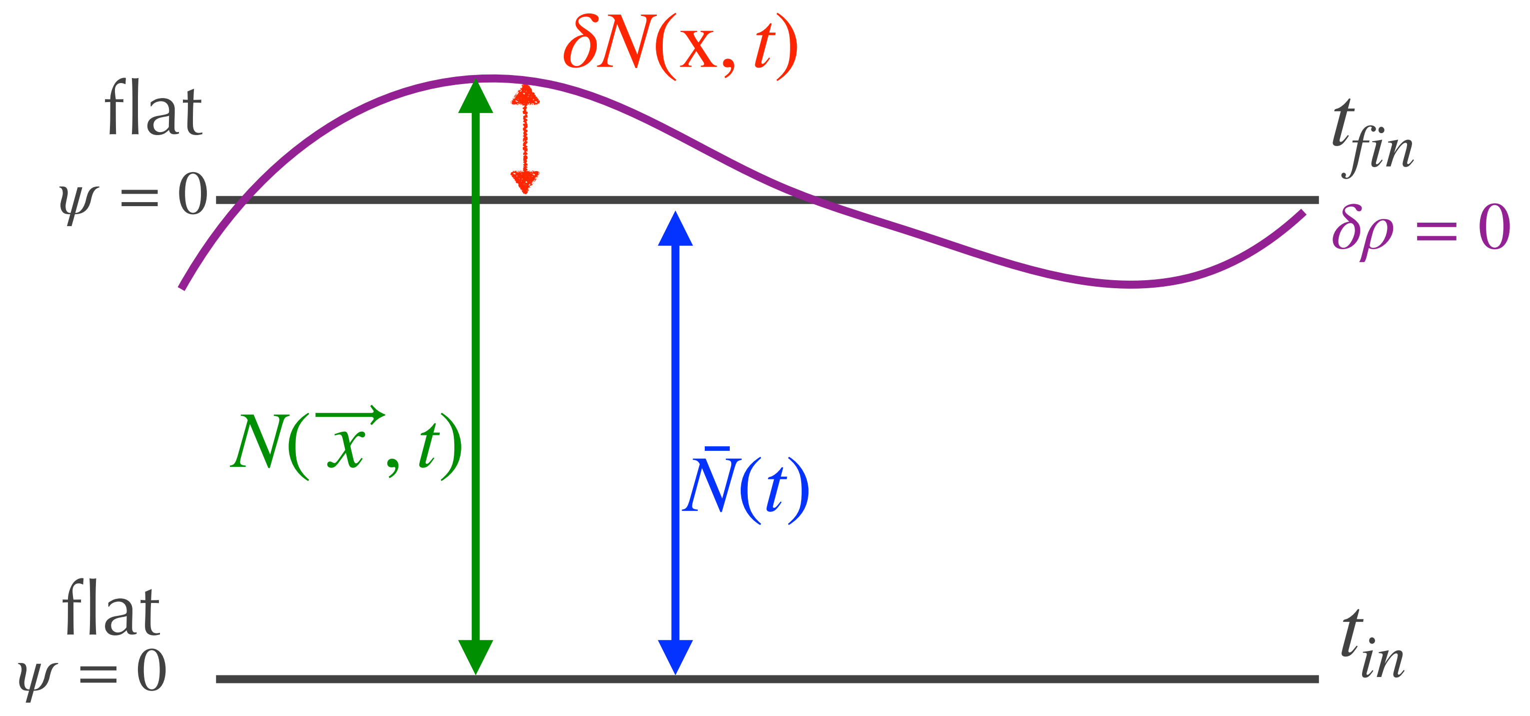
→ Given the stochastic framework, how properties of cosmological perturbations (distribution functions, power spectrum etc.) are affected?

δN formalism

At large scales, the curvature perturbation ζ corresponds to the number of e-folds realised between a spatially flat hypersurface and a final hypersurface of uniform energy density



$$\zeta(t, \mathrm{x}) = N(t, \vec{x}) - \overline{N}(t) \equiv \delta N$$

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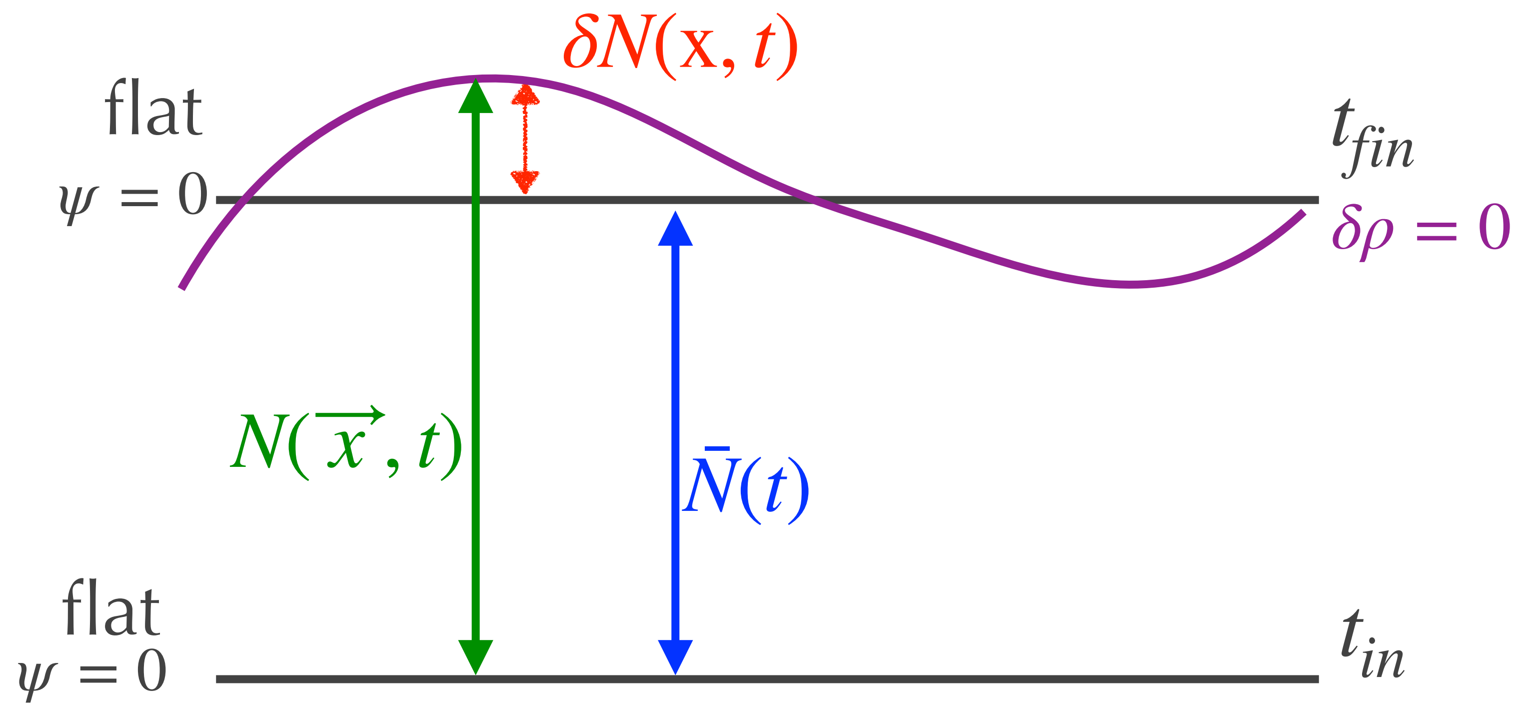

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Separate universe approach

Sasaki, Stewart [1996]

Salopek, Bond [1990]

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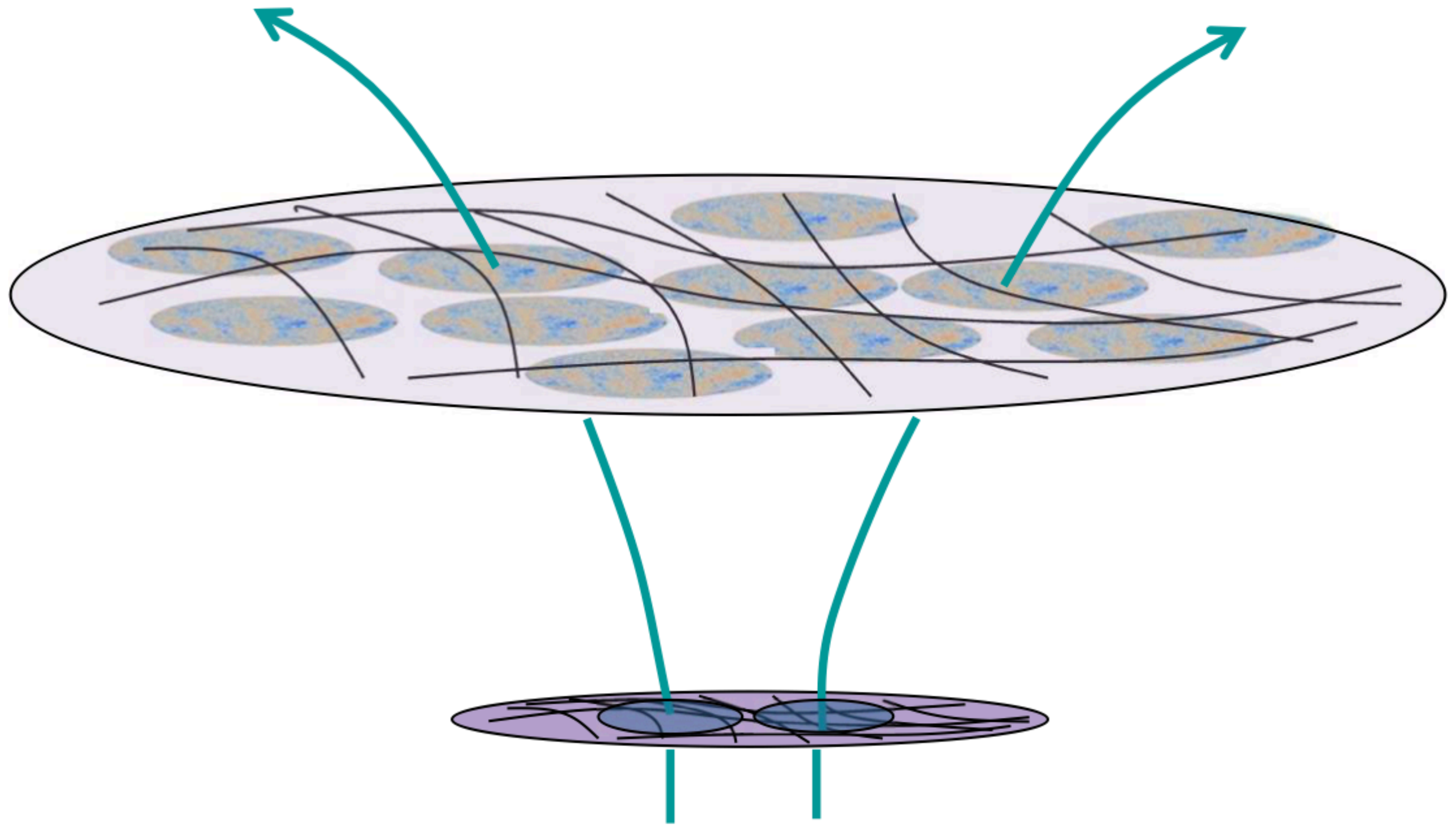

Universe : at large scales, collection of independent, locally homogeneous and isotropic patches

$N(t, \vec{x})$: amount of expansion in such universes

Stochastic δN-formalism

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Duration of inflation becomes a stochastic variable: $\mathcal{N}$

[Enqvist, Nurmi, Podolsky, Rigopoulos [2008]

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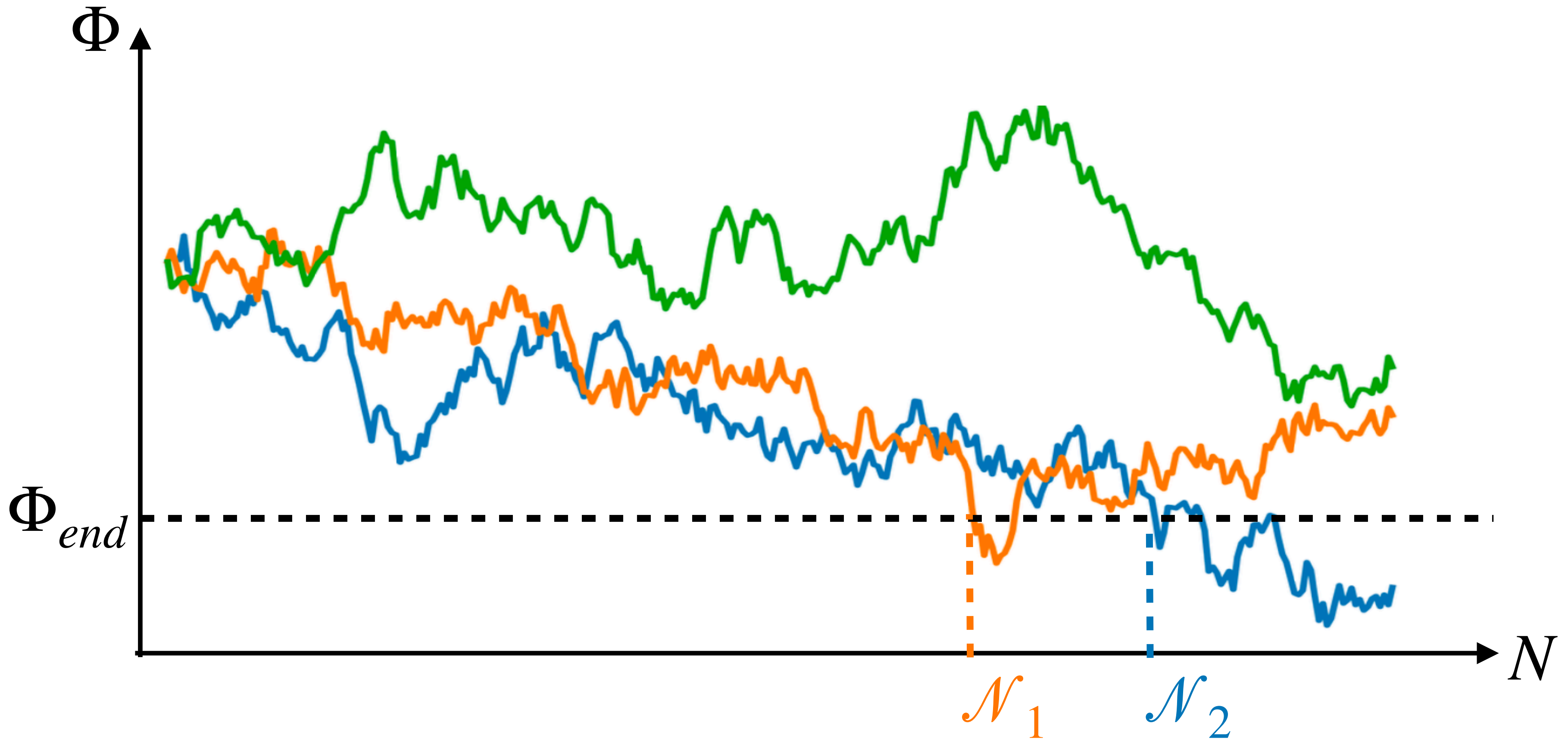

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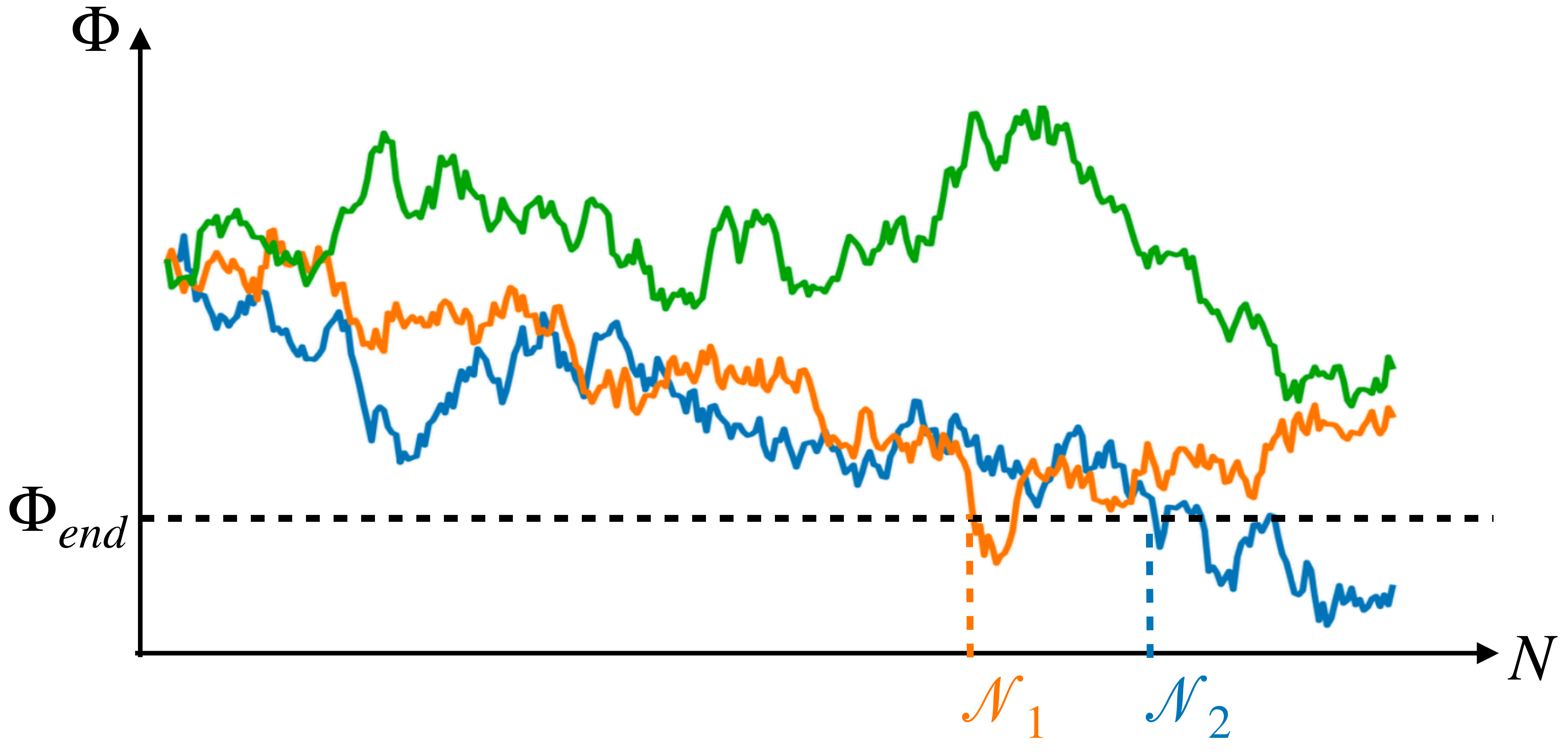

Distribution function for the duration of inflation (first-passage time)

$$\frac{\partial}{\partial \mathcal{N}} P(\mathcal{N}, \Phi) = \mathcal{L}^{\dagger}_{FP}(\Phi) \cdot P(\mathcal{N}, \Phi)$$

$$\mathcal{L}^{\dagger}_{\mathrm{FP}}(\Phi) = F^i \frac{\partial}{\partial \Phi^i} + \frac{1}{2} G^i_k G^{ki} \frac{\partial^2}{\partial \Phi^i \partial \Phi^j}$$

$$P_{\mathrm{FPT}, \Phi = \Phi_{\mathrm{end}}}(\mathcal{N}) = \delta(\mathcal{N})$$

Stochastic $\delta\mathcal{N}$-formalism

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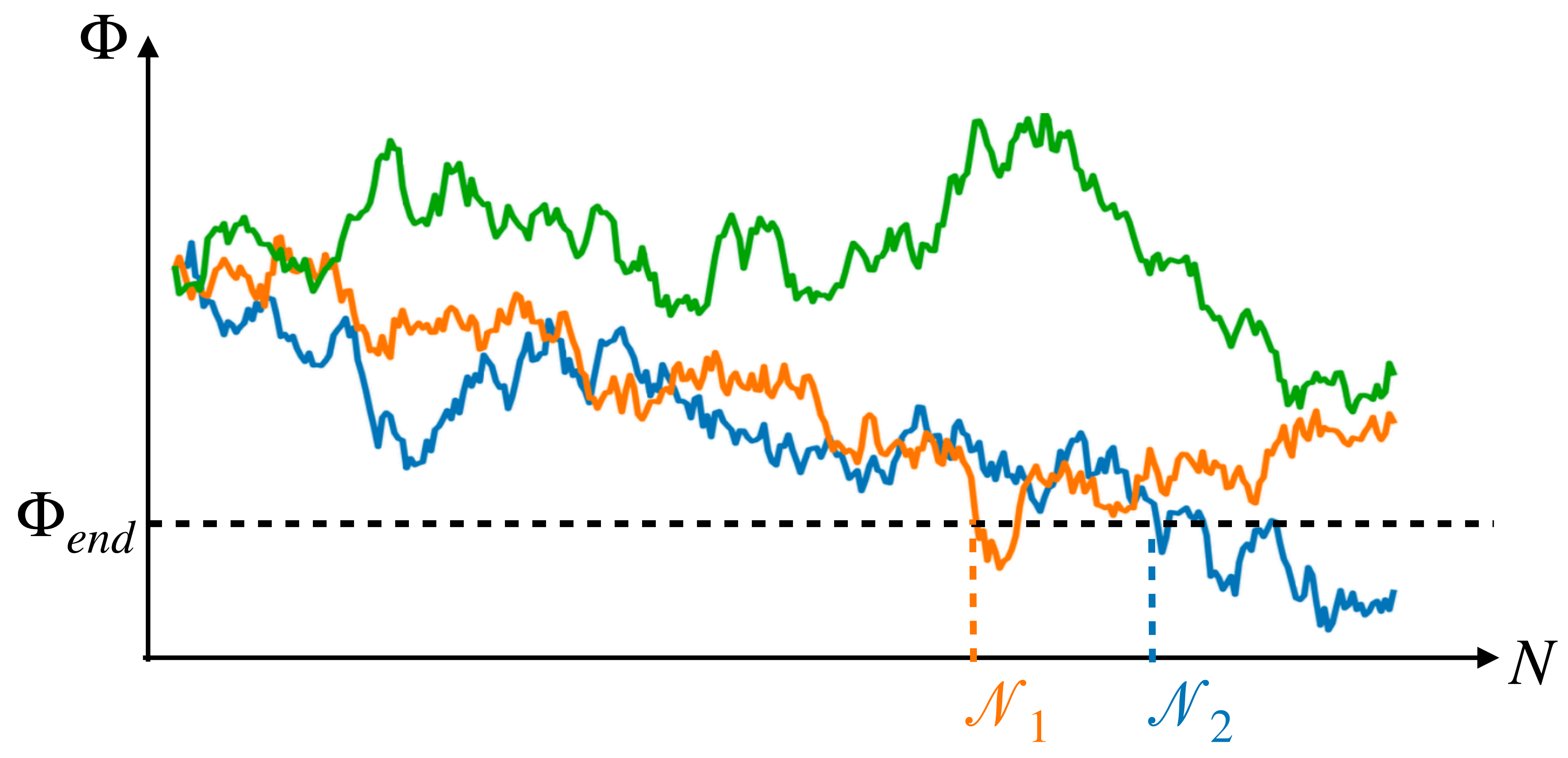

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Statistics of ζ from the statistics of $\mathcal{N}$: $\zeta_{cg}(\vec{x}) = \mathcal{N}(\vec{x}) - \langle \mathcal{N} \rangle$

Stochastic-δN formalism: exponential tails

Full PDF of the first passage time

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Characteristic function (includes all moments)

$$\chi(t,\Phi) \equiv \langle e^{it\mathcal{N}} \rangle = \int_{-\infty}^{\infty} e^{it\mathcal{N}} P(\mathcal{N},\Phi)\, d\mathcal{N} \longrightarrow \mathcal{L}^{\dagger}_{FP} \cdot \chi(t,\Phi) = -\, i\, t\, \chi(t,\Phi) \longrightarrow P(\mathcal{N},\Phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-it\mathcal{N}} \chi(t,\Phi)\, dt$$

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Useful trick: pole expansion

Ezquiaga, Garcia-Bellido, Vennin (2020)

$$\chi(t,\Phi) = \sum_n \frac{a_n(\Phi)}{\Lambda_n - i\,t} + g(t,\Phi)$$

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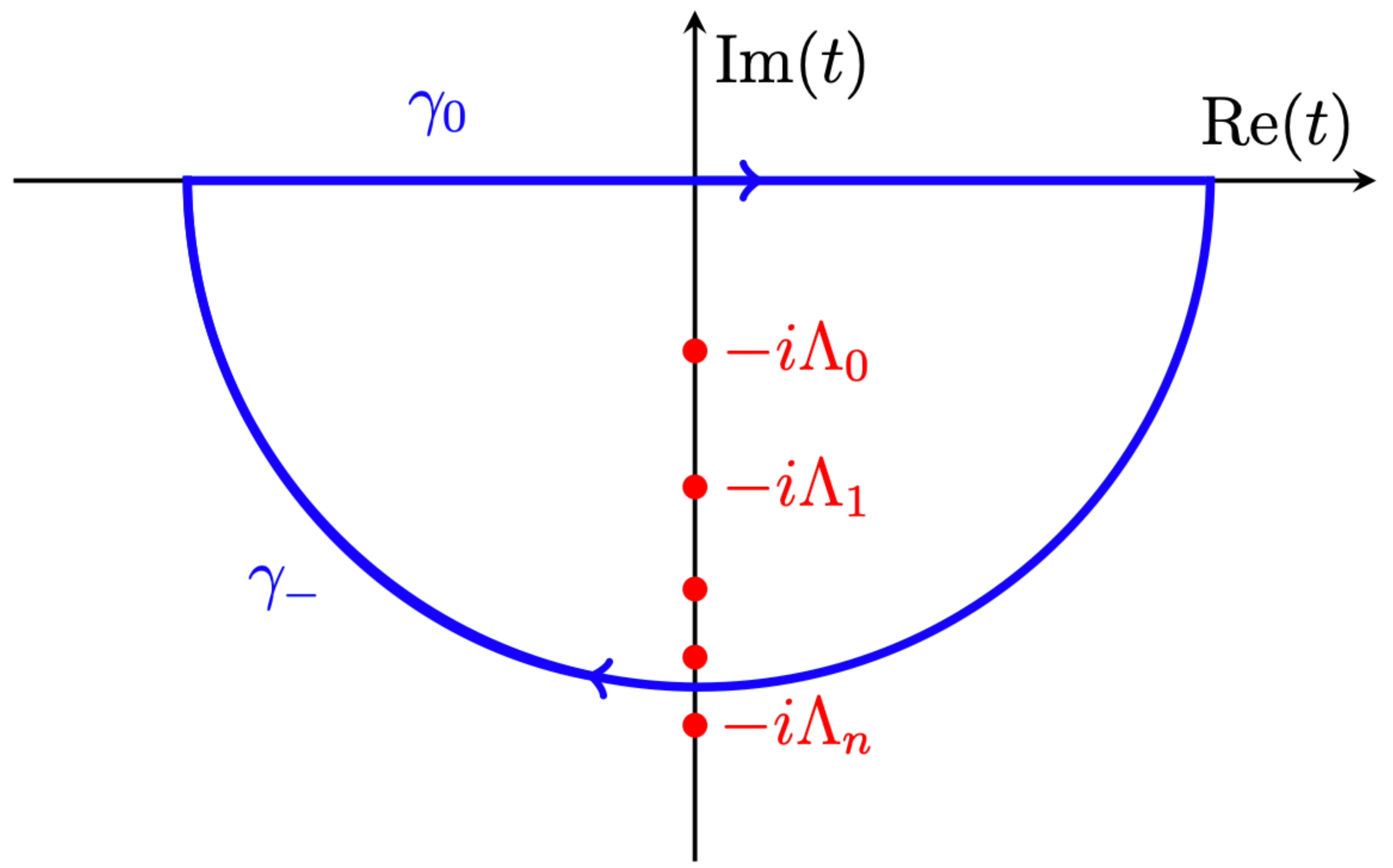

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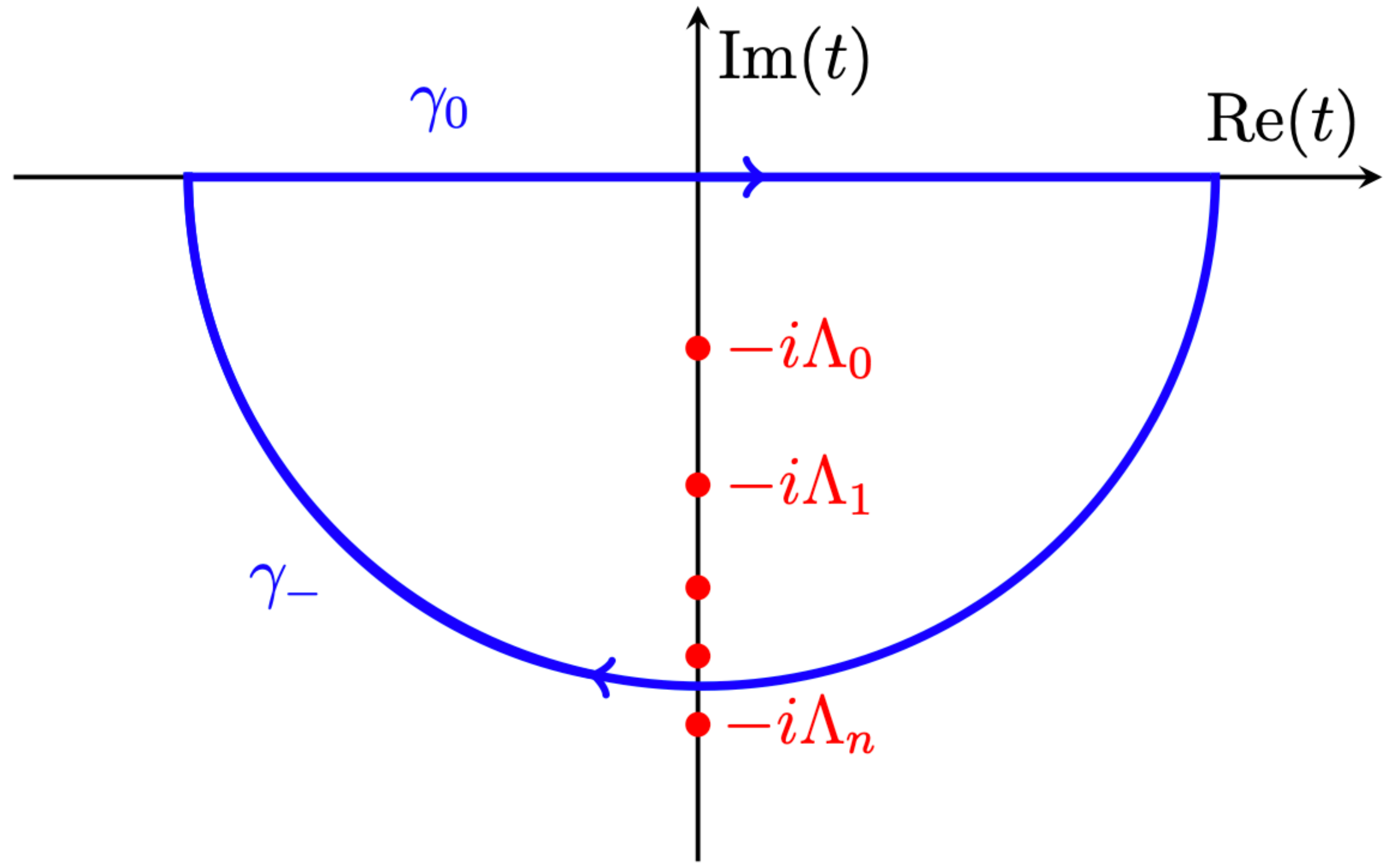
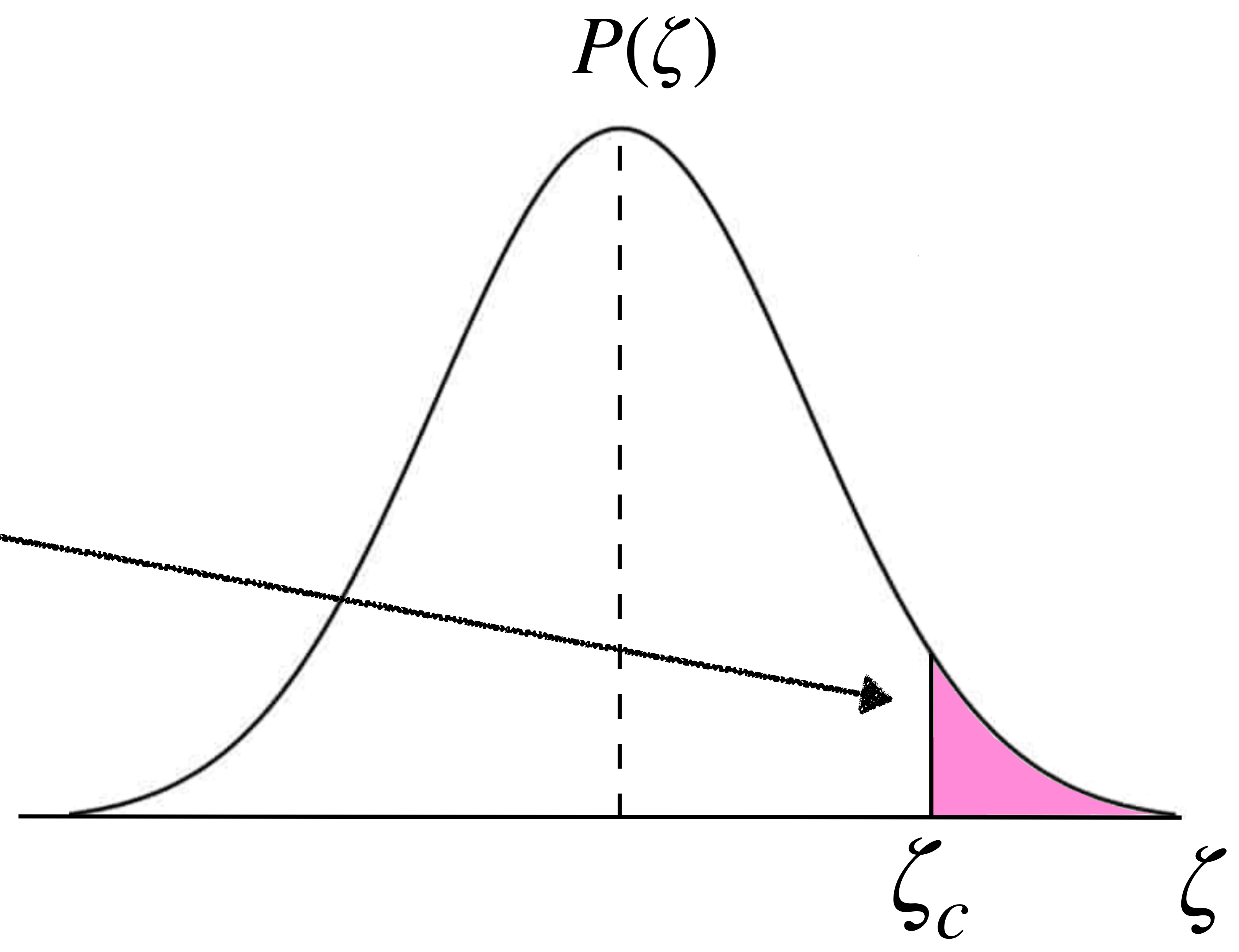


Tail of the PDF of $\mathcal{N}$ (hence ζ) has an exponential fall-off behaviour

This type of non-Gaussianities cannot be captured by perturbative parametrisations (such as the $f_{\rm NL}$, $g_{\rm NL}$ expansion)

Implications for PBHs

PBHs are sensitive to the tails of distribution functions



Abundance of PBHs at formation:

(Press-Schechter method)

$$\beta(M) = \int_{\zeta_c}^{\infty} P(\zeta)\, d\zeta$$

Mass fraction underestimated of orders of magnitude

Implications for inflationary models (amount of fine tuning, overproduction,….)

Consequences in light of observational constraints

Stochastic-δN formalism: exponential tails

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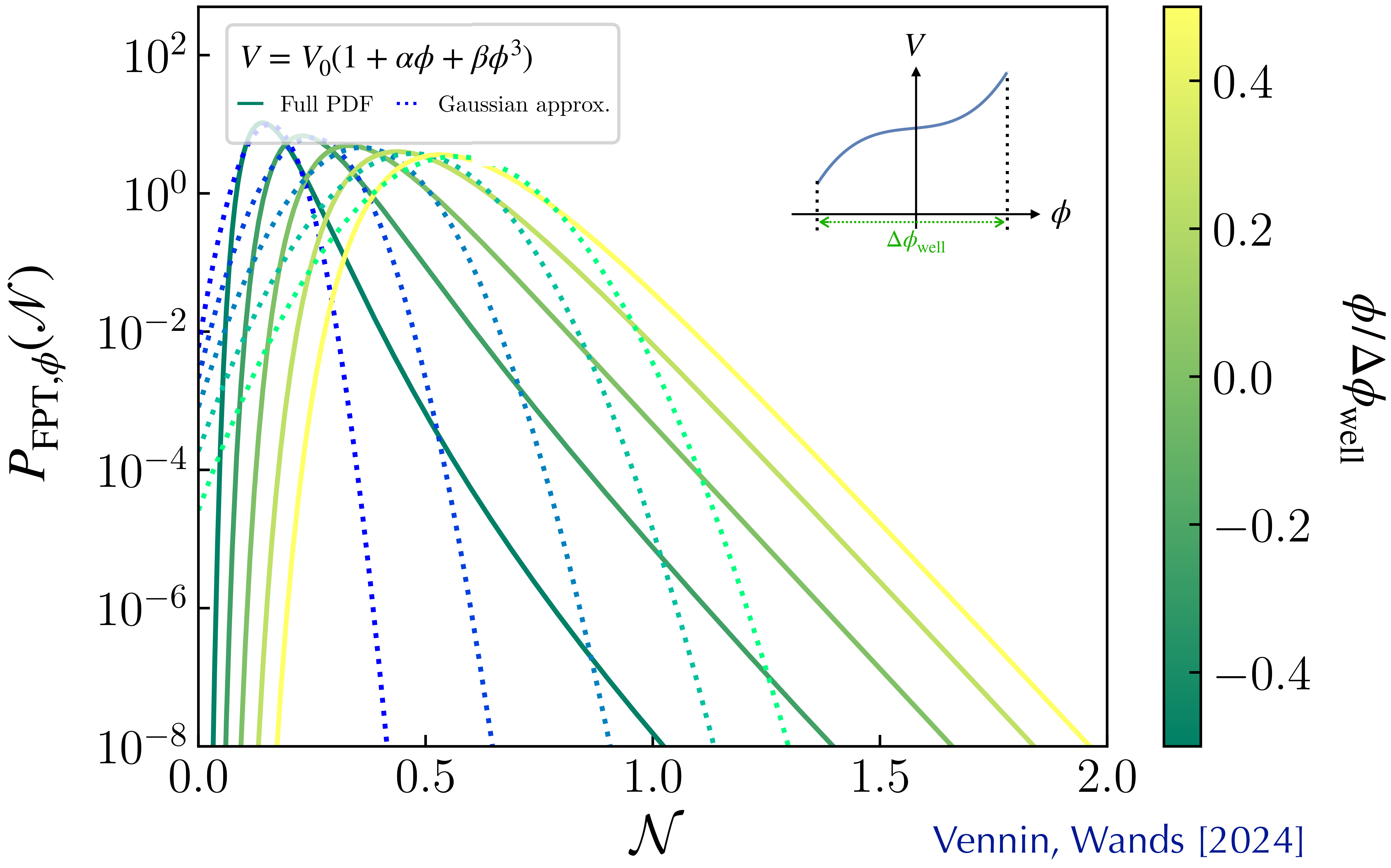

Vennin, Wands [2024]

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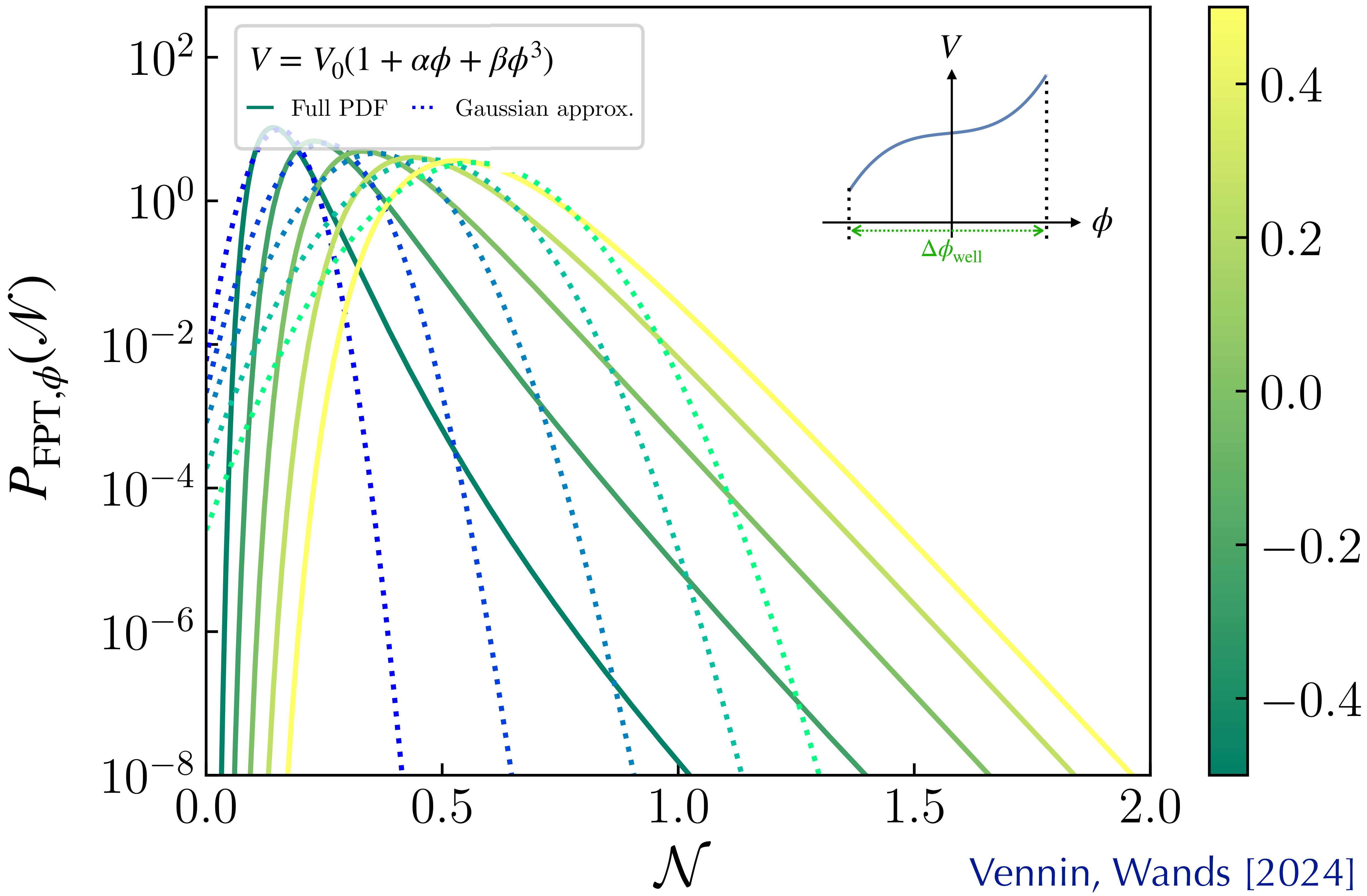

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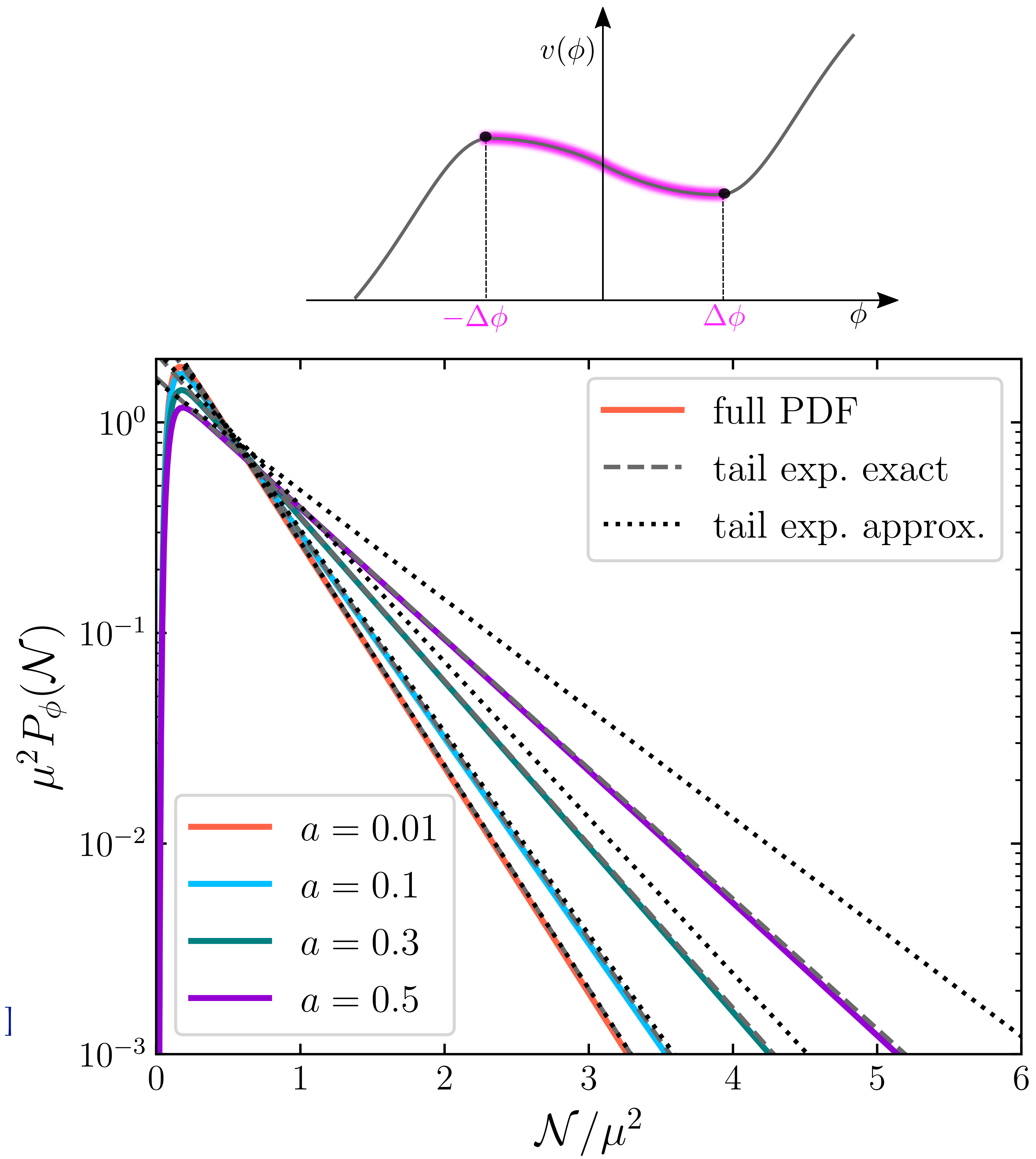


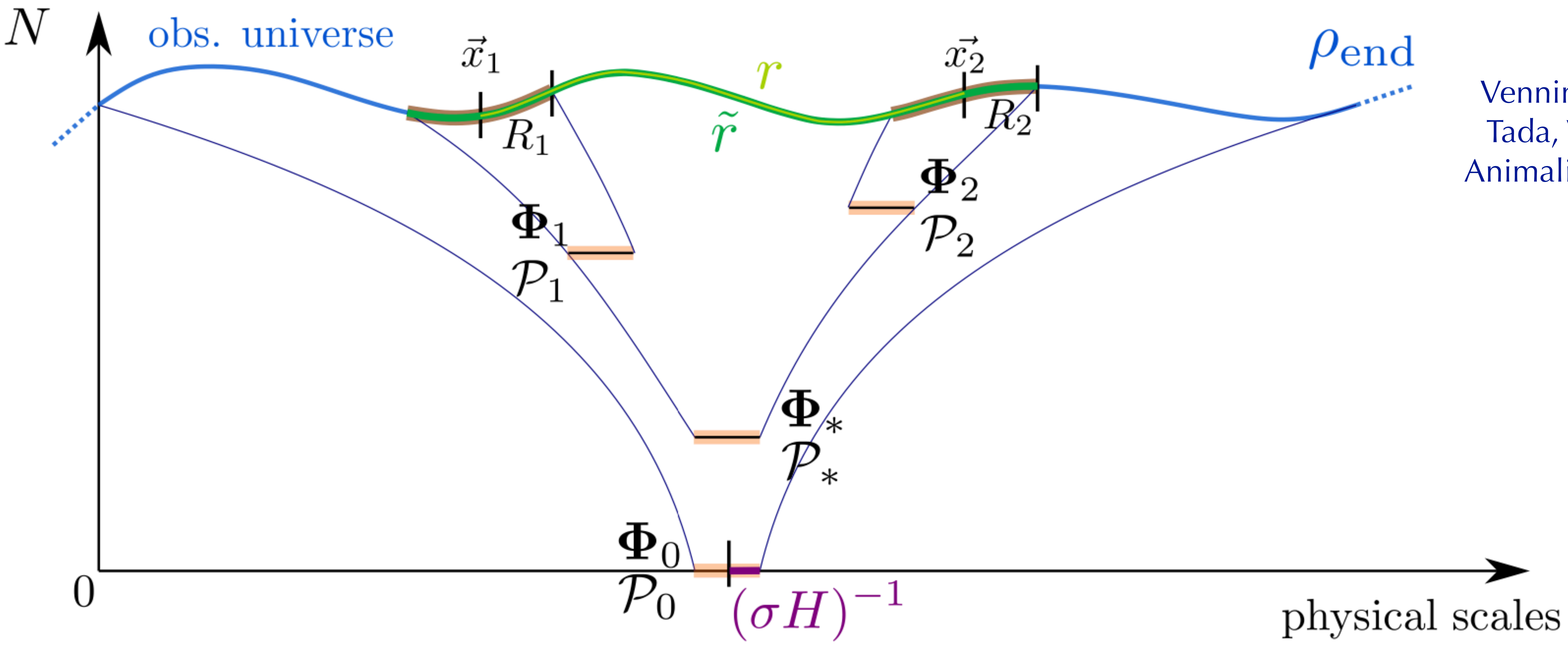
Animali, Vennin [2023]

Beyond one-point distributions

In the separate-universe framework, distance between two final Hubble patches encoded in the time at which their worldlines became independent

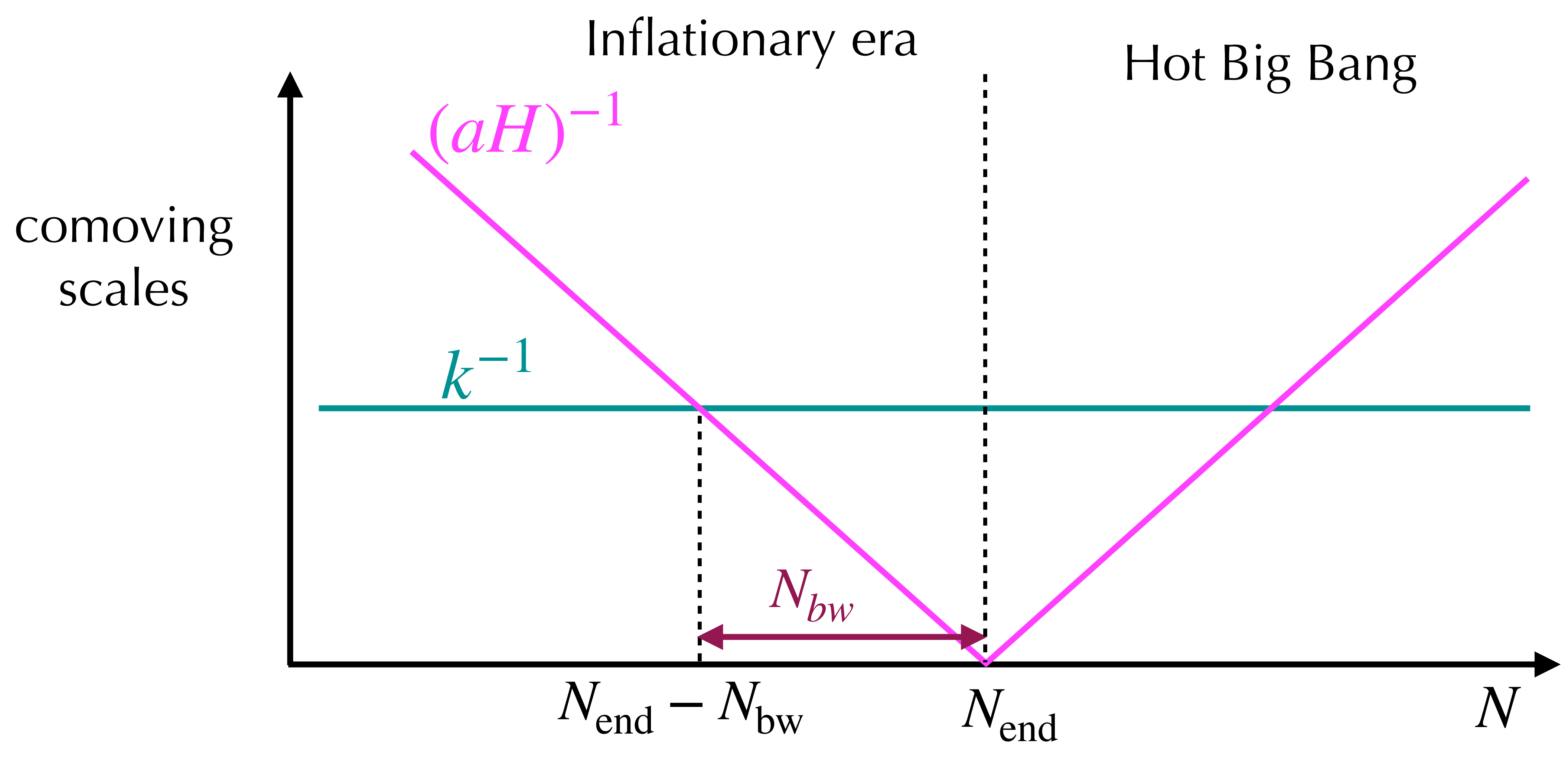
We can calculate correlations between durations of inflation (between curvature perturbation) at fixed physical distance

We can extract multiple-point statistics



Vennin, Ando [2021]
Tada, Vennin [2021]
Animali, Vennin [2024]

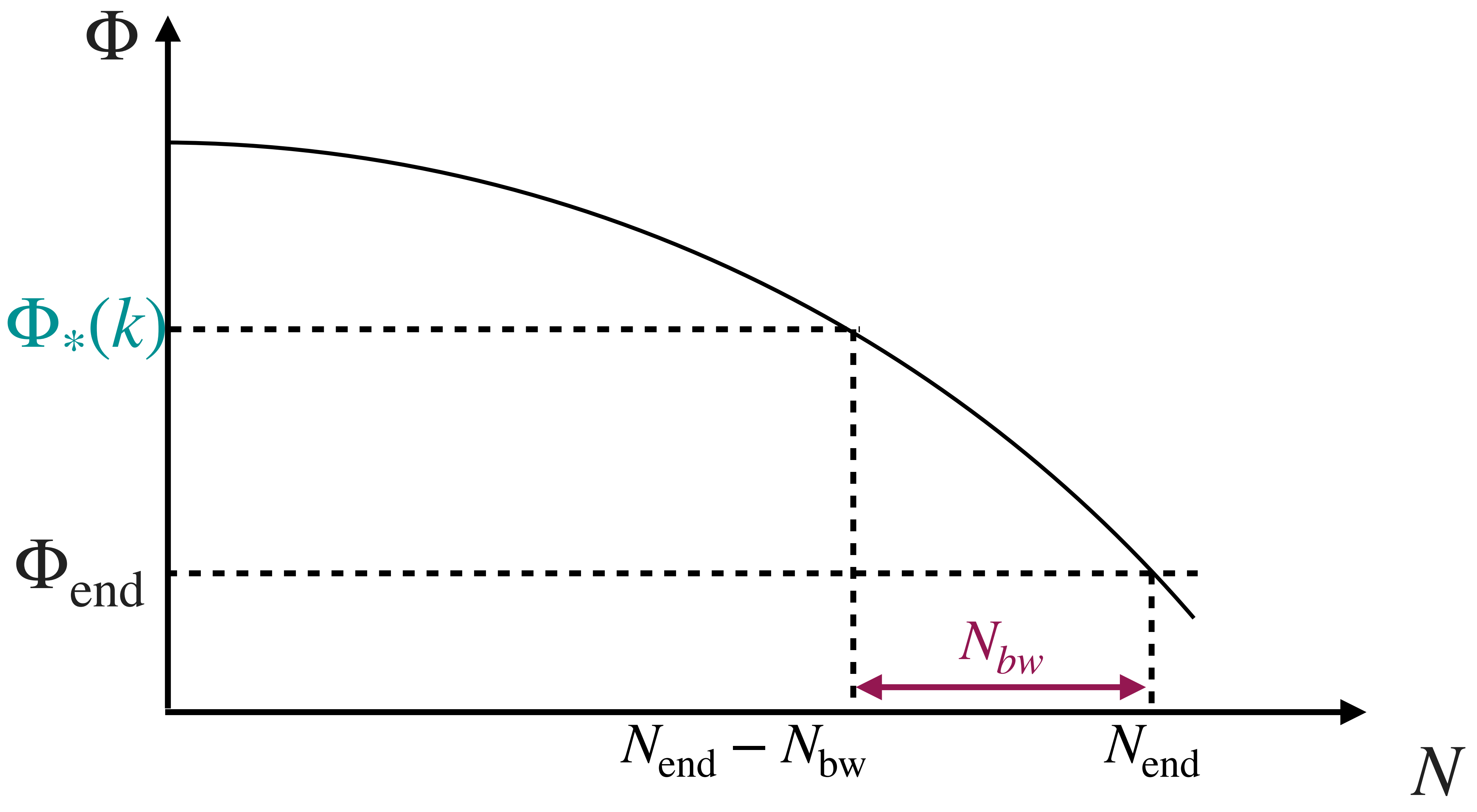
Extracting cosmological observables



Scale k crosses the Hubble radius at

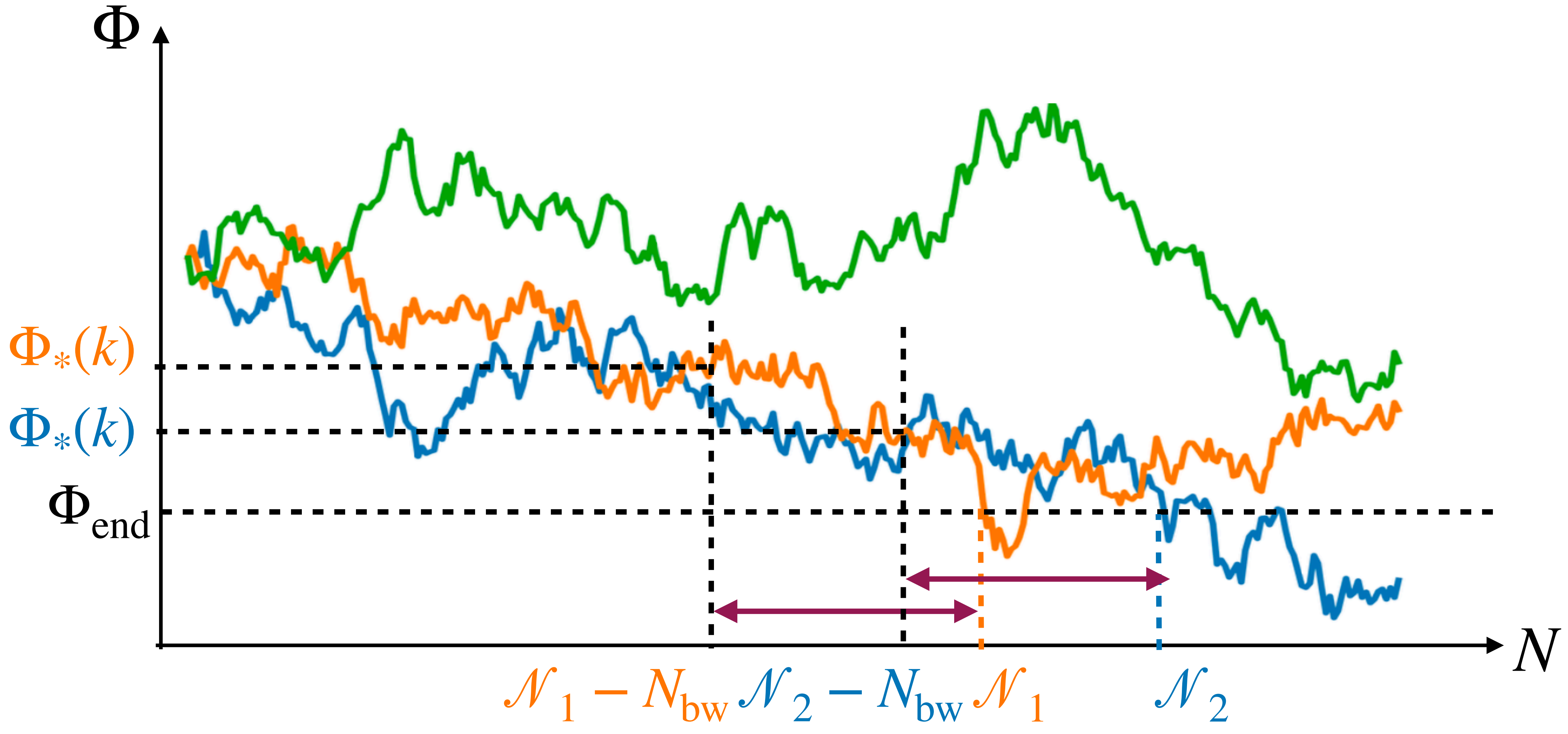
$$N_* = N_{\rm end} - N_{\rm bw} = N_{\rm end} - \log(a_{\rm end} H/k)$$

classical problem



one-to-one correspondence between k and $\Phi_*(k)$

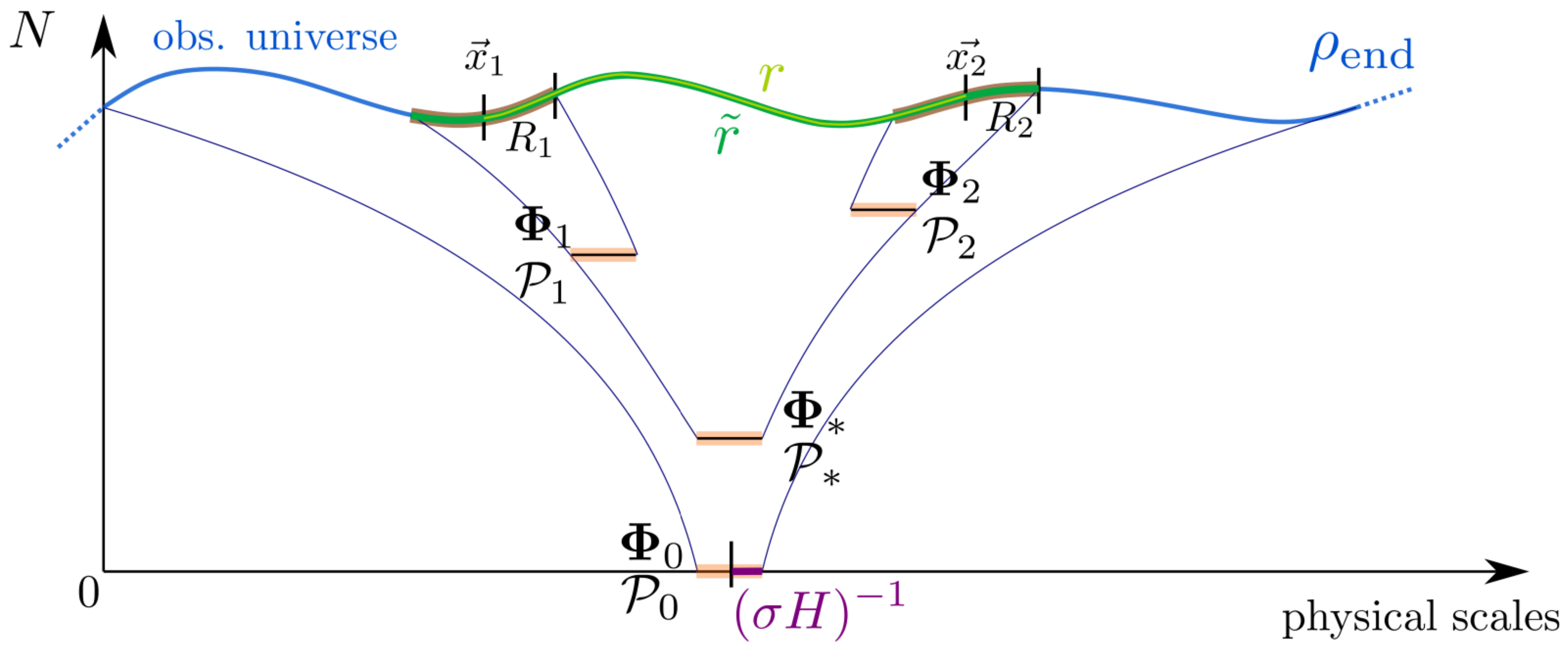
stochastic problem



$\Phi_*(k)$ is a stochastic quantity

Power spectrum in the backward approximation

"Power spectrum in stochastic inflation"
Vennin, Ando [2021]



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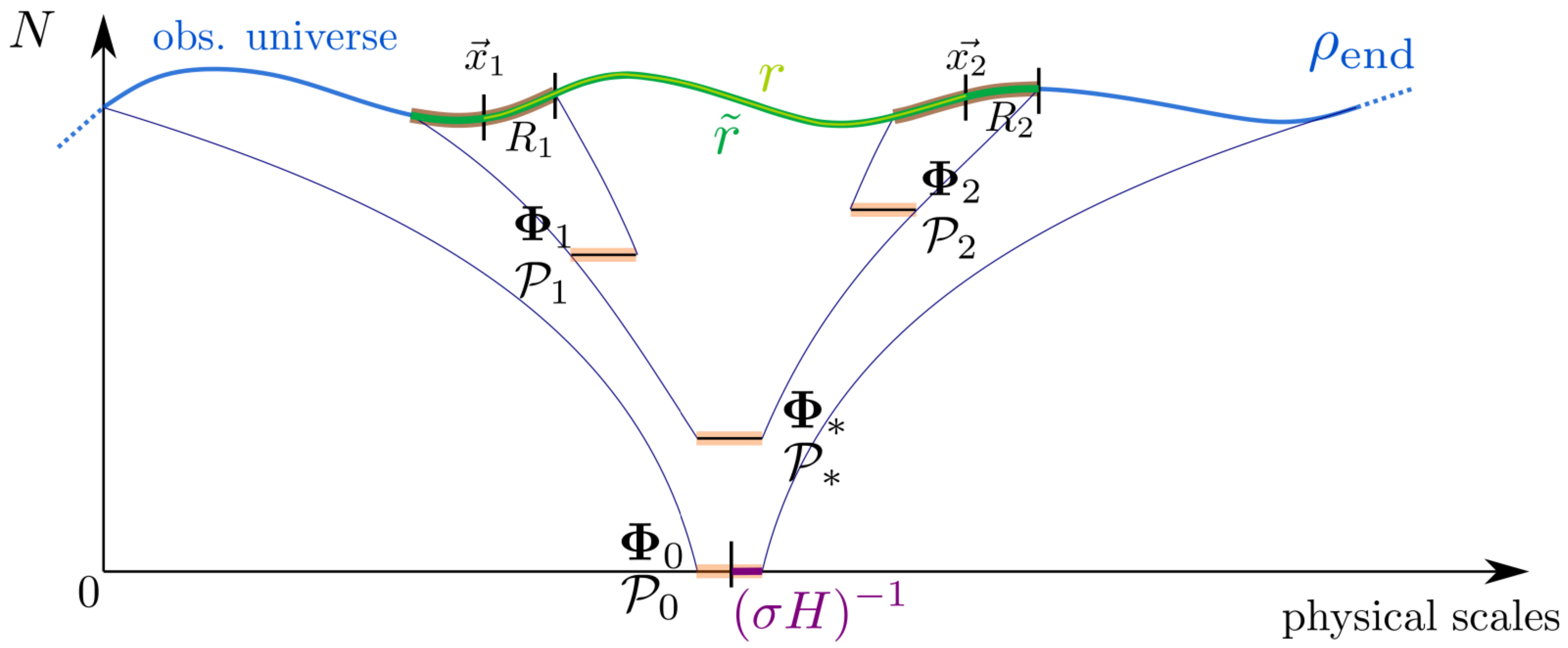
Split the expansion in shared and independent expansion: $\zeta(\overrightarrow{x}_i) = \mathcal{N}_i[\Phi_0 \to \Phi_*(\overrightarrow{x}_i, \overrightarrow{x}_j)] + \mathcal{N}_i[\Phi_*(\overrightarrow{x}_i, \overrightarrow{x}_j)] - \overline{\mathcal{N}}(\Phi_0)$



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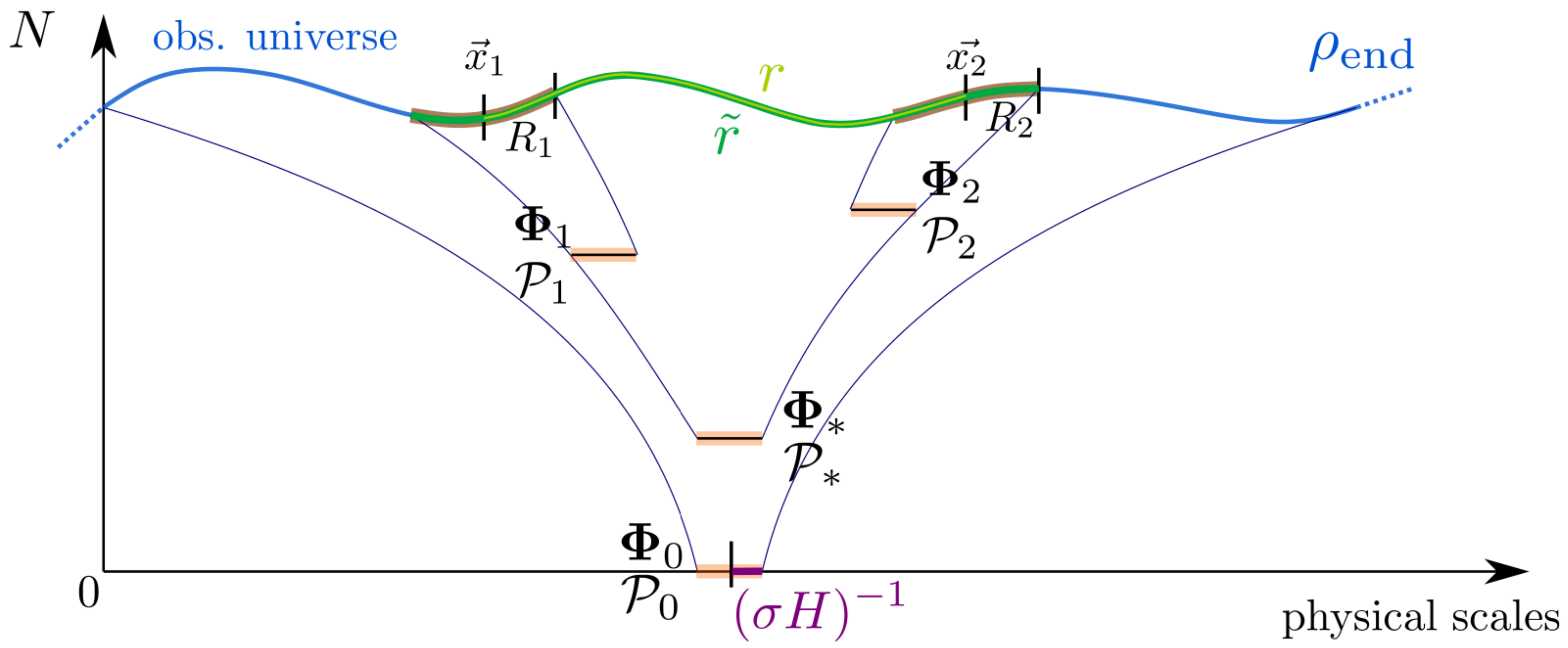
Compute the two point function: $\langle \zeta(\overrightarrow{x}_i)\, \zeta(\overrightarrow{x}_j) \rangle_{\tilde{r}} = \int \mathrm{d}\Phi_* P_{\tilde{r}}(\Phi_*) \langle \delta \mathcal{N}^2(\Phi_0 \to \Phi_*) \rangle$



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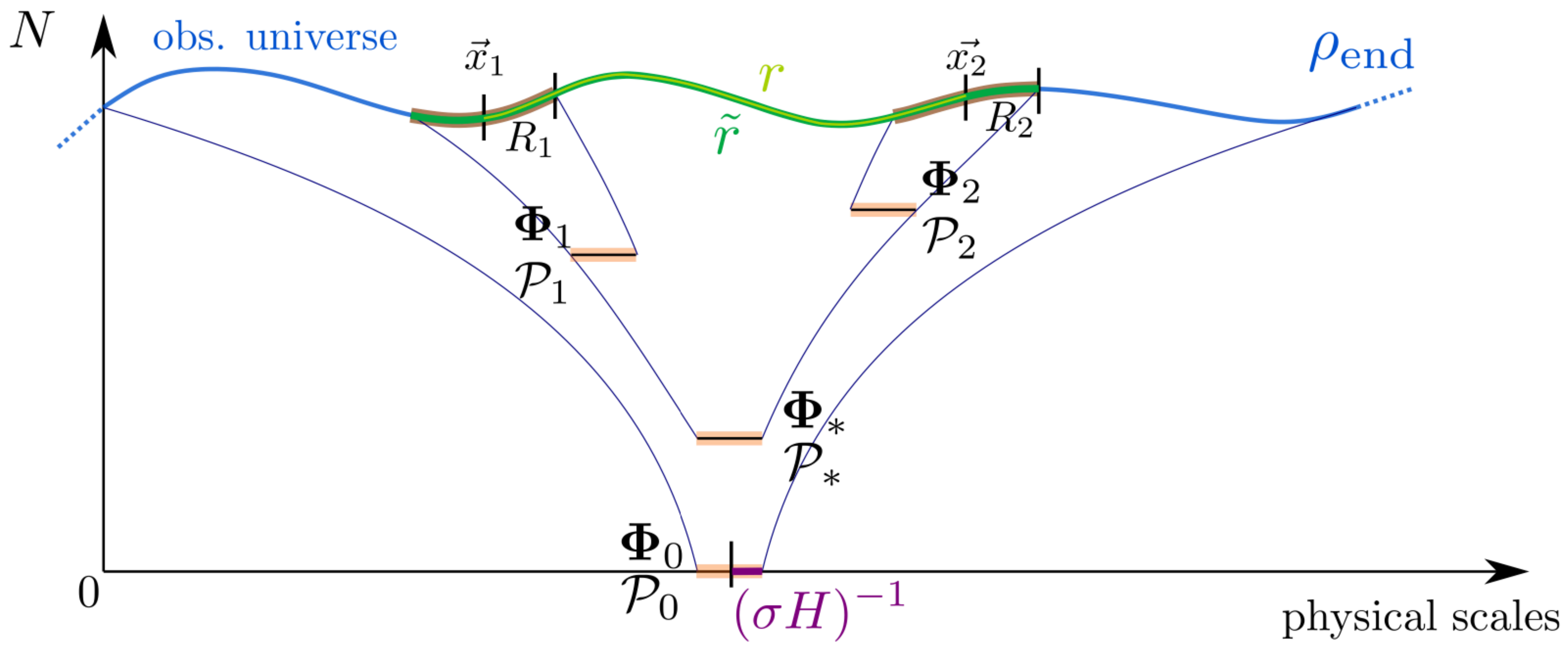

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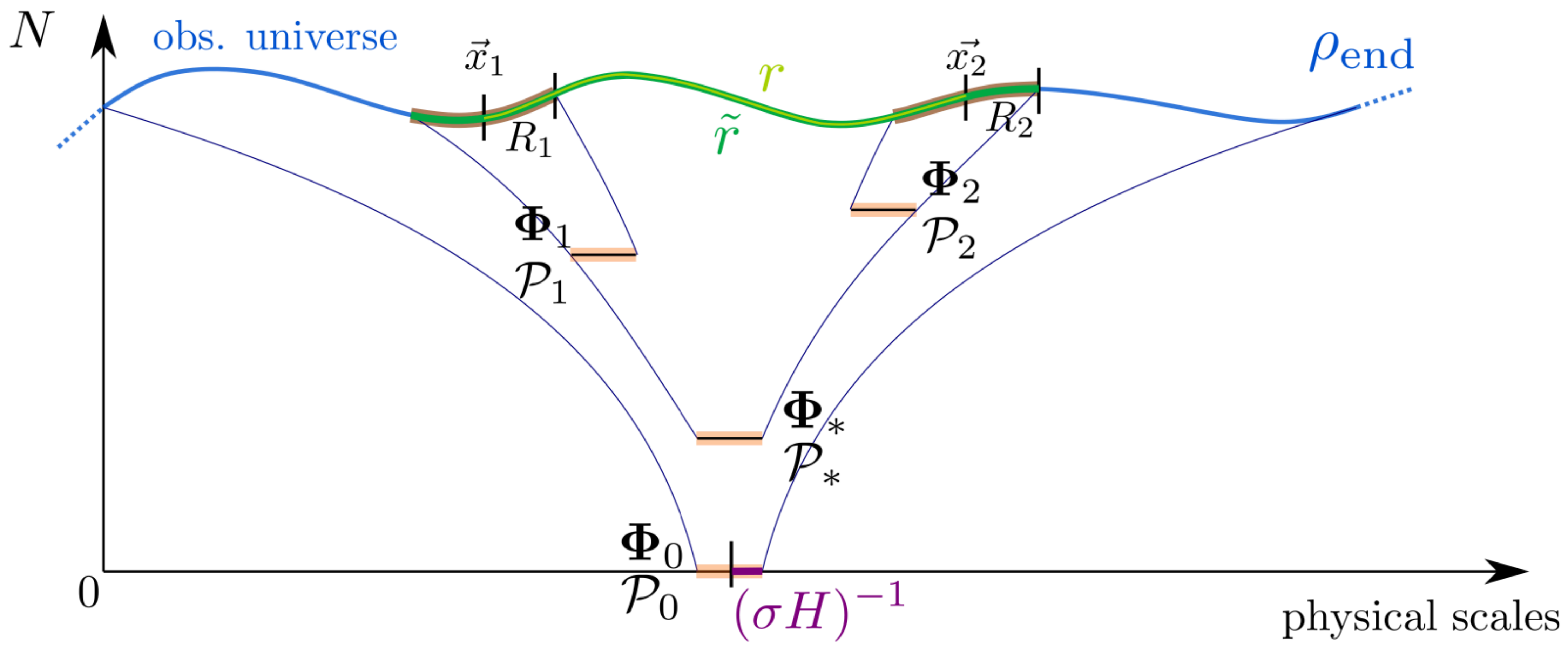
PDF of field values in the splitting patch → Pdf of field values at a fixed backward *e*-folds number

$$P_r(\Phi_*) \simeq P_{\rm bw}[\Phi_*, N_{\rm bw}(r)]$$



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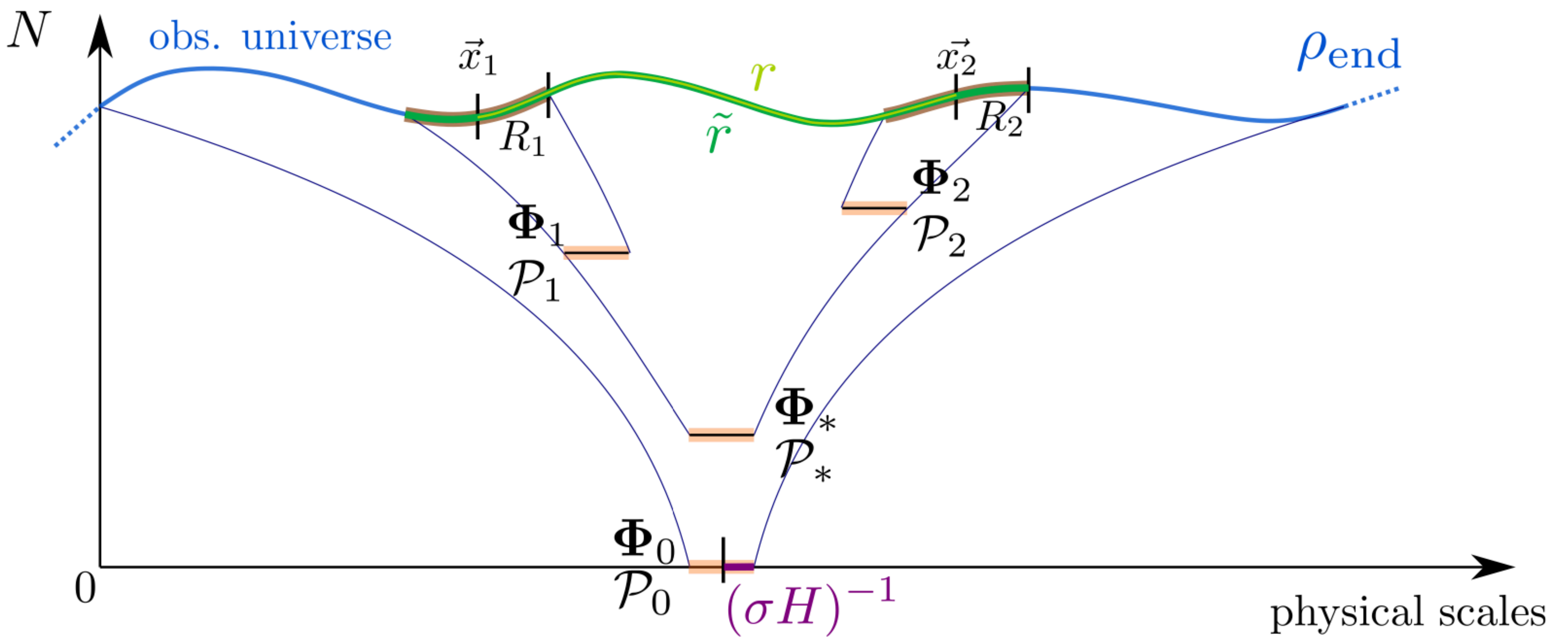
Compute the two point function: $\langle \zeta(\vec{x}_i)\,\zeta(\vec{x}_j)\rangle_{\tilde{r}} = \int \mathrm{d}\Phi_* P_{\tilde{r}}(\Phi_*) \langle \delta\mathcal{N}^2(\Phi_0 \to \Phi_*)\rangle$

PDF of field values in the splitting patch → Pdf of field values at a fixed backward *e*-folds number

$$P_r(\Phi_*) \simeq P_{\rm bw}[\Phi_*, N_{\rm bw}(r)]$$

Comoving lines separated by r become independent when: $e^{N_{\rm bw,*}} = rH(\Phi_*)$

Quasi de-Sitter limit: $N_{\rm bw}(r) = \log(rH_{\rm end}) = -\log\left(\dfrac{k}{k_{\rm end}}\right)$



Power spectrum in the backward approximation

“Power spectrum in stochastic inflation”
Vennin, Ando [2021]

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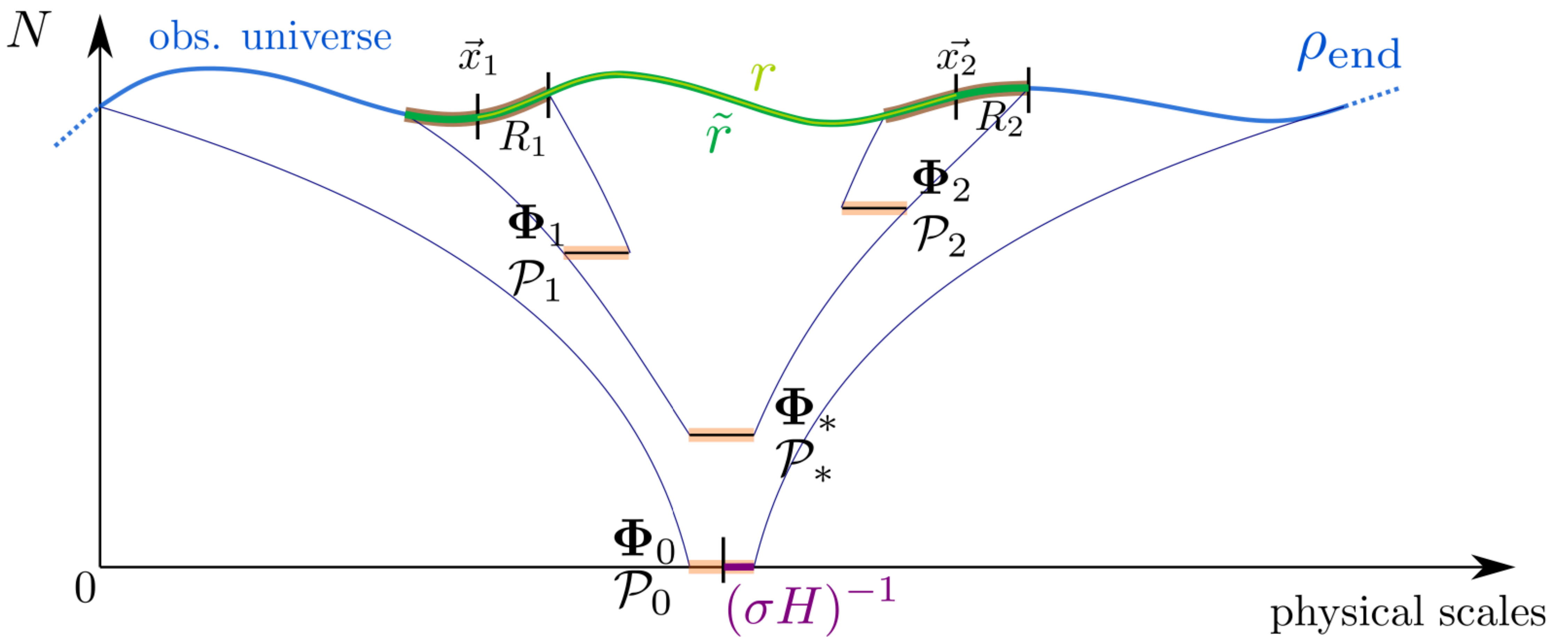

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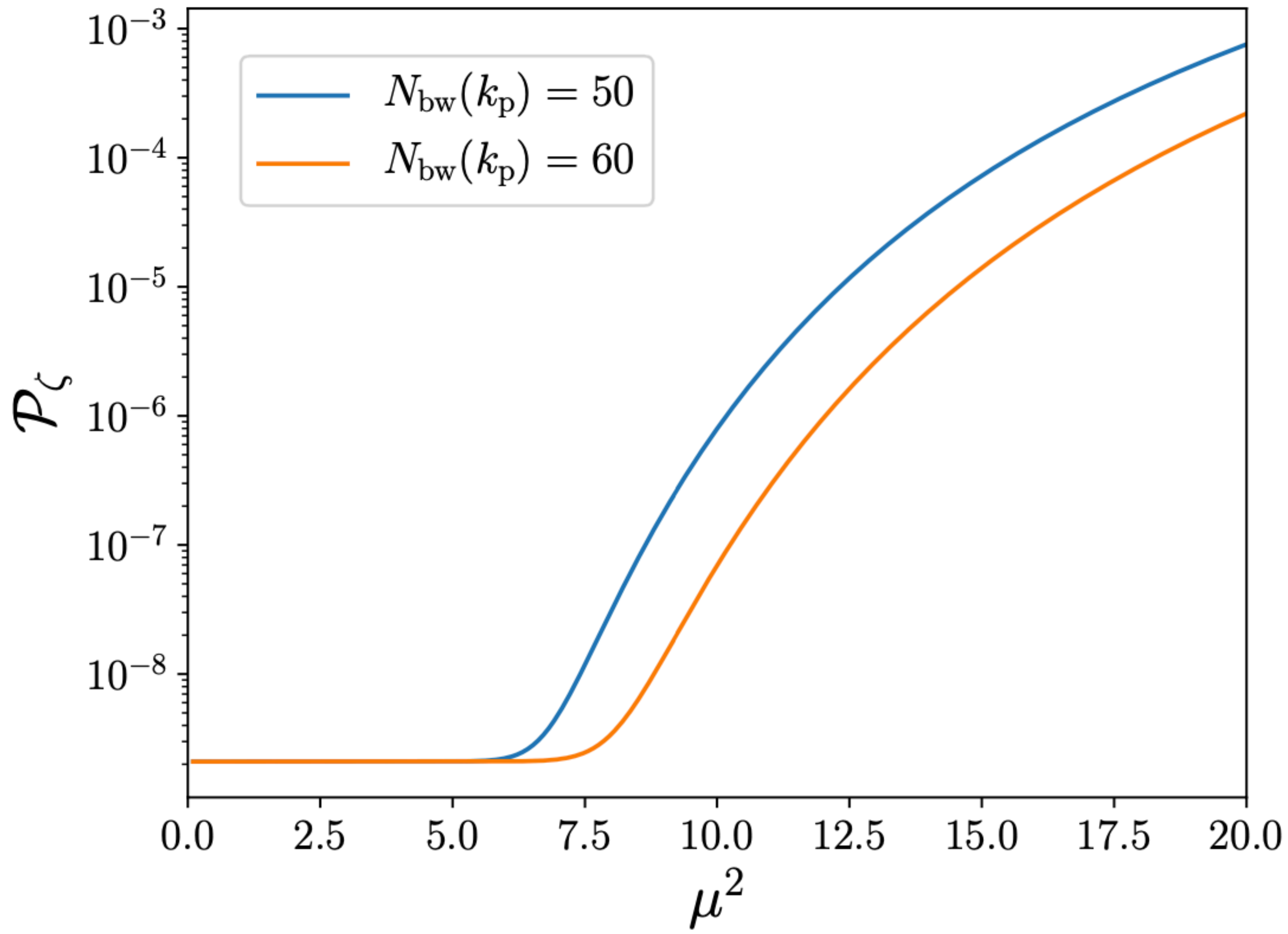

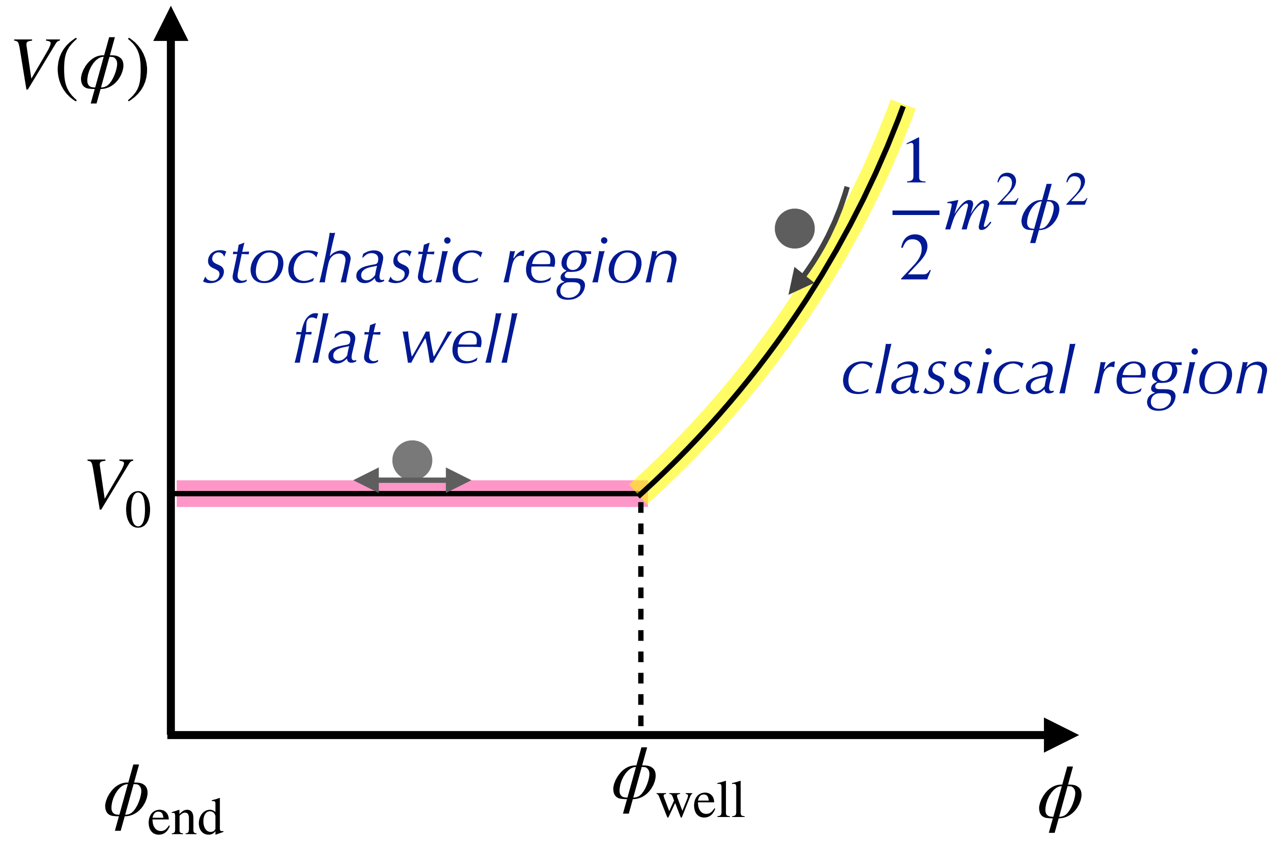

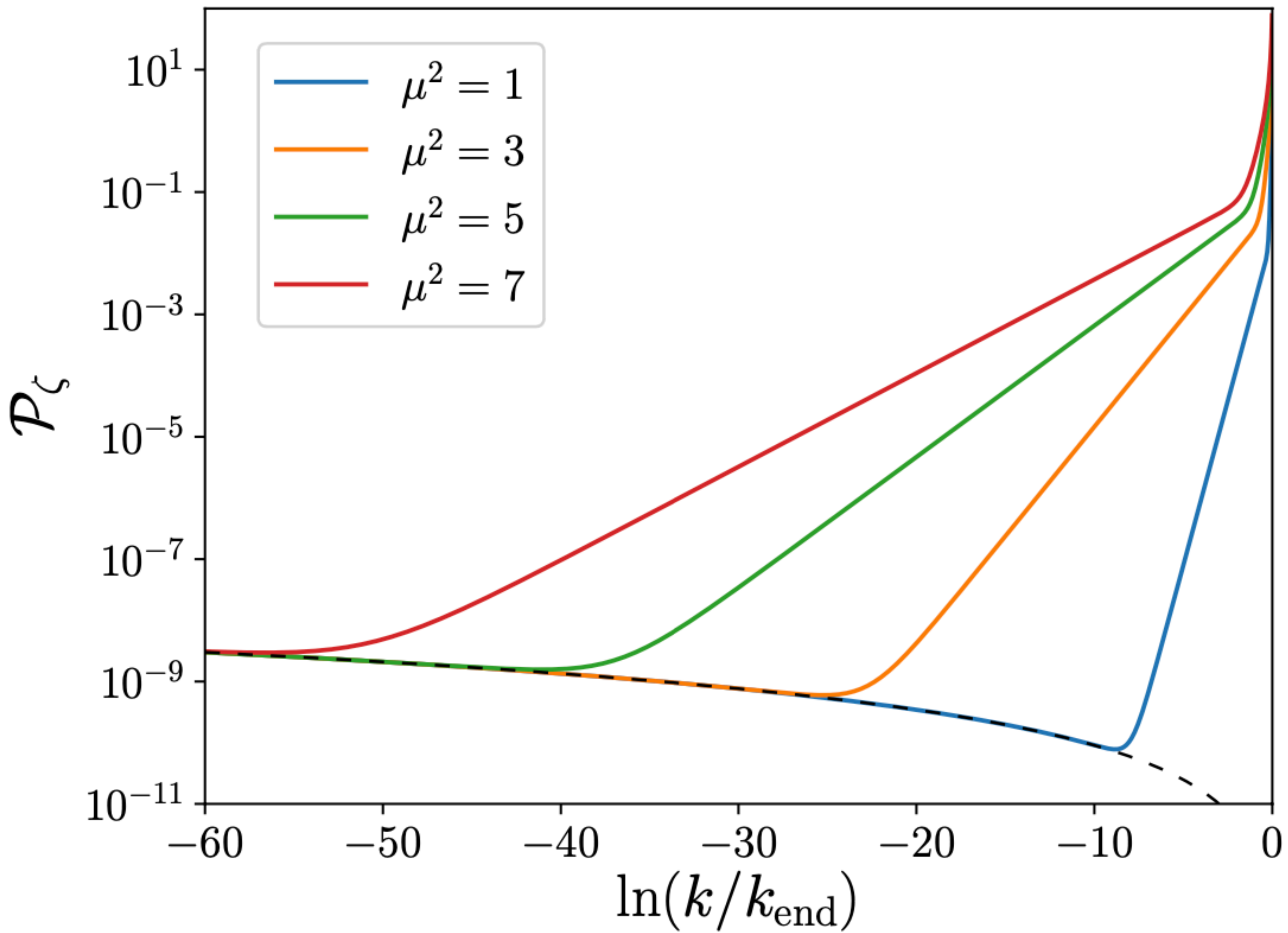

Stochastic trees

"Clustering of primordial black holes from quantum diffusion during inflation"
Animali, Vennin [2024]

Stochastic trees

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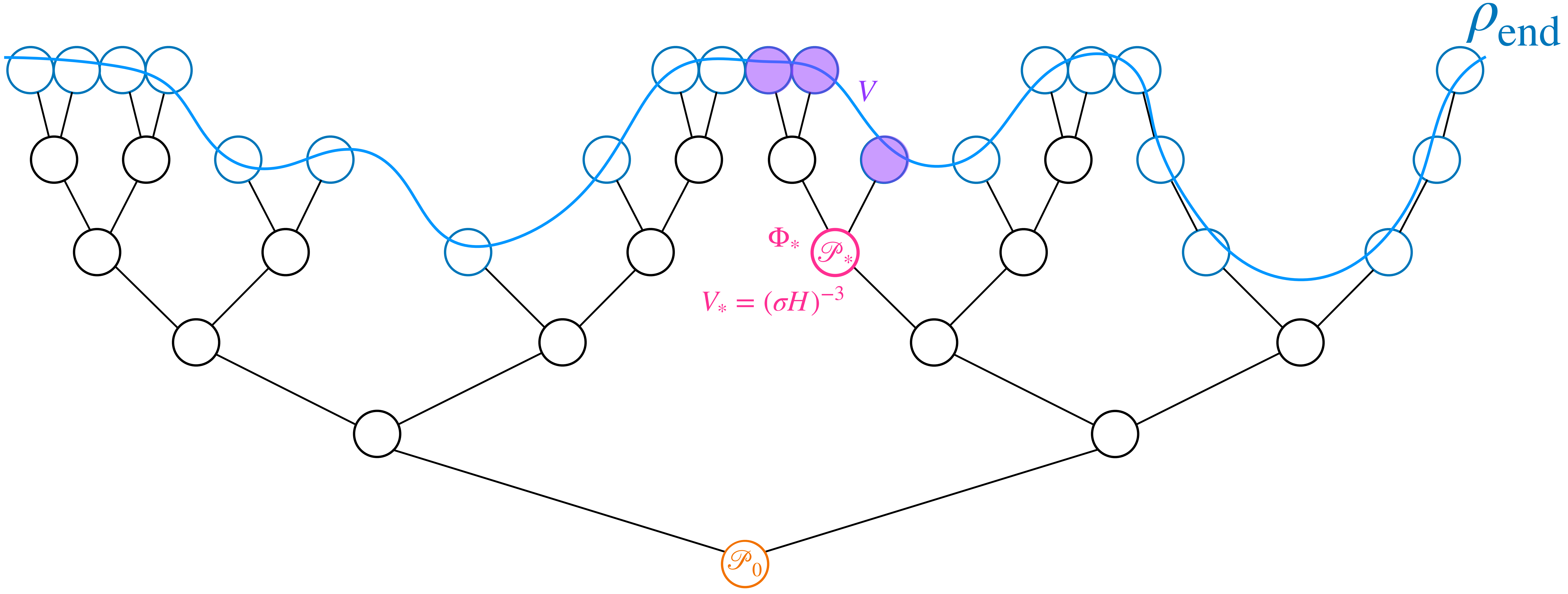
Relation between field values and physical distances encoded in the structure of a universe which inflates stochastically



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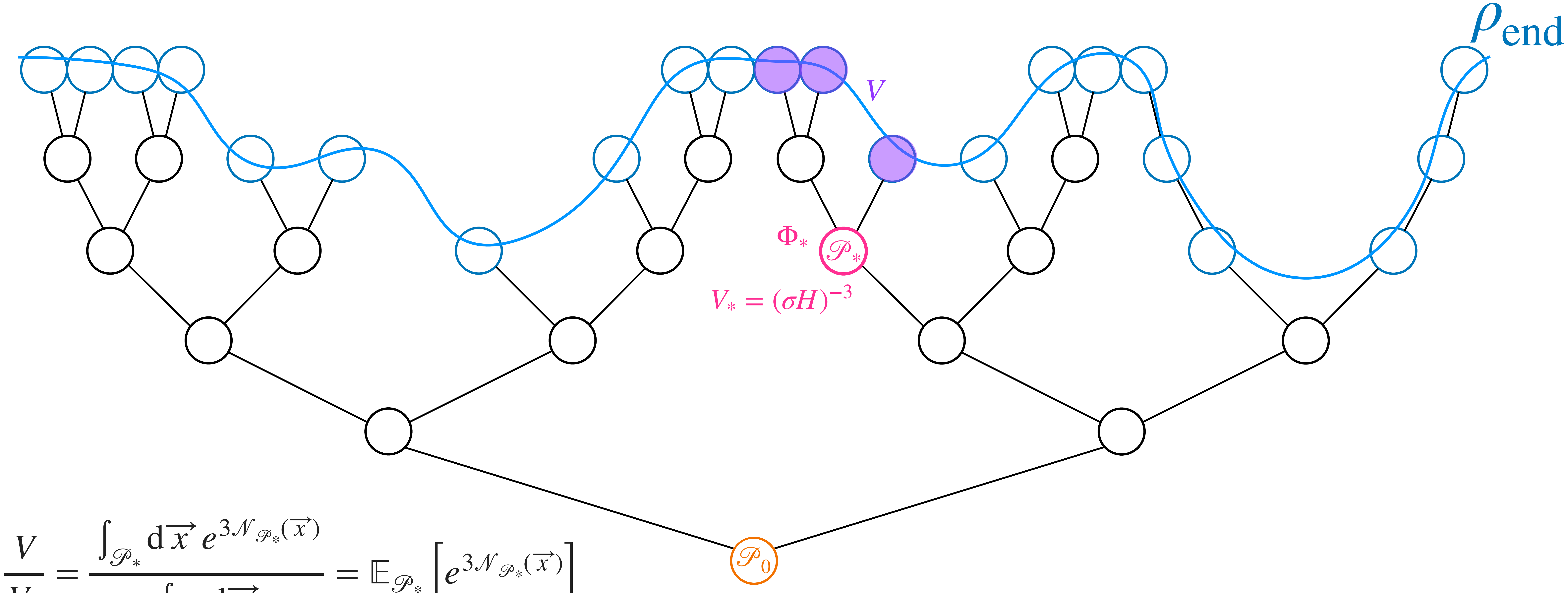

Final volume: $\dfrac{V}{V_*} = \dfrac{\int_{\mathscr{P}_*} \mathrm{d}\vec{x}\, e^{3\mathcal{N}_{\mathscr{P}_*}(\vec{x})}}{\int_{\mathscr{P}_*} \mathrm{d}\vec{x}} = \mathbb{E}_{\mathscr{P}_*}\left[e^{3\mathcal{N}_{\mathscr{P}_*}(\vec{x})}\right]$

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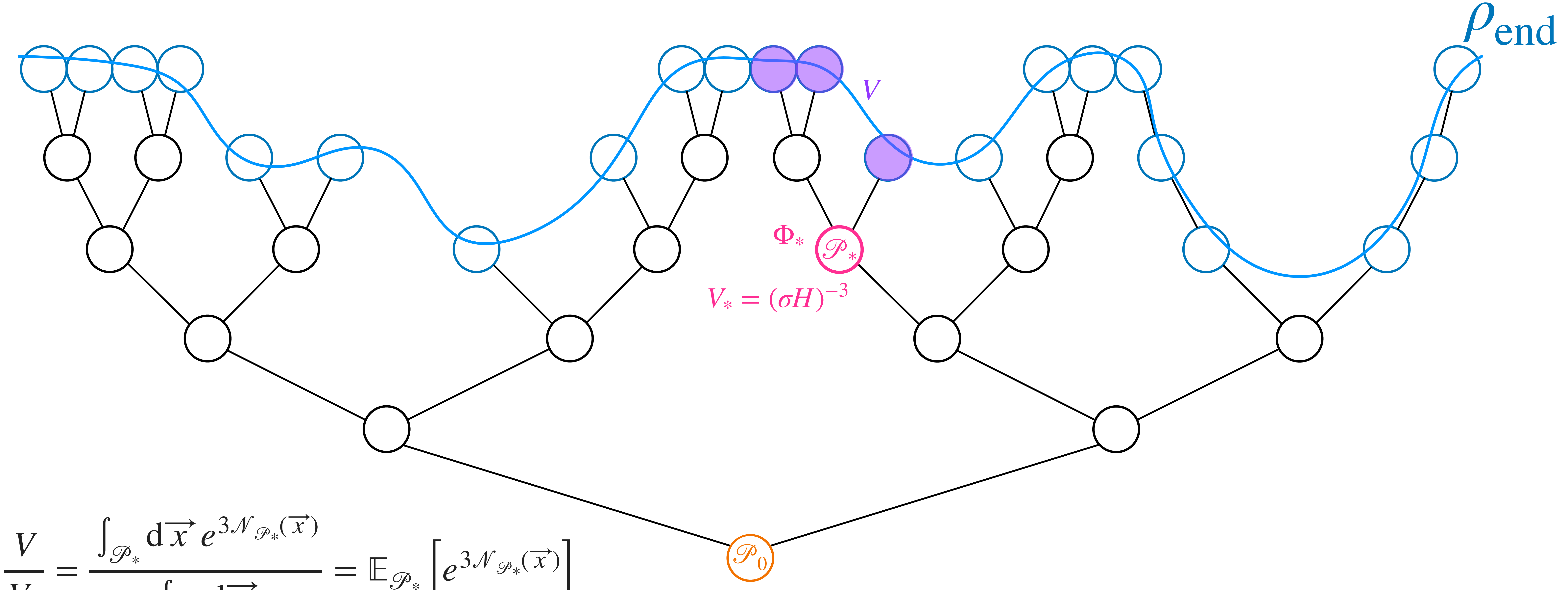

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Volume-averaged number of e-folds: $W \equiv \mathbb{E}^V_{\mathscr{P}_*}\left[\mathscr{N}_{\mathscr{P}_*}(\vec{x})\right] = \frac{\int_{\mathscr{P}_*} e^{3\mathscr{N}_{\mathscr{P}_*}(\vec{x})}\mathscr{N}_{\mathscr{P}_*}(\vec{x})\mathrm{d}\vec{x}}{\int_{\mathscr{P}_*} e^{3\mathscr{N}_{\mathscr{P}_*}(\vec{x})}\mathrm{d}\vec{x}} = \frac{V_*}{V}\mathbb{E}_{\mathscr{P}_*}\left[e^{3\mathscr{N}_{\mathscr{P}_*}(\vec{x})}\mathscr{N}_{\mathscr{P}_*}(\vec{x})\right]$

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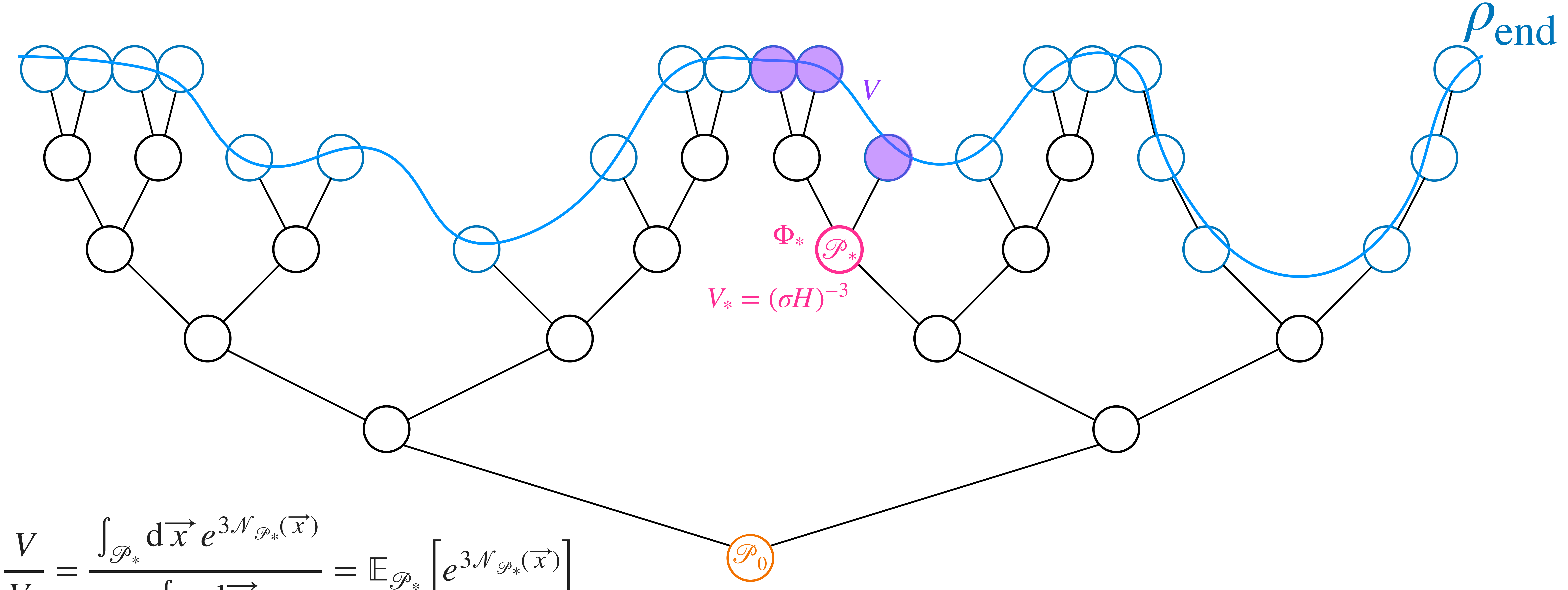

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Distributions $P(V\,|\,\Phi_*)$ and $P(V, W\,|\,\Phi_*)$ can be numerically sampled

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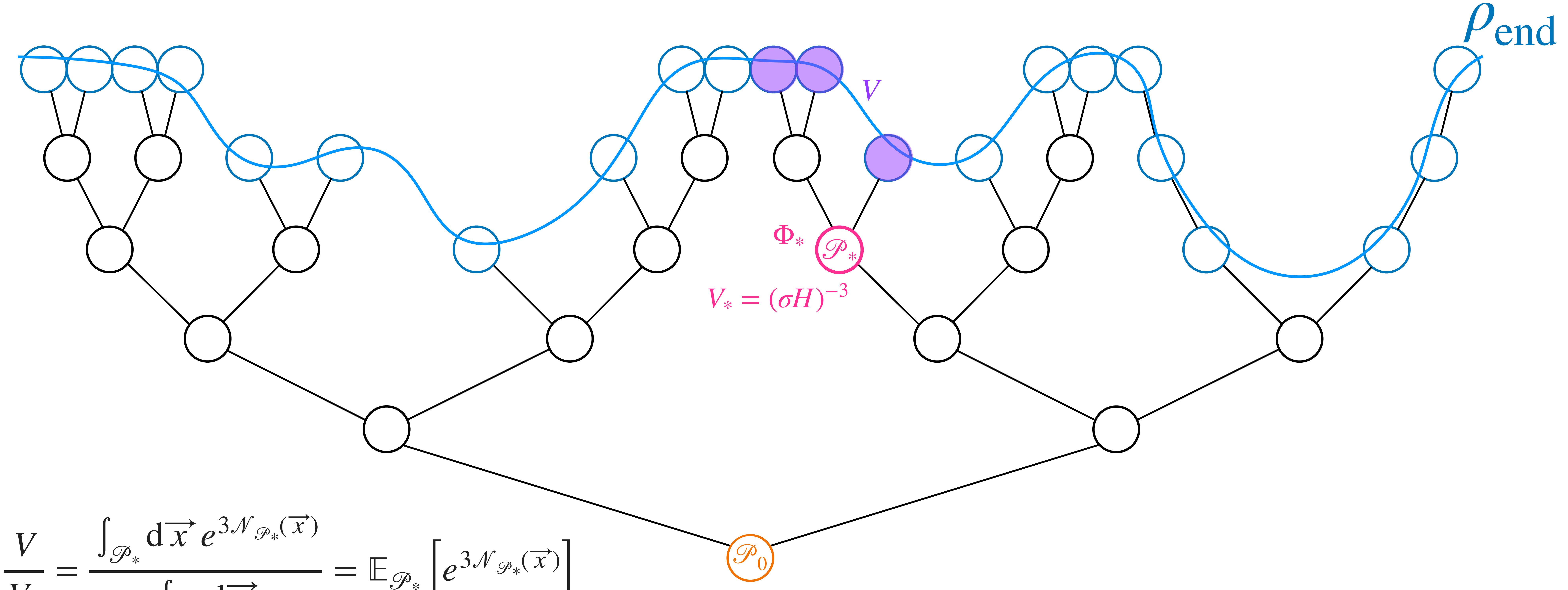

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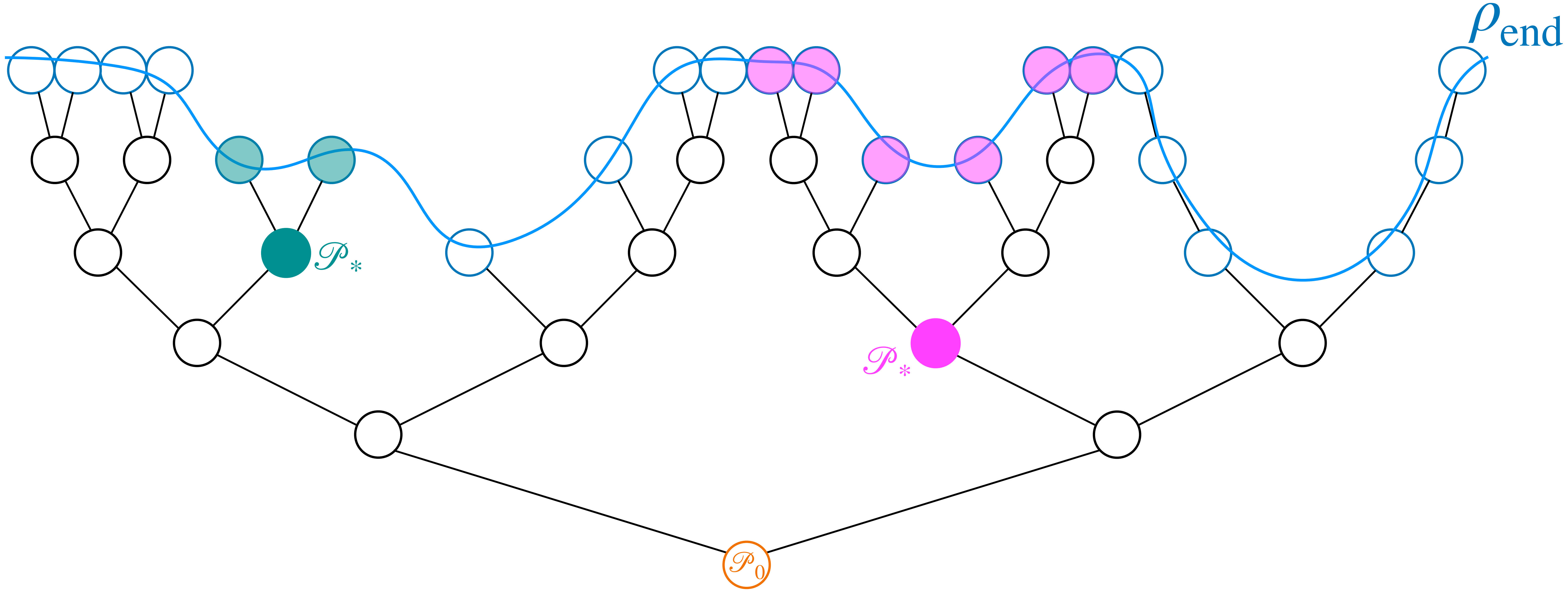
Distributions $P(V\,|\,\Phi_*)$ and $P(V, W\,|\,\Phi_*)$ can be numerically sampled

Backward distribution: $P(\Phi_*\,|\,V, \Phi_0) = \frac{P(V\,|\,\Phi_*)P(\Phi_*\,|\,\Phi_0)}{P(V)} = \frac{P(V\,|\,\Phi_*)P(\Phi_*\,|\,\Phi_0)}{\int \mathrm{d}\Phi_*\, P(V\,|\,\Phi_*)P(\Phi_*\,|\,\Phi_0)}$

Volume weighting

Volume weighting

Different regions of the universe inflate by different amounts $\mathcal{N}$:
they contribute differently to ensemble averages computed by local observers on the end-of-inflation hypersurface



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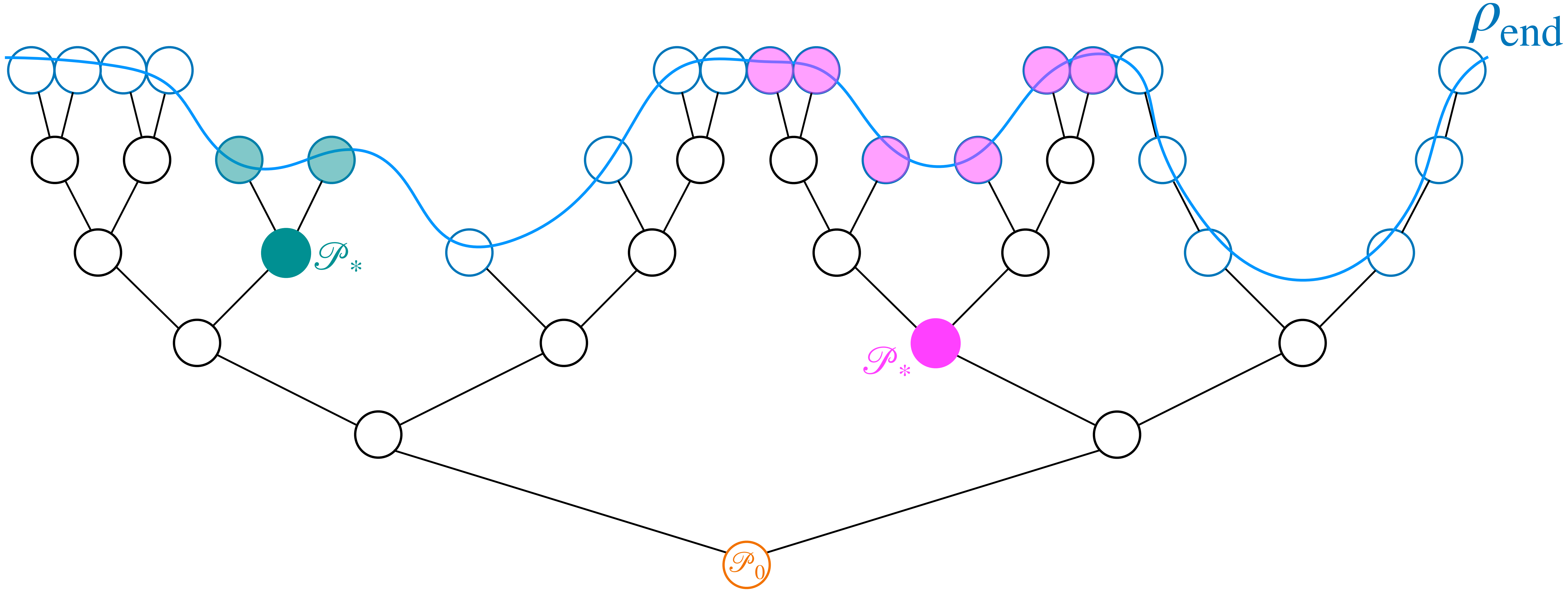

Distributions with respect to which observable quantities are defined should be **volume weighted**

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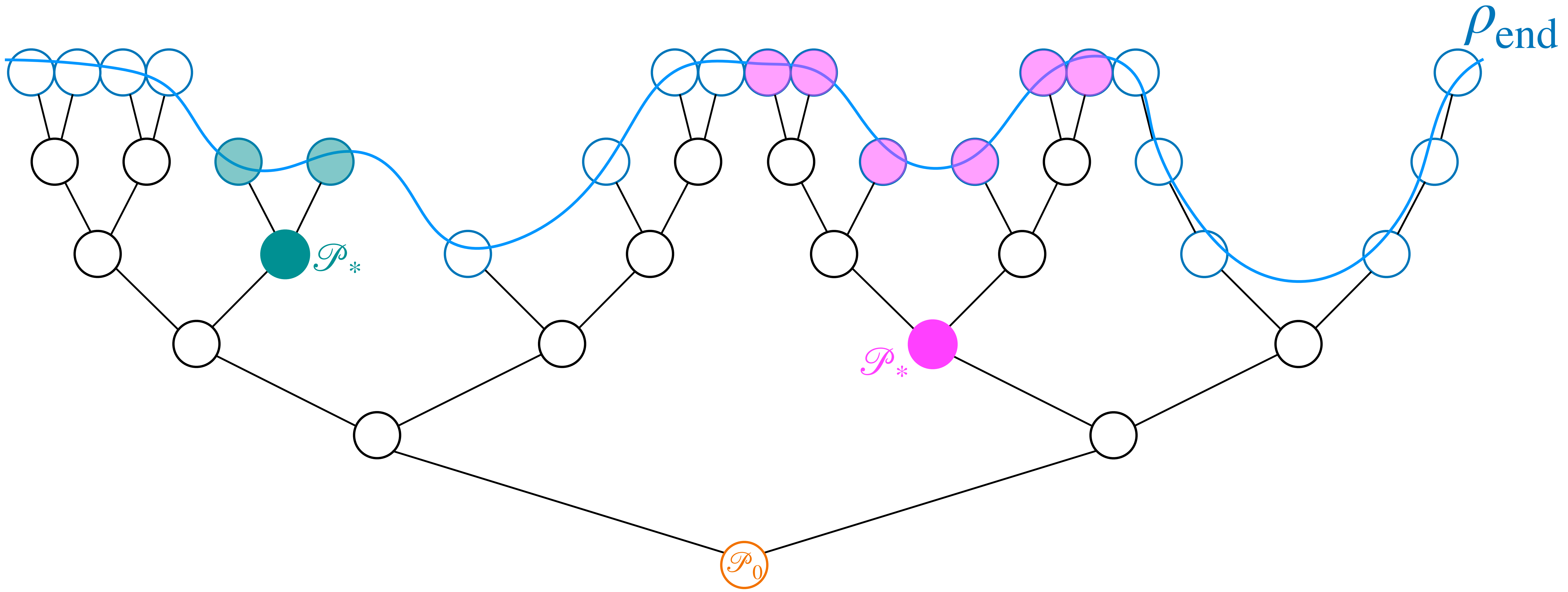

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$$P^V_{\mathrm{FPT},\Phi_0}(\mathcal{N}) = \frac{P_{\mathrm{FPT},\Phi_0}(\mathcal{N})\, e^{3\mathcal{N}}}{\int_0^\infty \mathrm{d}\mathcal{N} P_{\mathrm{FPT},\Phi_0}(\mathcal{N})\, e^{3\mathcal{N}}}$$

$$\zeta_{\rm cg}(\vec{x}) = \mathcal{N}_{\mathcal{P}_0}(\vec{x}) - \mathbb{E}^V_{\mathcal{P}_0}(\mathcal{N}_{\mathcal{P}_0}) \qquad P(\zeta_{\rm cg}\,|\,\Phi_0) = P^V_{\mathrm{FPT},\Phi_0}(\zeta_{\rm cg} + \mathbb{E}^V_{\mathcal{P}_0}(\mathcal{N}_{\mathcal{P}_0}))$$

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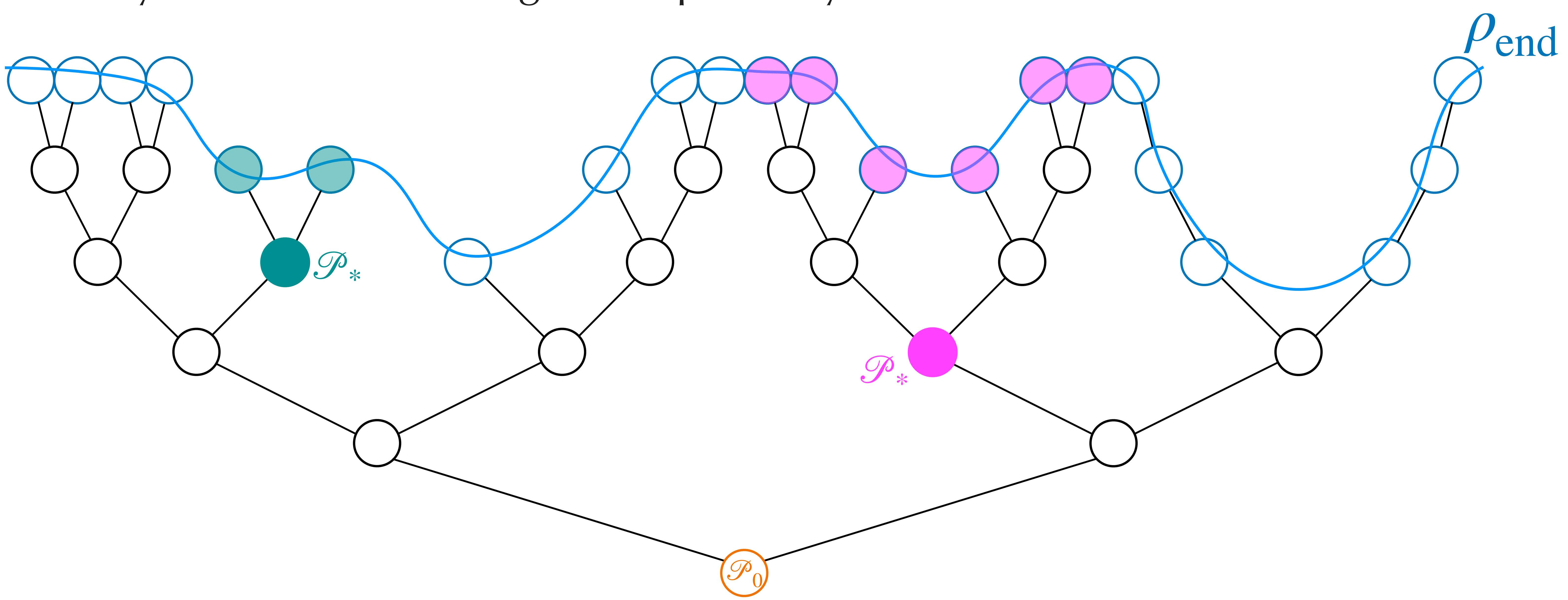

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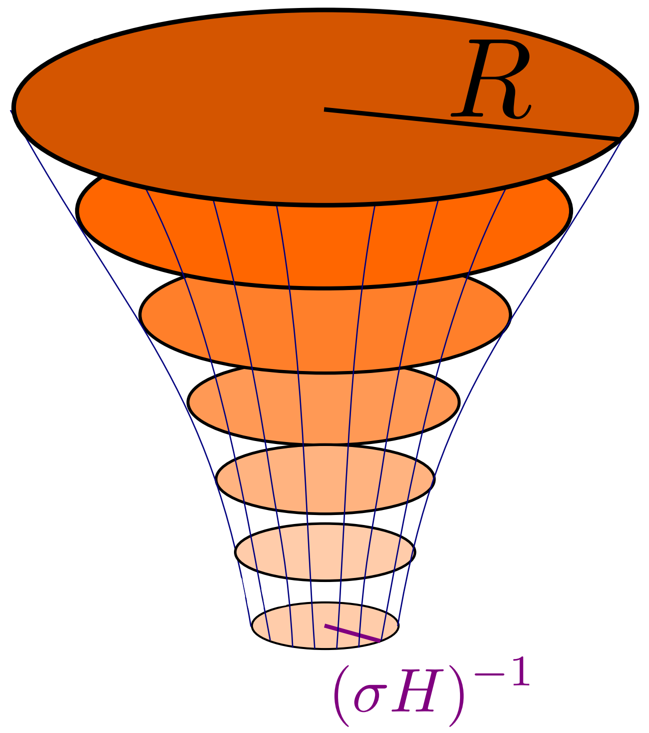
For $P_{\mathrm{FPT},\Phi_0}(\mathcal{N}) \propto e^{-\Lambda\mathcal{N}}$ and $\Lambda \leq 3$ the volume-weighted distribution is not well-defined "eternal inflation"

Large-volume approximation



$R^3 \gg (\sigma H)^{-3}$

Large-volume approximation

Ensemble average over the set of final leaves → Stochastic average of a single element within the ensemble

$$V \to \langle V \rangle \qquad P(V\,|\,\Phi_*) \simeq \delta_{\rm D}\left(V - V_* \langle e^{3\mathcal{N}_{\Phi_*}} \rangle\right)$$

$$W \to \langle W \rangle \qquad W \simeq \langle \mathcal{N}_{\Phi_*} \rangle_V = \frac{\langle \mathcal{N}_{\Phi_*} e^{3\mathcal{N}_{\Phi_*}} \rangle}{\langle e^{3\mathcal{N}_{\Phi_*}} \rangle}$$

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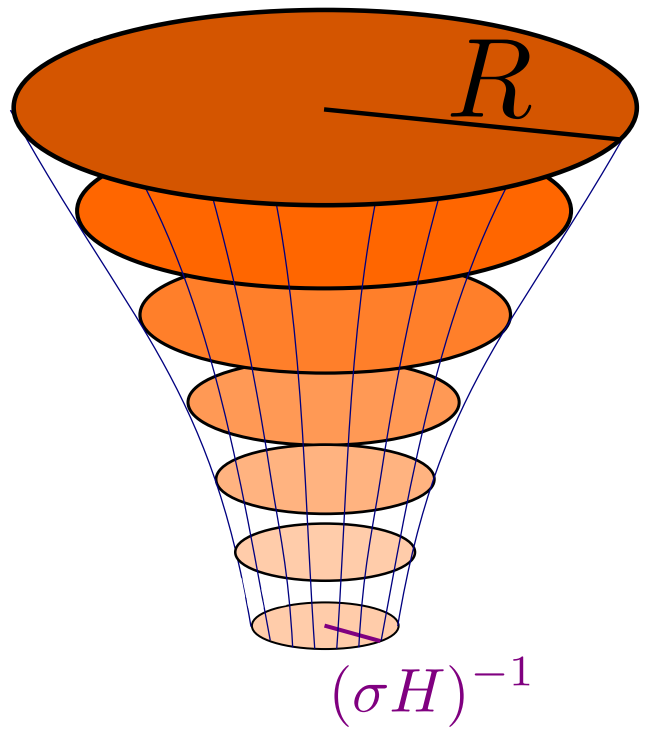

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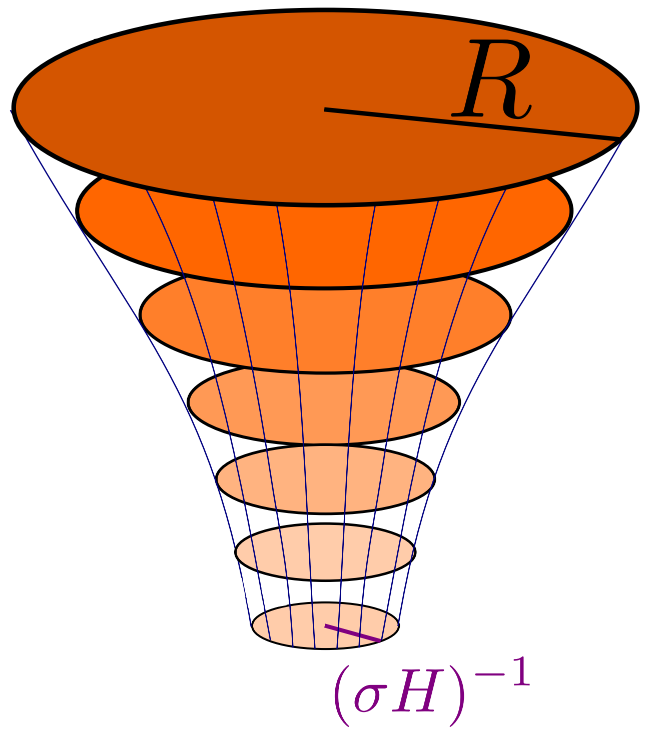

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Parent patches become hypersurfaces of fixed final volume, approximated by the mean volume

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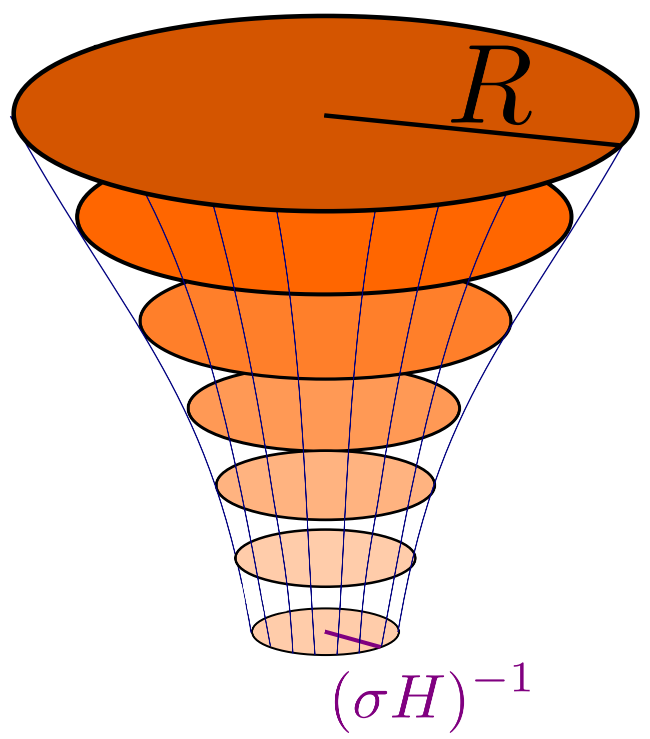

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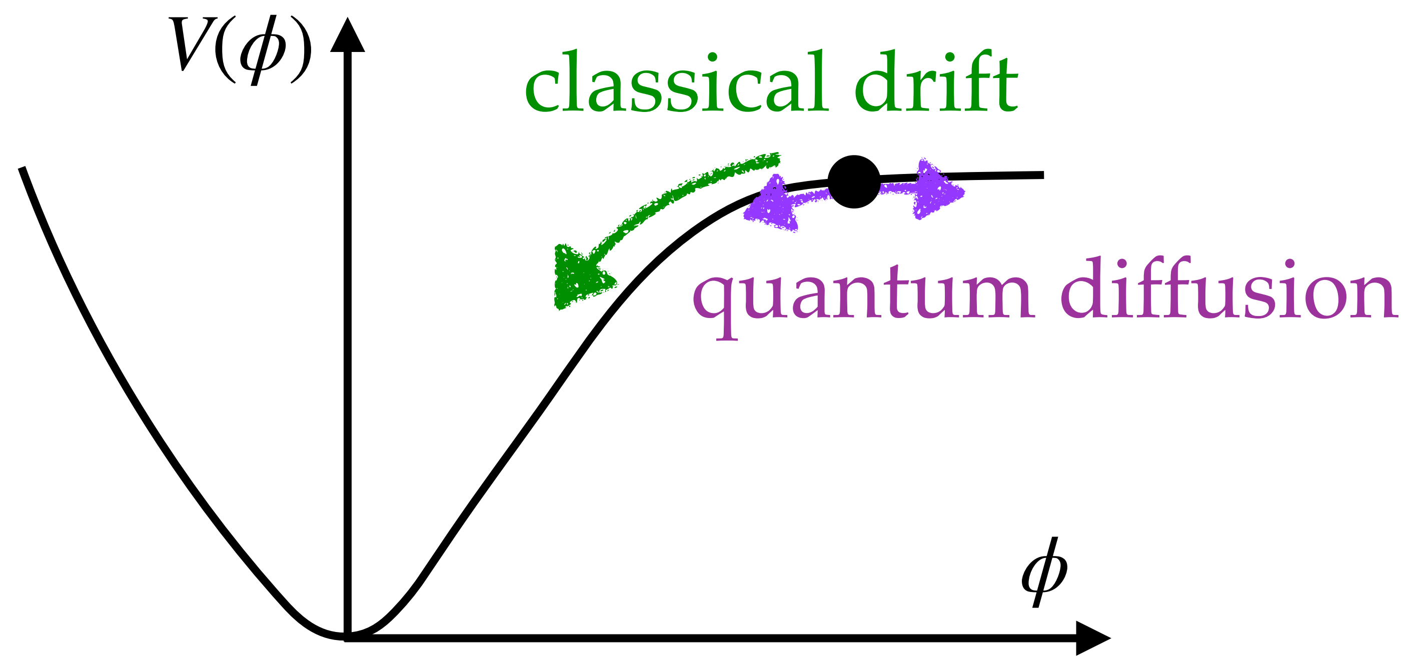
Single-clock models

$\Phi \to \phi$: single-field models of inflation along a dynamical attractor (slow roll)



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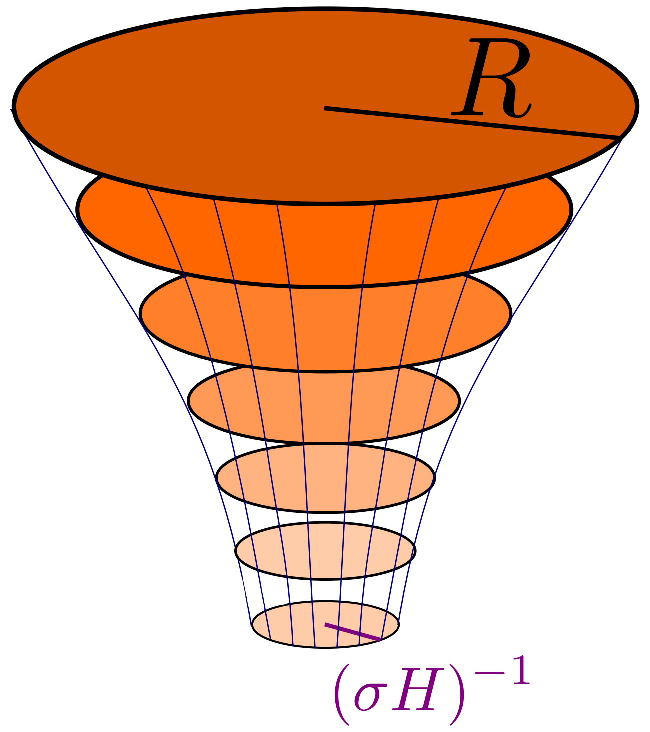

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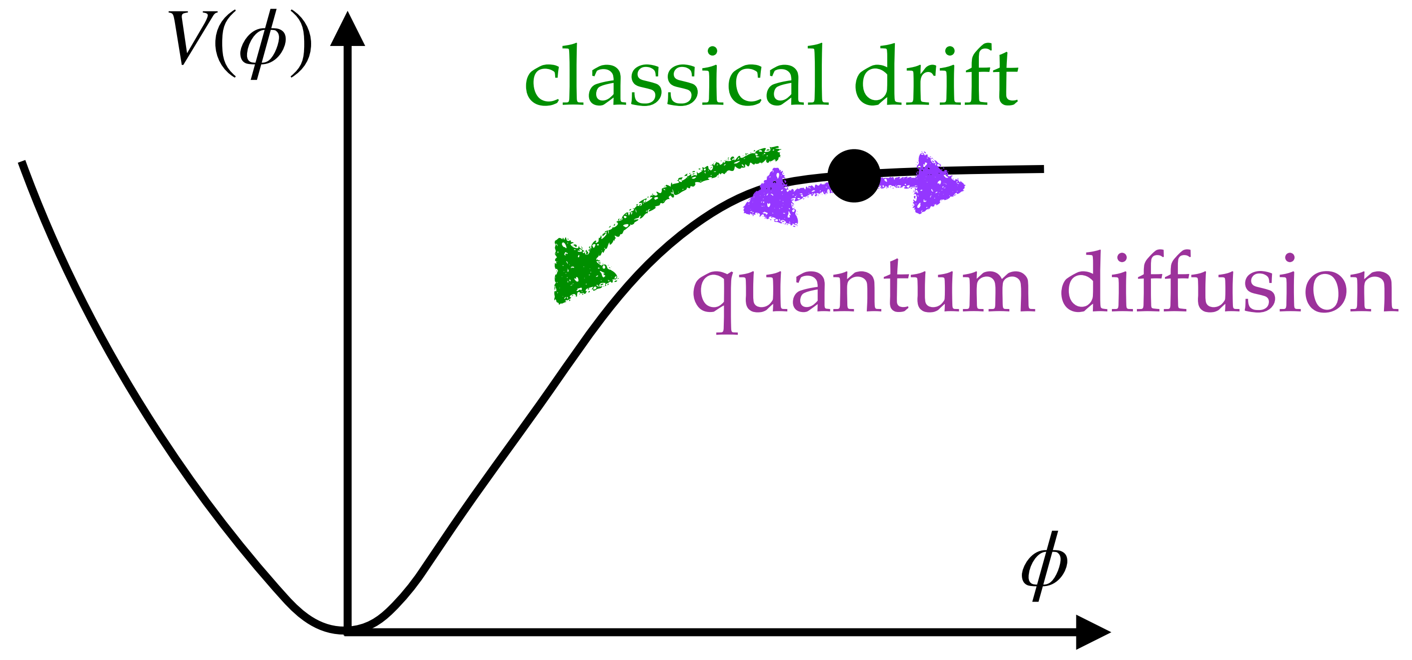
$\Phi \to \phi:$ single-field models of inflation along a dynamical attractor (slow roll)

Hypersurfaces of fixed mean final volume reduce to **single points**



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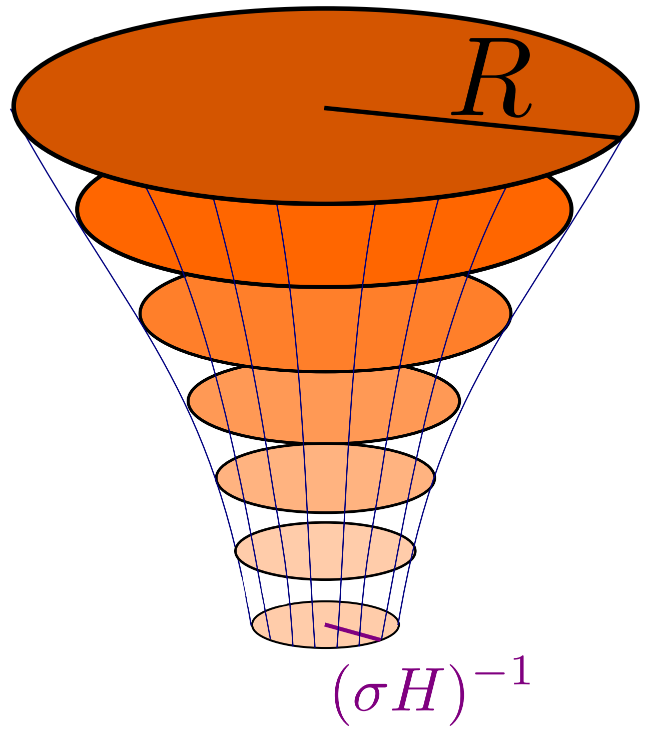

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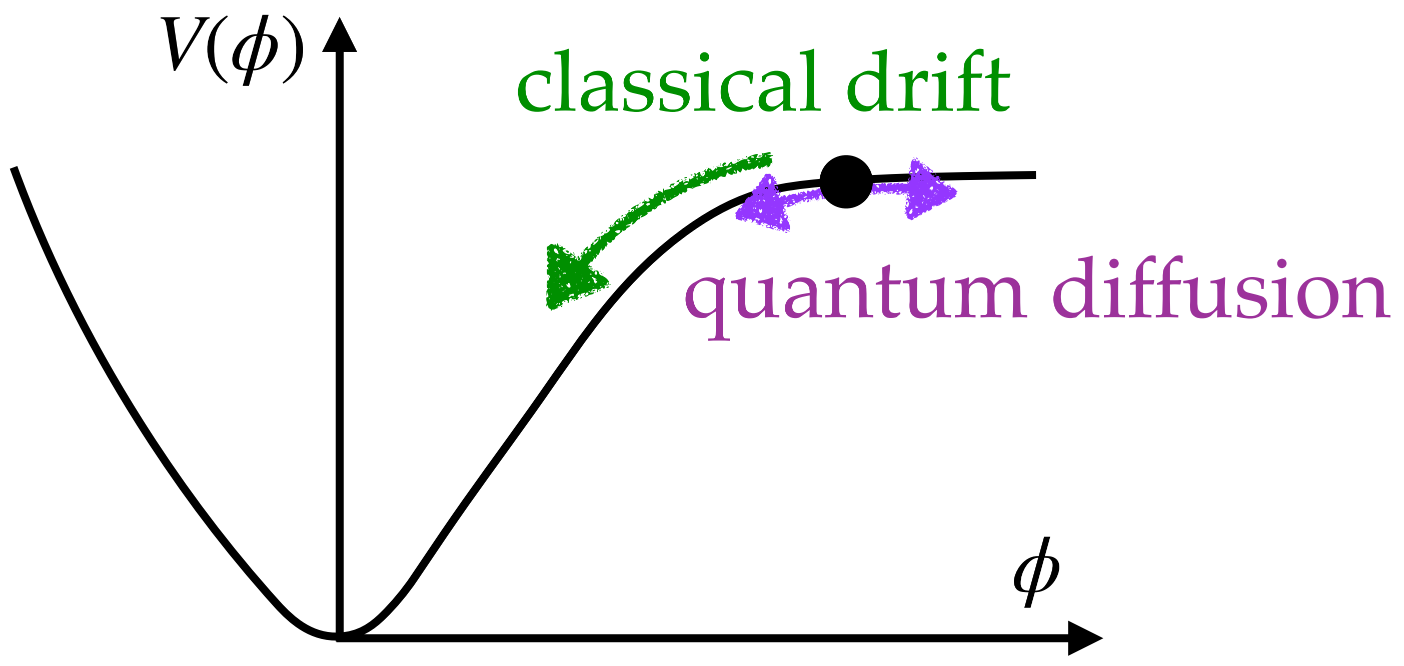
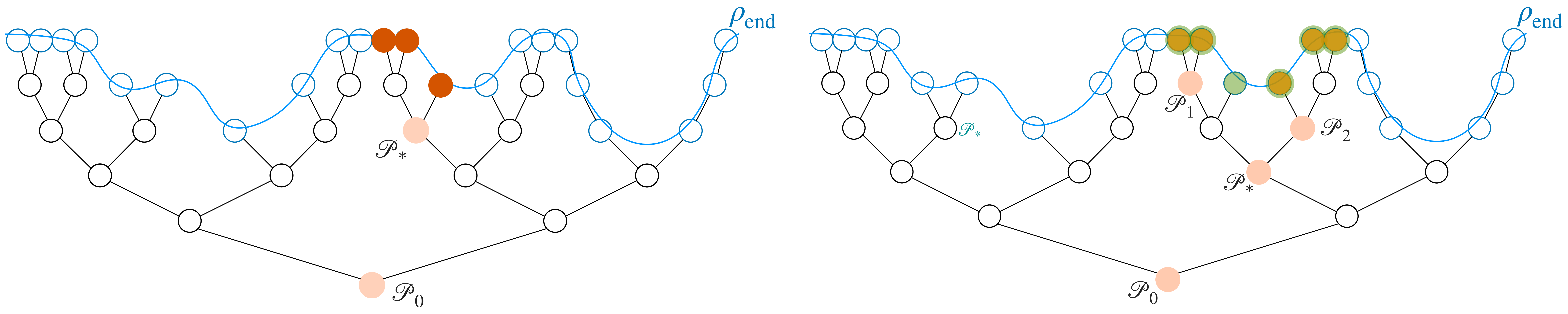
Hypersurfaces of fixed mean final volume reduce to **single points**

Backward fields become **deterministic** quantities



Statistics of coarse-grained fields in the large-volume approximation

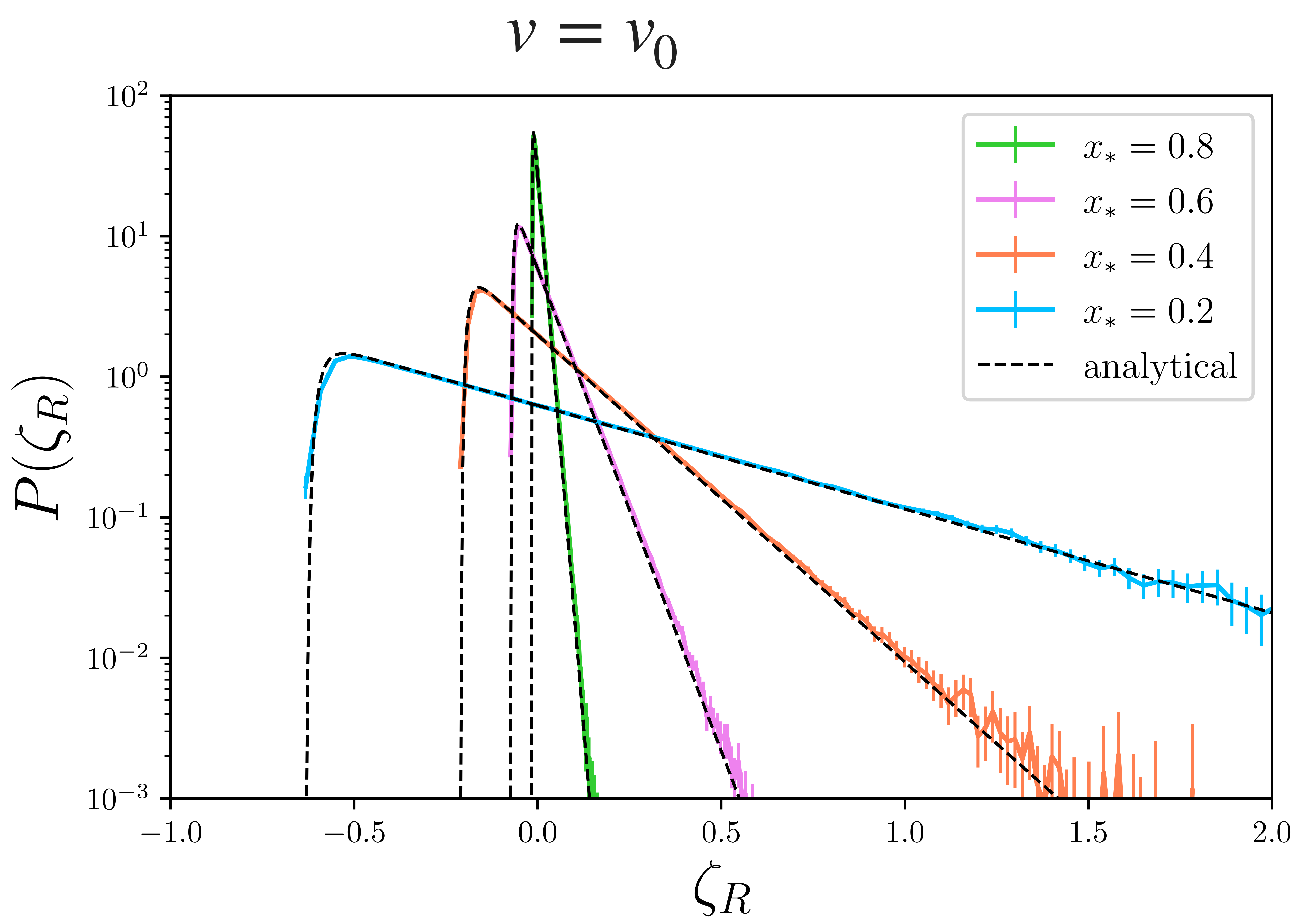


$$P(\zeta_R) = P^V_{\mathrm{FPT},\phi_0\to\phi_*}\left(\zeta_R - \langle \mathcal{N}_{\phi_*}\rangle_V + \langle \mathcal{N}_{\phi_0}\rangle_V\right)$$

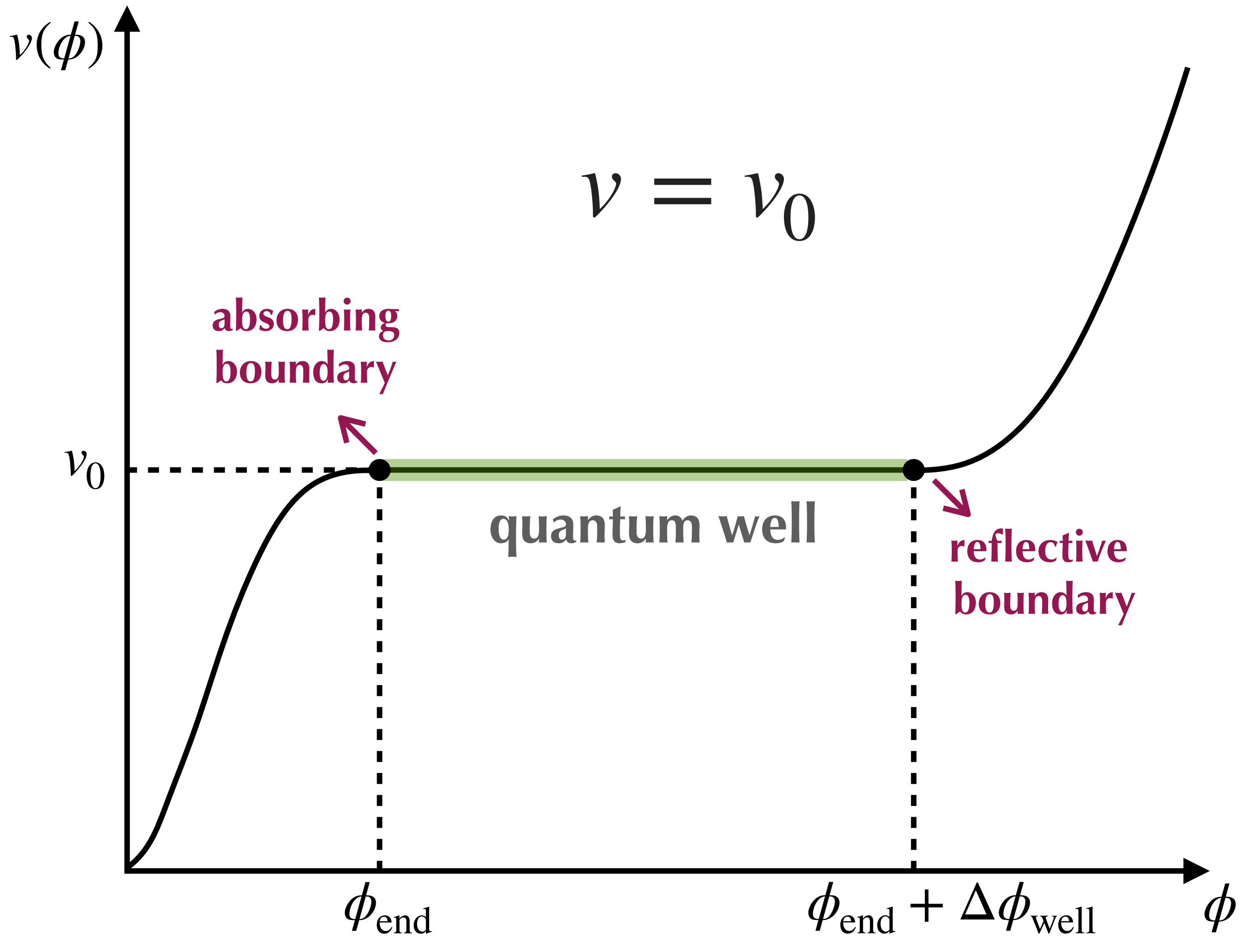
$$P(\zeta_{R_1},\zeta_{R_2}) = \int \mathrm{d}\mathcal{N}_{\phi_0\to\phi_*}(\mathcal{N}_{\phi_0\to\phi_*}) P^V_{\mathrm{FPT},\phi_*\to\phi_1}\left(\zeta_{R_1} - \mathcal{N}_{\phi_0\to\phi_*} + \langle \mathcal{N}_{\phi_0}\rangle_V - \langle \mathcal{N}_{\phi_1}\rangle_V\right) P^V_{\mathrm{FPT},\phi_*\to\phi_2}\left(\zeta_{R_2} - \mathcal{N}_{\phi_0\to\phi_*} + \langle \mathcal{N}_{\phi_0}\rangle_V - \langle \mathcal{N}_{\phi_2}\rangle_V\right)$$

Large-volume approximation

One-point distribution of curvature perturbation coarse grained at scale R

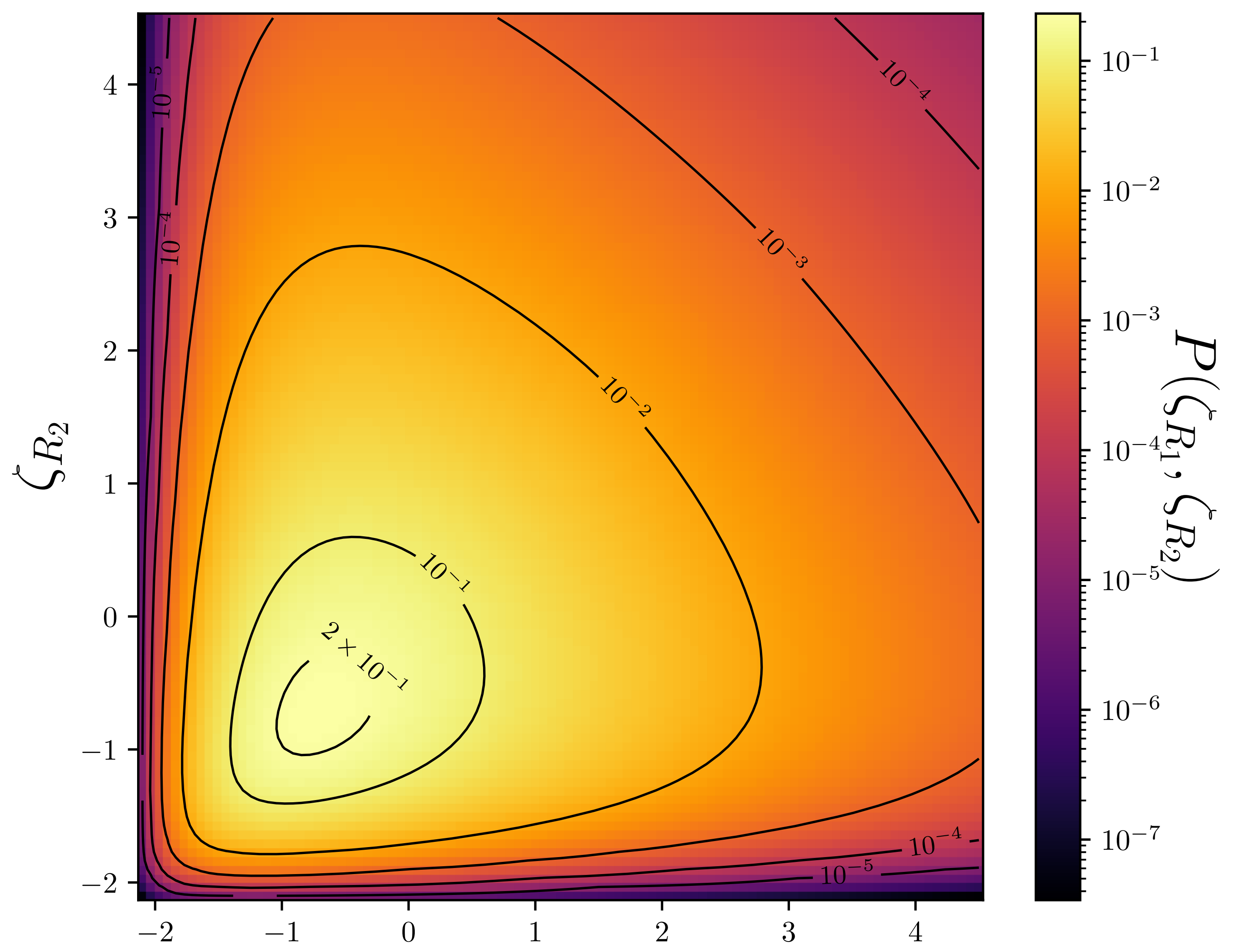


quantum well:

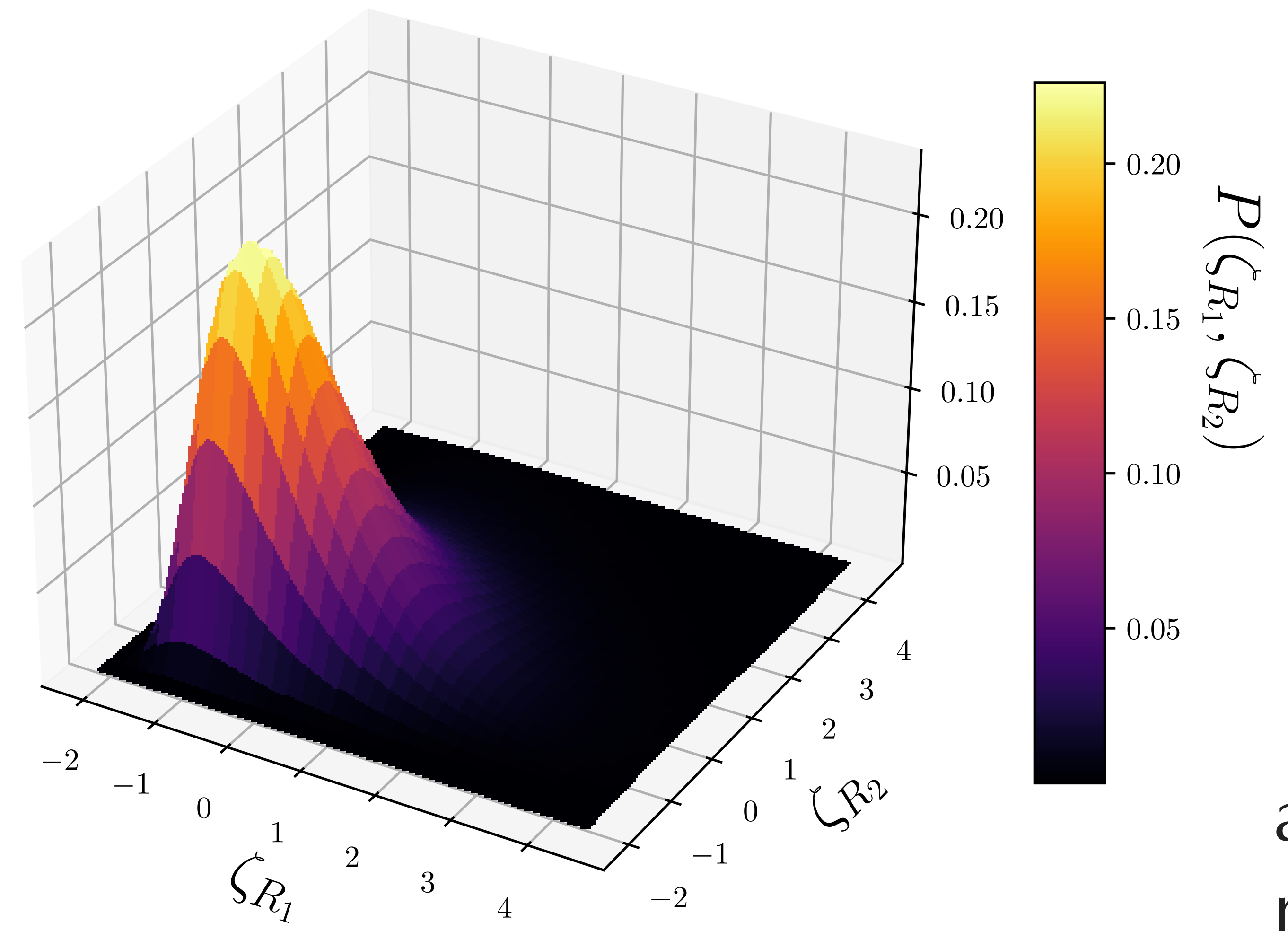


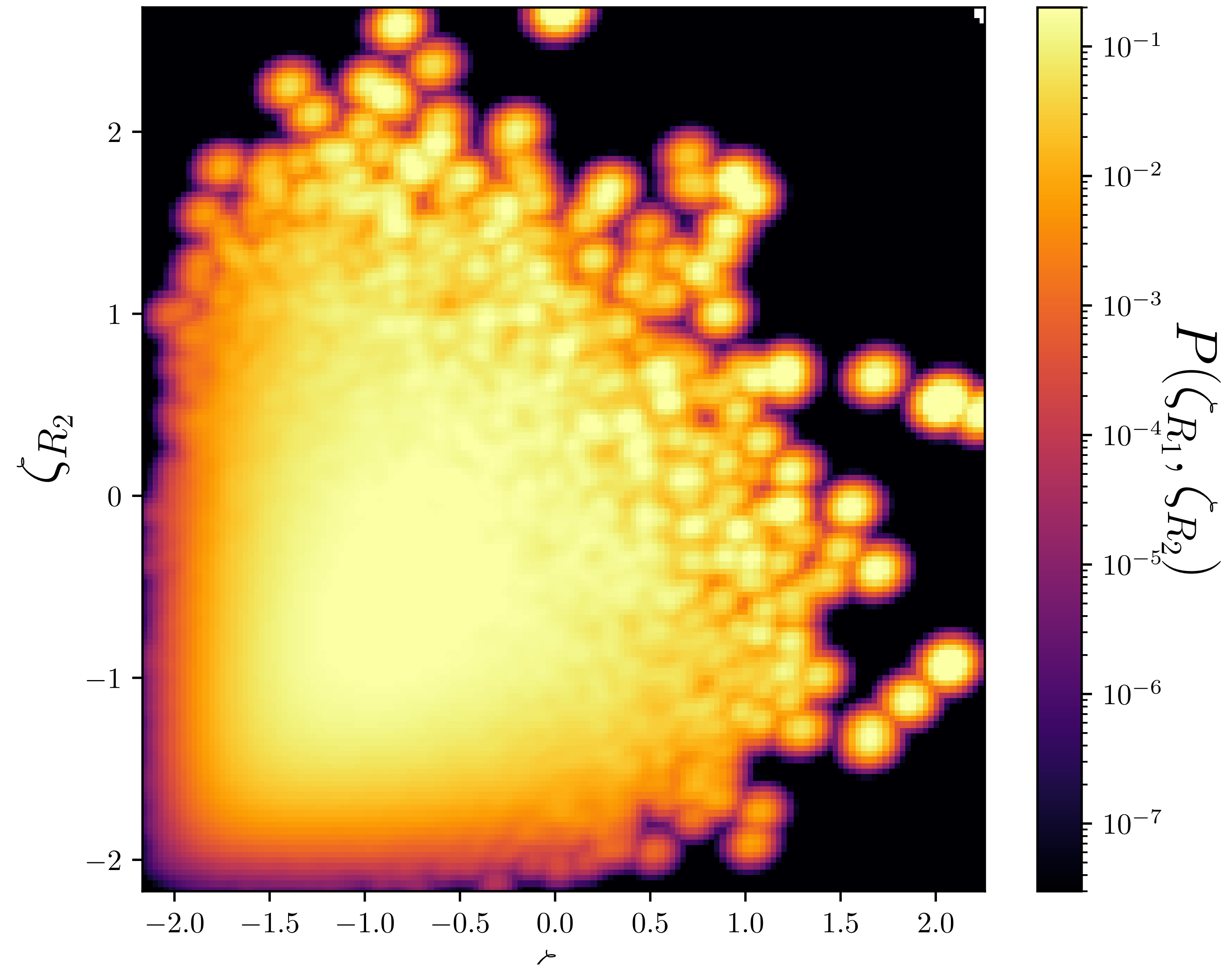
Large-volume approximation

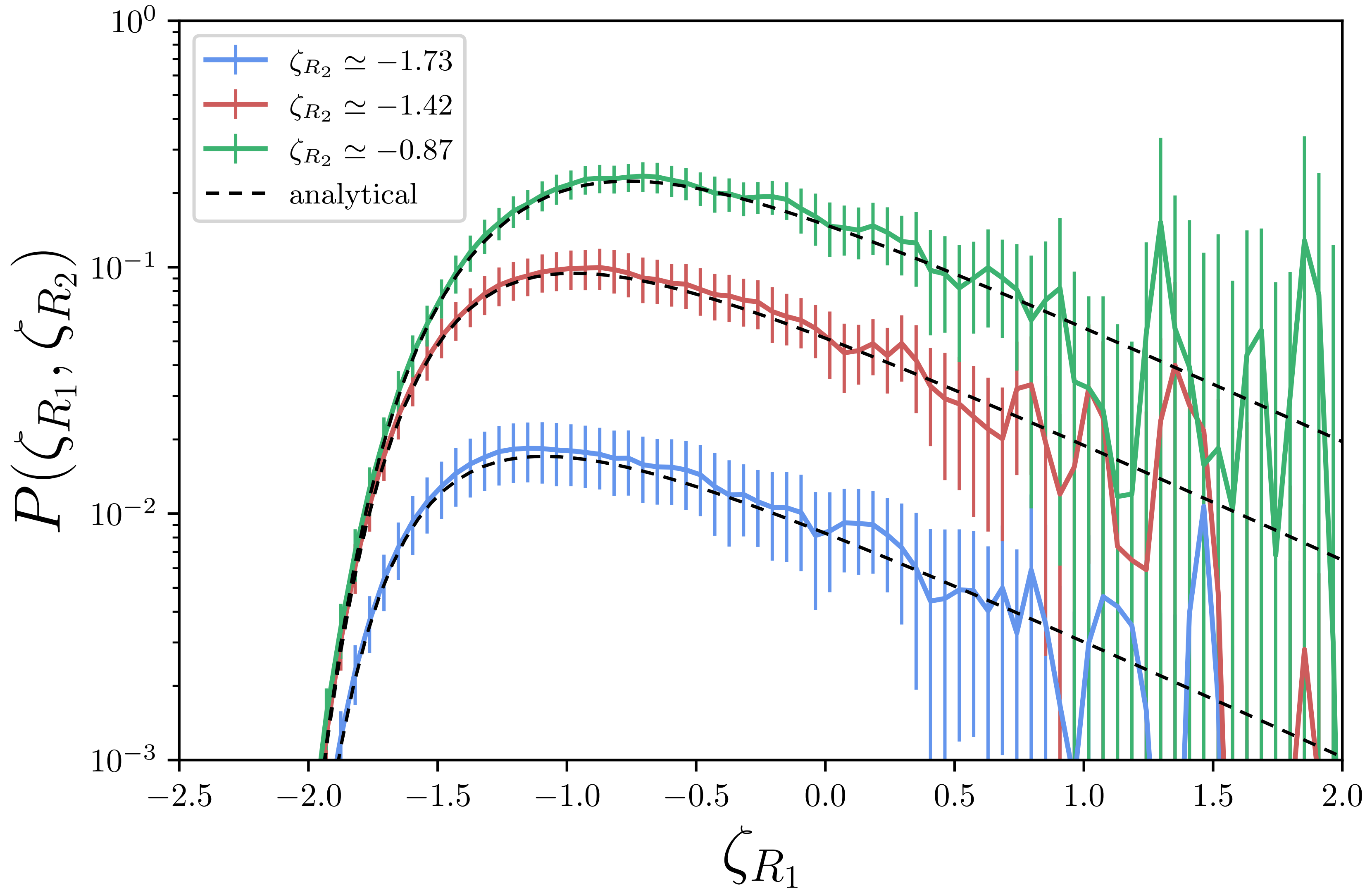
Two-point distribution of coarse grained fields



analytical approx.
results

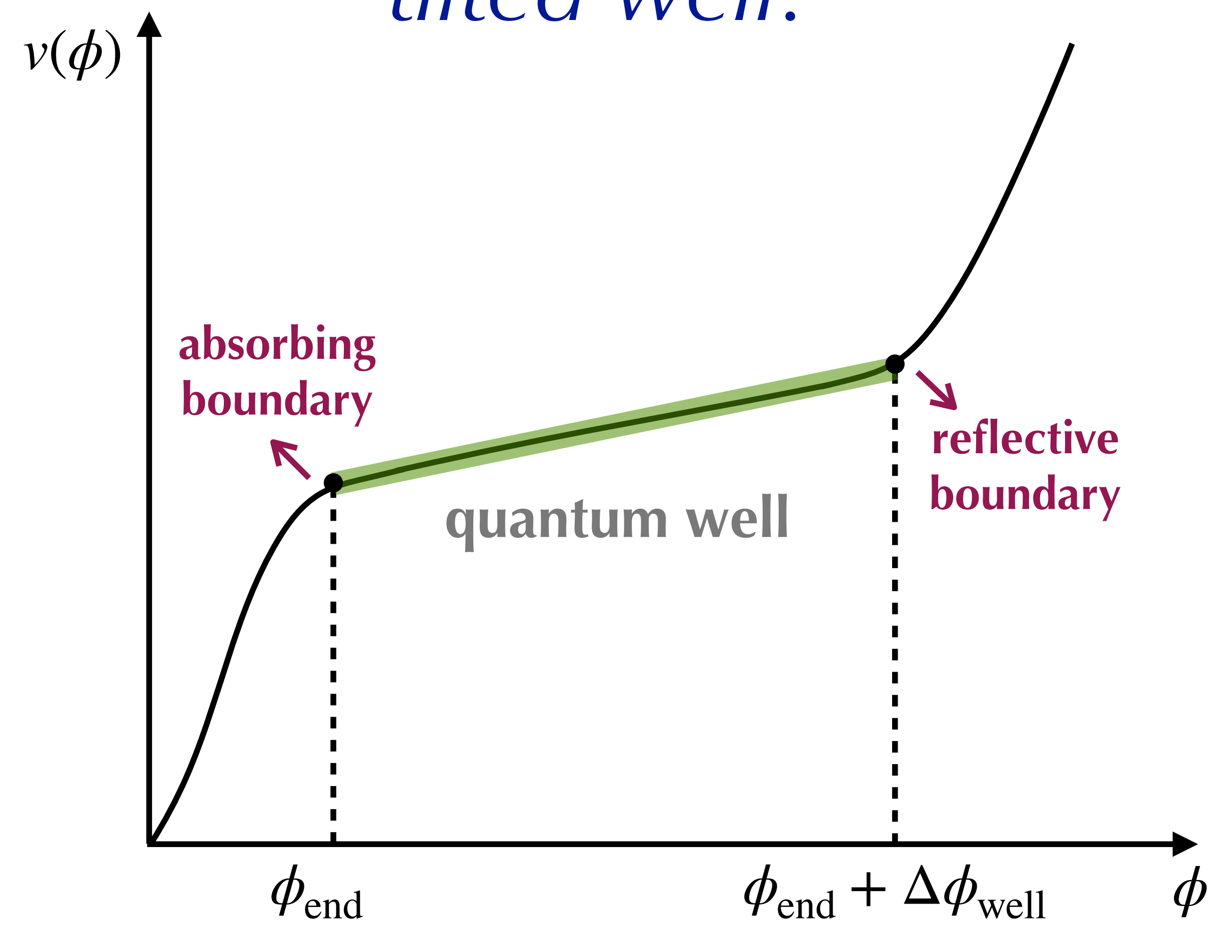






analytical approx.
results

tilted well:



numerical simulations

Power spectrum in the large-volume approximation

Two-point correlation function of coarse-grained fields:

$$\langle \zeta_{R_1}\zeta_{R_2}\rangle = \int \mathrm{d}\zeta_{R_1}\int \mathrm{d}\zeta_{R_2} P(\zeta_{R_1},\zeta_{R_2})\zeta_{R_1}\zeta_{R_2} = \langle \mathcal{N}^2_{\phi_0\to\phi_*}\rangle_V - \langle \mathcal{N}_{\phi_0\to\phi_*}\rangle_V^2 \equiv \langle \delta\mathcal{N}^2_{\phi_0\to\phi_*}\rangle_V = \langle \delta\mathcal{N}^2_{\phi_0}\rangle_V - \langle \delta\mathcal{N}^2_{\phi_*}\rangle_V$$

no dependence on the coarse-graining scales R_1, R_2

In Fourier space: $\zeta_{R_i}(\vec{x}_i) = \int \frac{\mathrm{d}\vec{k}}{(2\pi)^{3/2}} \zeta_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}_i}\, \widetilde{W}\left(\frac{kR_i}{a}\right)$

$$\langle \zeta_{R_1}\zeta_{R_2}\rangle = \int_0^\infty \mathrm{d}\ln k\, \mathcal{P}_\zeta(k)\widetilde{W}\left(\frac{kR_1}{a}\right)\widetilde{W}\left(\frac{kR_2}{a}\right)\widetilde{W}\left(\frac{kr}{a}\right)$$

$r > R_1, R_2$ → $\langle \zeta_{R_1}\zeta_{R_2}\rangle = \int_0^\infty \mathrm{d}\ln k \mathcal{P}_\zeta(k)\widetilde{W}\left(\frac{kr}{a}\right)$

Differentiation w.r.t. r:

$$\mathcal{P}_\zeta(k) = -\frac{\partial}{\partial \ln r}\langle \zeta_{R_1}\zeta_{R_2}\rangle \Big|_{r=a_{\mathrm{end}}/k} = \frac{\partial}{\partial \ln r}\langle \delta\mathcal{N}_{\phi_*}\rangle^2 \Big|_{r=a_{\mathrm{end}}/k}$$

$$\tilde{r} = r + R_1 + R_2$$

$$r \gg R_1, R_2 \to \frac{r}{\tilde{r}} \simeq 1$$

$$\partial \ln N/\partial\phi \simeq \sqrt{\epsilon_1/2}/M_{\mathrm{Pl}}$$

$$\mathcal{P}_\zeta(k) = \frac{r}{\tilde{r}}\left[\frac{1}{3}\frac{\partial}{\partial\phi_*}\ln\langle e^{3\mathcal{N}_{\phi_*}}\rangle - \frac{\partial}{\partial\phi_*}\ln H(\phi_*)\right]^{-1}\frac{\partial}{\partial\phi_*}\langle\delta\mathcal{N}^2_{\phi_*}\rangle_V\Big|_{\langle e^{3\mathcal{N}_{\phi_*}}\rangle^{1/3}=\frac{1}{2}\frac{r}{\tilde{r}}\frac{a_{\mathrm{end}}\sigma H(\phi_*)}{k}}$$

c.f.r. V. Vennin and A. A. Starobinsky [2015]
T. Fujita, M. Kawasaki, Y. Tada and T. Takesako [2013]

Same expression at l.o. in slow roll neglecting volume weighting and defining ϕ_* via $\langle\mathcal{N}\rangle$ and not via $\langle e^{3\mathcal{N}}\rangle$

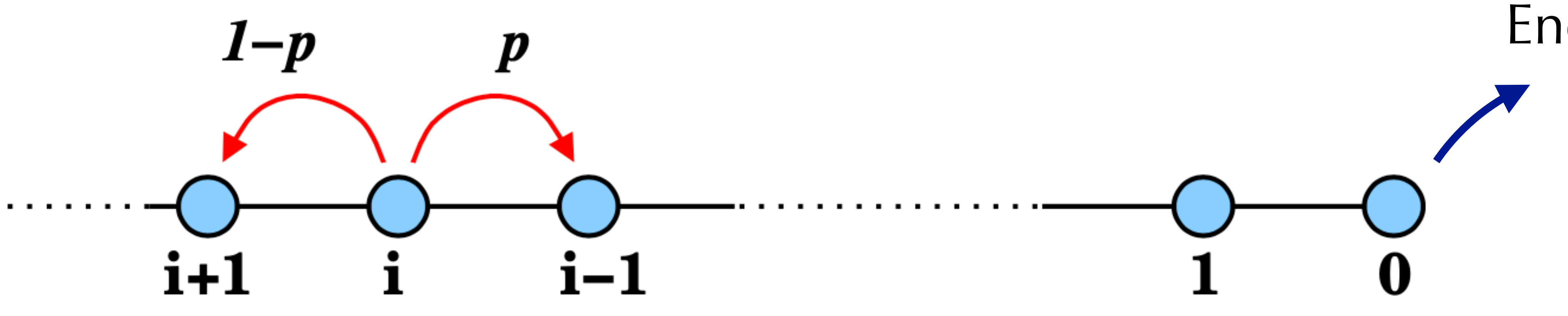
Going beyond

Bacteria model of inflation

Creminelli, Dubovsky, Nicholas, Senatore, Zaldarriaga [2008]
Dubovsky, Senatore, Villadoro [2009]

Discretisation of the inflationary dynamics

End site : bacterium dies



Galton-Watson process

Bacteria live on discrete set of positions along a line, replicating into N copies at each time step

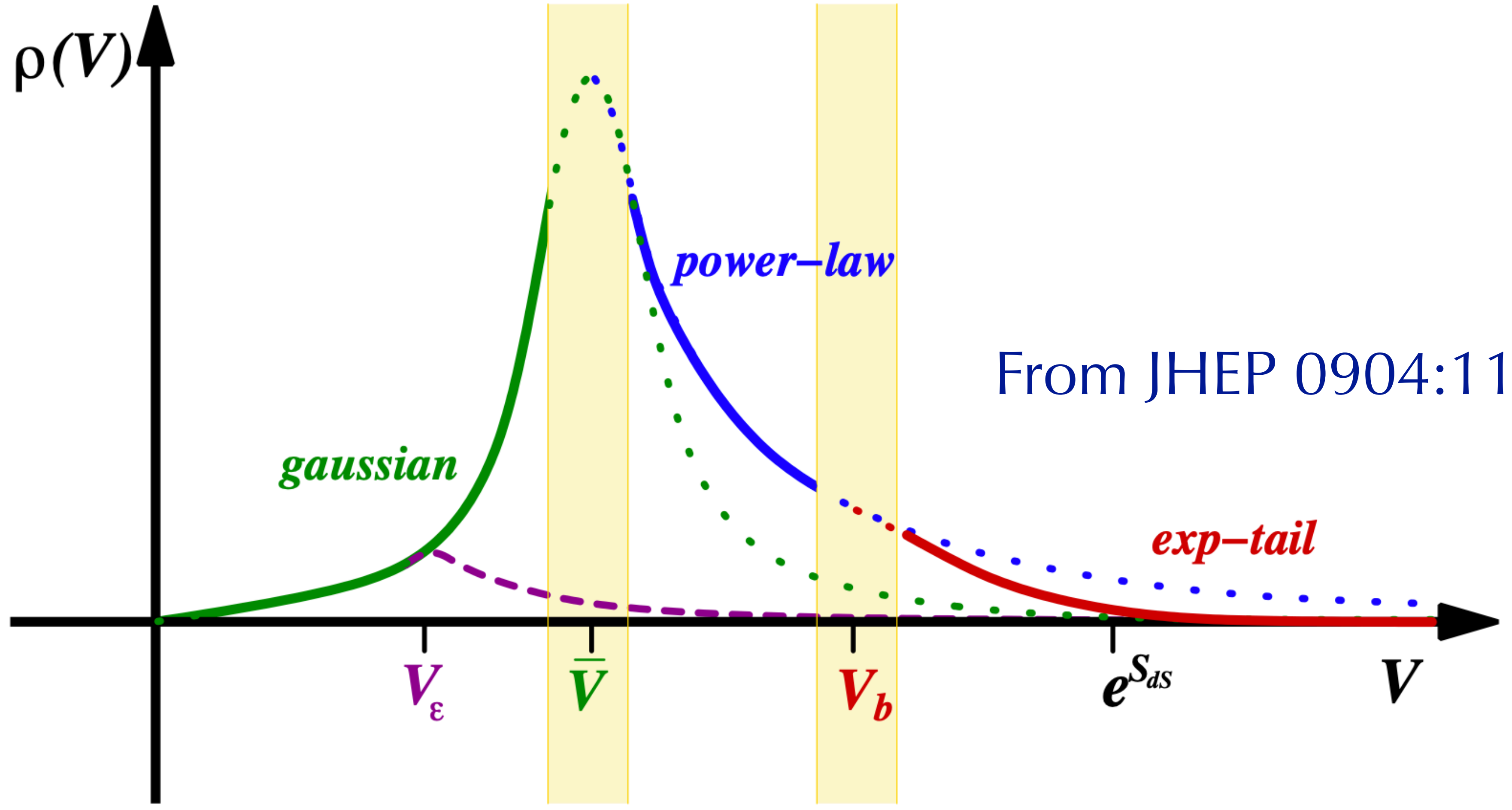
Bacteria → Hubble patches

Sites → Inflaton values

Difference in $(1-p)$ and p → Drift

Random hopping → Quantum diffusion

Number of dead bacteria → Final volume



From JHEP 0904:118,2009

Stochastic-δN program on stochastic trees

Ongoing works with
Pierre Auclair, Baptiste Blachier, Vincent Vennin

Conclusions

Stochastic inflation and stochastic-δN formalisms are powerful framework to compute non perturbative results for cosmological observables

For small noise amplitude, standard results are recovered, but for regimes of large perturbations, it gives specific imprints in cosmological observables

It can be extended beyond one-point statistics: power spectrum

At observable scales, the power spectrum seems not affected by quantum diffusion

Still several assumptions: single field, slow roll, toy models, backward approximation, large-volume approximation: not definitive results

Promising directions: more results are yet to come!