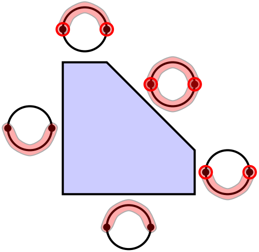
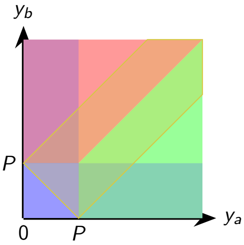
Looping with Combinatorics





Paolo Benincasa

Instituto Galego de Física de Altas Enerxías

28 October 2024 – Looping in the Primordial Universe

Based on works in collaboration with:
F. Vazão,
G. Brunello, M. K. Mandal, P. Mastrolia, F. Vazão.

Why loops?

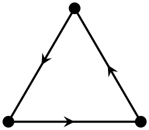
1. Phenomenology

2. (Quantum) consistency of the theory

Why loops?

1. Phenomenology

   E.g.: Coupling inflaton-fermions



2. (Quantum) consistency of the theory
   a. Infra-red effects
   b. Renormalisation
   c. Perturbation theory breakdown
   d. General consistency conditions

Why loops? A flat-space lesson

Guiding principles

Symmetries

Quantum Mechanics

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Quantum Mechanics

Flat
Space

Poincaré
invariance

Unitarity

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Poincaré invariant operator: $\hat{\mathcal{O}}$

$\langle 3,4|\hat{\mathcal{O}}|1,2\rangle := \delta\left(p_1+p_2-p_3-p_4\right)$

Unitary S-matrix: $\hat{S}\,|\,\hat{S}\hat{S}^\dagger = \hat{\mathbb{I}} = \hat{S}^\dagger\hat{S}$

$\hat{S} := e^{i\lambda\hat{\mathcal{O}}}$

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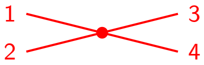

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Anything wrong?

?

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Anything wrong? → **Wrong analytic structure Non causal!**

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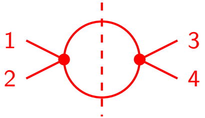

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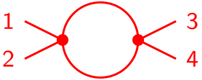


Why loops? A flat-space lesson

It is important to understand the analytic structure of observables on general grounds

Constraints from first principles → consistency conditions on any reasonable theory

What is the relevant space of functions? → Avoid the difficulties of direct computation.

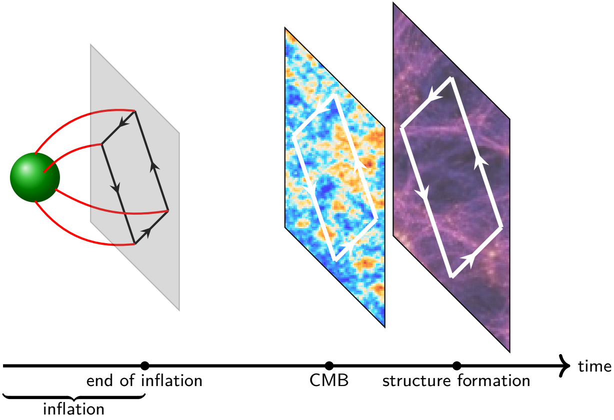
What is the best way of asking these questions? → A new language needed?

Why Combinatorics?

Deeper understanding of the physics encoded
into cosmological observables

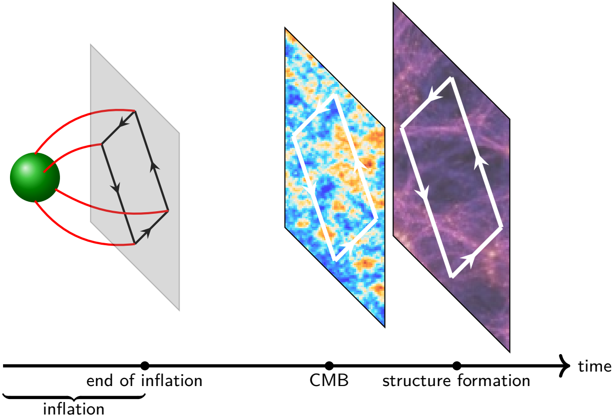
Novel rules which can allow to go beyond the regime
in which the combinatorial description has been formulated

Observables & Fundamental Principles



$\langle \Phi\Phi \cdots \rangle$ $\quad$ $\langle \delta T \delta T \cdots \rangle$ $\quad$ $\langle \delta\rho_g \delta\rho_g \cdots \rangle$

Observables & Fundamental Principles



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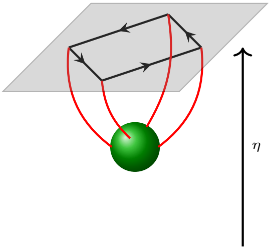
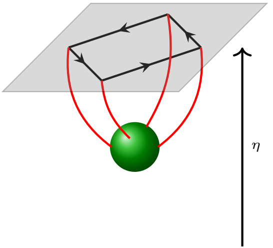
Observables & Fundamental Principles



$$\langle \Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)\rangle \;=\; \int \mathcal{D}\Phi\,\mathfrak{P}[\Phi]\Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)$$

Observables & Fundamental Principles



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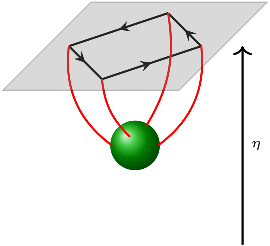
Probability distribution

Observables & Fundamental Principles

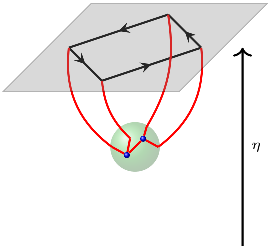


$$\langle \Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)\rangle \;=\; \frac{\int \mathcal{D}\Phi\,\Psi^\dagger[\Phi]\Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)\Psi[\Phi]}{\int \mathcal{D}\Phi\,|\Psi[\Phi]|^2}$$

$$\Psi[\Phi] := \langle\Phi|\hat{\mathcal{T}}\exp\left\{-i\int_{-\infty}^{0} d\eta\, H(\eta)\right\}|0\rangle$$

Wavefunction of the universe
(transition amplitude from $|0\rangle$ to $\langle\Phi|$)

Observables & Fundamental Principles



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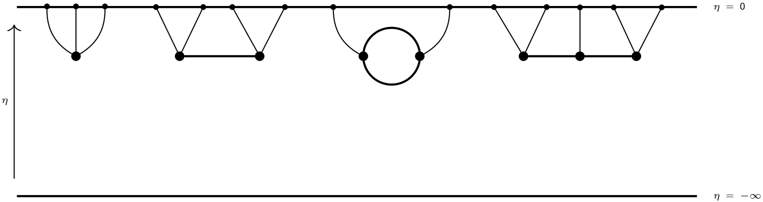
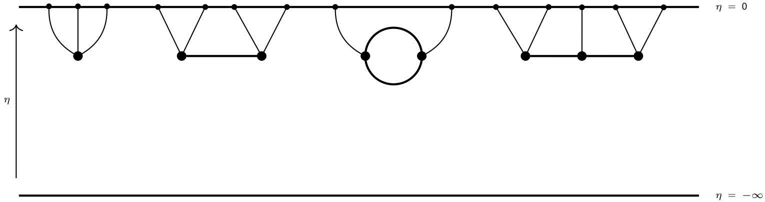
Perturbation theory



Cosmological Integrals



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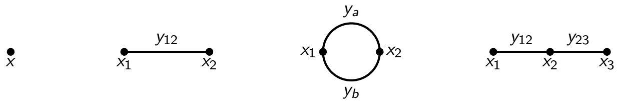

[N. Arkani-Hamed, P.B., A.Postnikov; '17]
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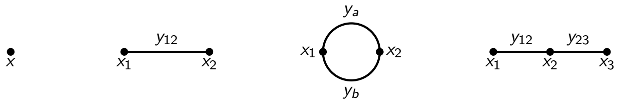
$$\mathcal{I}_{\mathcal{G}} = \prod_{s\in\mathcal{V}} \left[\int_{X_s}^{+\infty} dx_s\, \tilde{\lambda}(x_s - X_s)\right] \int_{\Gamma} \prod_{e\in\mathcal{E}^{(L)}} \left[\frac{dy_e}{y_e}\, y_e^{\beta_e}\right] \mu_d(y)\, \frac{\mathfrak{n}_{\delta}(x,y)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} q_{\mathfrak{g}}(x,y)}$$

Cosmology Loop integration Universal integrand

Cosmological Integrals



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Cosmology → Power-law FRW

$\tilde{\lambda}(x_s - X_s) \sim (x_s - X_s)^{\alpha-1}$

Loop integration

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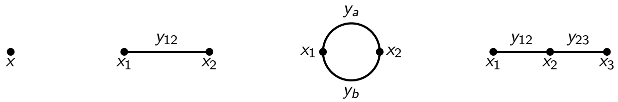

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External kinematics: $X_s := \sum_{j\in s} |\vec{p}^{(j)}|,\; y_e := \left|\sum_{j\in s_e} \vec{p}^{(j)}\right| \; (e\in\mathcal{E}\setminus\{\mathcal{E}^{(L)}\})$

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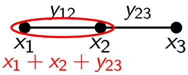


In this talk

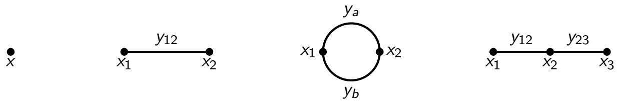
1. Cosmological integrands: Singularities, combinatorics & computation

2. The IR/UV structure of cosmological integrals

3. One loop corrections without integration

Cosmological integrands: Singularities, combinatorics & computation

Cosmological Integrals



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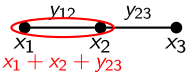

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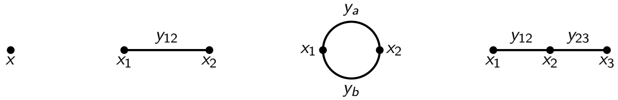

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Cosmology

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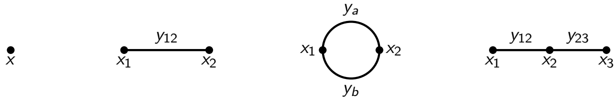
Universal integrand

↑

(weighted) cosmological polytope

[N. Arkani-Hamed, P.B., A. Postnikov; '17]
[P.B.; '19]
[P.B., G. Dian; '24]

Cosmological Integrals



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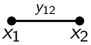
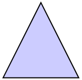
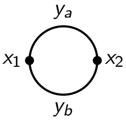


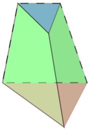
$$\omega\left(\mathcal{Y},\mathcal{P}_{\mathcal{G}}\right) = \frac{\mathfrak{n}_{\delta}(x,y)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} q_{\mathfrak{g}}(x,y)}\;\frac{\prod_{s\in\mathcal{V}} dx_s \prod_{e\in\mathcal{E}} dy_e}{\text{Vol}\{GL(1)\}}$$



(weighted) cosmological polytope



(Weighted) cosmological polytopes capture the singularity structure of $\mathcal{I}_{\mathcal{G}}$



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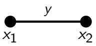
From Graphs to Polytopes

A flavour of cosmological polytopes



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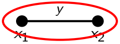

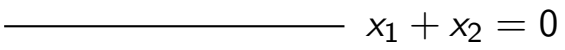

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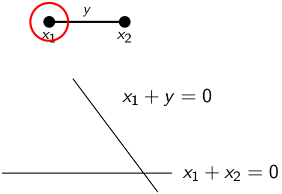

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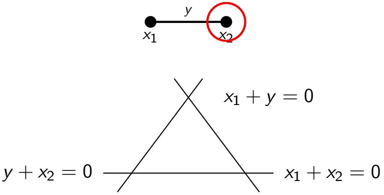

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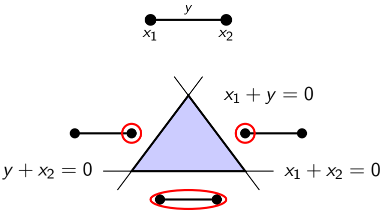
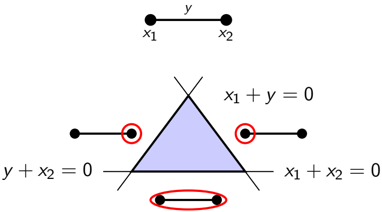


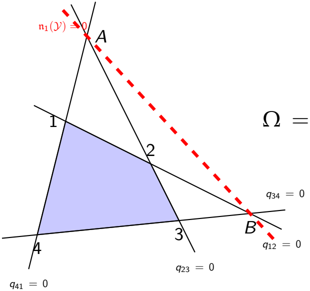
The singularities form a bounded region to which a function Ω is naturally associated

$$\Omega = \frac{1}{(x_1 + x_2)(x_1 + y)(y + x_2)} \equiv \frac{\mathfrak{n}_\delta}{q_{\mathcal{G}}\, q_{\mathfrak{g}_1}\, q_{\mathfrak{g}_2}}$$

From Graphs to Polytopes

A flavour of cosmological polytopes





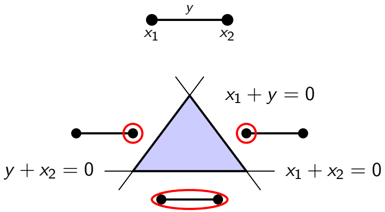
$$\Omega = \frac{\mathfrak{n}_1}{q_{12}q_{23}q_{34}q_{41}} = \frac{1}{q_{12}q_{34}}\left[\frac{1}{q_{23}} + \frac{1}{q_{41}}\right]$$

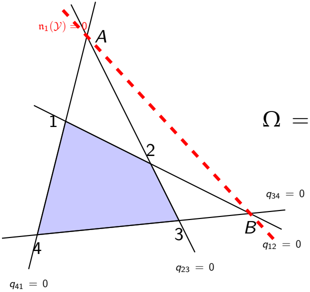
Linear relation
$q_{12} + q_{34} = q_{23} + q_{41}$

Triangulation of the polytope
≡
Representation for the integrand

From Graphs to Polytopes

A flavour of cosmological polytopes





Point B: $\begin{cases} q_{12} = 0 \\ q_{34} = 0 \end{cases}$

$$\Omega = \frac{\mathfrak{n}_1}{q_{12}q_{23}q_{34}q_{41}} = \frac{1}{q_{12}q_{34}}\left[\frac{1}{q_{23}} + \frac{1}{q_{41}}\right]$$

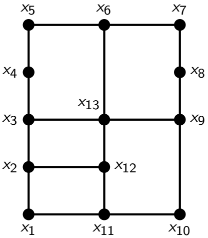
Linear relation
$q_{12} + q_{34} = q_{23} + q_{41}$

Compatibility conditions:

$$\text{Res}_{q_{12}=0}\text{Res}_{q_{34}=0}\Omega = 0 = \text{Res}_{q_{23}=0}\text{Res}_{q_{41}=0}\Omega$$

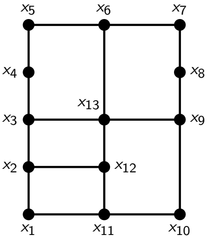
From Graphs to Polytopes

[P.B., W. Torres Bobadilla; '21]



From Graphs to Polytopes

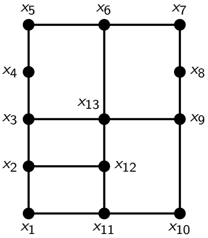
[P.B., W. Torres Bobadilla; '21]



$$\Omega = \prod_{\mathfrak{g}\in\mathfrak{G}_\circ} \frac{1}{q_{\mathfrak{g}}(x,y)} \sum_{\{\mathfrak{G}_c\}} \prod_{\mathfrak{g}'\in\mathfrak{G}_c} \frac{1}{q_{\mathfrak{g}'}(x,y)}$$

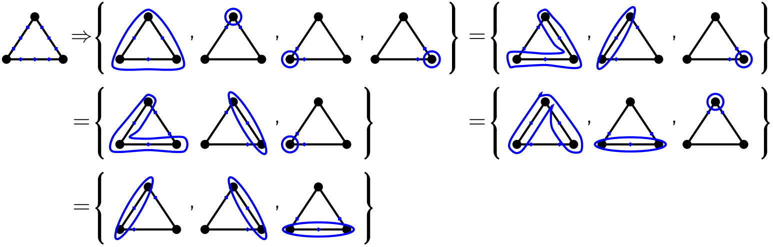
From Graphs to Polytopes

[P.B., W. Torres Bobadilla; '21]



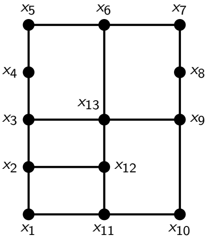
$$\Omega = \prod_{\mathfrak{g}\in\mathfrak{G}_\circ} \frac{1}{q_{\mathfrak{g}}(x,y)} \sum_{\{\mathfrak{G}_c\}} \prod_{\mathfrak{g}'\in\mathfrak{G}_c} \frac{1}{q_{\mathfrak{g}'}(x,y)}$$

Example: $\{\mathfrak{G}_\circ\}$ for a triangle graph



From Graphs to Polytopes

[P.B., W. Torres Bobadilla; '21]



$$\Omega = \prod_{\mathfrak{g}\in\mathfrak{G}_\circ} \frac{1}{q_{\mathfrak{g}}(x,y)} \sum_{\{\mathfrak{G}_c\}} \prod_{\mathfrak{g}'\in\mathfrak{G}_c} \frac{1}{q_{\mathfrak{g}'}(x,y)}$$

Compatibility conditions allow to:

1. write all the possible representations without spurious singularities
2. make manifest the symmetries that maps a simplex into another one
3. improve analytical/numerical efficiency of the integration.

(Weighted) Cosmological Polytopes & $\mathcal{I}_{\mathcal{G}}$: A dictionary

Cosmological Polytope $\mathcal{P}_{\mathcal{G}}$	Cosmological Integral $\mathcal{I}_{\mathcal{G}}$
Canonical form ω	Integrand of $\mathcal{I}_{\mathcal{G}}$
Triangulations	Representations for the intgrand
Boundaries (Faces)	Residues of the integrands
Canonical form preserving transformations	Symmetries of the integrand
Paths along contiguous vertices	Symbols for $\mathcal{I}_{\mathcal{G}}$

(Weighted) Cosmological Polytopes & $\mathcal{I}_{\mathcal{G}}$: A dictionary

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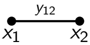
Paths along contiguous vertices	Symbols for $\mathcal{I}_{\mathcal{G}}$

Trascendental function	$f_k = \int_a^b d\log R_1 \circ \ldots \circ d\log R_k$	Iterated integral
Symbols	$\mathcal{S}(f_k) := R_1 \otimes \ldots \otimes R_k$	

Towards a combinatorial RG:
The IR/UV structure of cosmological integrals

Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

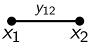
[P.B., F. Vazão; 24]



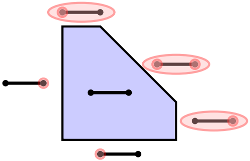
$$= \int_{\mathbb{R}_+} \prod_{j=1}^{2} \left[\frac{dx_j}{x_j} \, x_j^{\alpha} \right] \frac{1}{(x_1 + x_2 + \mathcal{X}_{\mathcal{G}})(x_1 + \mathcal{X}_{\mathfrak{g}_1})(x_2 + \mathcal{X}_{2_2})}$$

Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

[𝐏.𝐁., F. Vazão; 24]



$$= \int_{\mathbb{R}_+} \prod_{j=1}^{2} \left[\frac{dx_j}{x_j} \, x_j^{\alpha} \right] \frac{1}{(x_1^{1}x_2^{0} + x_1^{0}x_2^{1} + \mathcal{X}_{\mathcal{G}} x_1^{0}x_2^{0})(x_1^{1}x_2^{0} + \mathcal{X}_{\mathfrak{g}_1} x_1^{0}x_2^{0})(x_1^{0}x_2^{1} + \mathcal{X}_{\mathfrak{g}_2} x_1^{0}x_2^{0})}$$

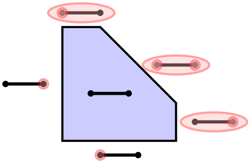


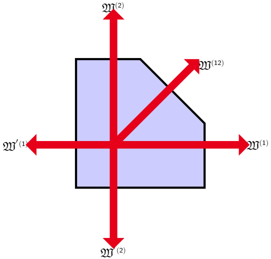
The integral converges for values of α that identifies points inside the Newton polytope

Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

[P.B., F. Vazão; 24]

$$\overset{y_{12}}{\underset{x_1 \qquad x_2}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet}} = \int_{\mathbb{R}_+} \prod_{j=1}^{2} \left[\frac{dx_j}{x_j}\, x_j^{\alpha} \right] \frac{1}{(x_1^1 x_2^0 + x_1^0 x_2^1 + \mathcal{X}_{\mathcal{G}} x_1^0 x_2^0)(x_1^1 x_2^0 + \mathcal{X}_{\mathfrak{g}_1} x_1^0 x_2^0)(x_1^0 x_2^1 + \mathcal{X}_{\mathfrak{g}_2} x_1^0 x_2^0)}$$





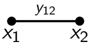
The integral converges for values of α that identifies points inside the Newton polytope

$$\mathfrak{W}^{(12)} = \begin{pmatrix} 2\alpha - 3 \\ 1 \\ 1 \end{pmatrix}, \quad \mathfrak{W}^{(1)} = \begin{pmatrix} \alpha - 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathfrak{W}^{(2)} = \begin{pmatrix} \alpha - 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathfrak{W}'^{(1)} = \begin{pmatrix} -\alpha \\ -1 \\ 0 \end{pmatrix}, \quad \mathfrak{W}'^{(2)} = \begin{pmatrix} -\alpha \\ 0 \\ -1 \end{pmatrix},$$

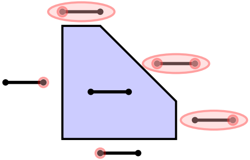
The integral diverges in the direction $\mathfrak{e}$ if the related λ is ≥ 0

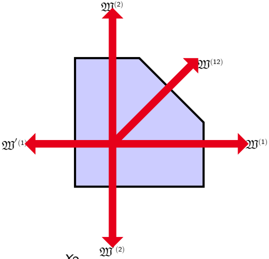
Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_\mathcal{G}$

[P.B., F. Vazão; 24]



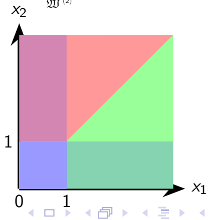
$$= \int_{\mathbb{R}_+} \prod_{j=1}^{2} \left[\frac{dx_j}{x_j}\, x_j^{\alpha} \right] \frac{1}{(x_1^1 x_2^0 + x_1^0 x_2^1 + \mathcal{X}_{\mathcal{G}} x_1^0 x_2^0)(x_1^1 x_2^0 + \mathcal{X}_{\mathfrak{g}_1} x_1^0 x_2^0)(x_1^0 x_2^1 + \mathcal{X}_{\mathfrak{g}_2} x_1^0 x_2^0)}$$





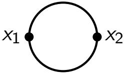
E.g.: if $\lambda^{(12)} \longrightarrow 0$: sector decomposition

$$\mathcal{I}^{\mathrm{div}}_{\Delta_{j,12}} = \int_0^1 \frac{d\zeta_j}{\zeta_j} \frac{(\zeta_j)^{-\lambda^{(j)}}}{1+\zeta_j} \times \int_0^1 \frac{d\zeta_{12}}{\zeta_{12}} \, (\zeta_{12})^{-\lambda^{(12)}}$$



Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

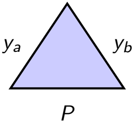
[P.B., F. Vazão; 24]



$$= \int_{\mathbb{R}_+} \prod_{j=1}^{2} \left[\frac{dx_j}{x_j} \, x_j^{\alpha} \right] \int_{\Gamma} \prod_{e \in \mathcal{E}} \left[\frac{dy_e}{y_e} \, y_e^{\beta} \right] \mu(y) \times$$

$$\times \frac{2(x_1 + x_2 + y_a + y_b + \mathcal{X}_{\mathcal{G}})}{(x_1 + x_2 + \mathcal{X}_{\mathcal{G}})(x_1 + x_2 + y_a + \mathcal{X}_{\mathcal{G}})(x_1 + x_2 + y_b + \mathcal{X}_{\mathcal{G}})(x_1 + y_a + y_b + \mathcal{X}_1)(x_2 + y_a + y_b + \mathcal{X}_2)}$$

1. $\mu(y) \sim \left[\frac{\text{Vol}^2 \Sigma_2(y_e^2, P^2)}{\text{Vol}^2 \Sigma_1(P^2)} \right]^{\frac{d-3}{2}}$



2. $\Gamma \implies$ Volume of the triangle, and all its side, are positive

$$(y_a + y_b + P)(y_a + y_b - P)(y_a - y_b + P)(-y_a + y_b + P) \geq 0,$$

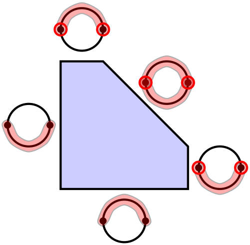
$$y_a \geq 0, \quad y_b \geq 0, \quad P \geq 0$$

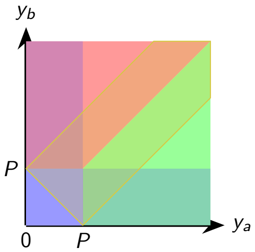
Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

[P.B., F. Vazão; 24]

$$x_1 \bigcirc x_2 = \int_{\mathbb{R}_+} \prod_{j=1}^{2} \left[\frac{dx_j}{x_j} \, x_j^{\alpha} \right] \int_{\Gamma} \prod_{e \in \mathcal{E}} \left[\frac{dy_e}{y_e} \, y_e^{\beta} \right] \mu(y) \times$$

$$\times \frac{2(x_1 + x_2 + y_a + y_b + \mathcal{X}_{\mathcal{G}})}{(x_1 + x_2 + \mathcal{X}_{\mathcal{G}})(x_1 + x_2 + y_a + \mathcal{X}_{\mathcal{G}})(x_1 + x_2 + y_b + \mathcal{X}_{\mathcal{G}})(x_1 + y_a + y_b + \mathcal{X}_1)(x_2 + y_a + y_b + \mathcal{X}_2)}$$





Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

[P.B., F. Vazão; 24]

$$\mathcal{I}_{\mathcal{G}} = \int_0^{+\infty} \prod_{s\in\mathcal{V}} \left[\frac{dx_s}{x_s}\, x_s^{\alpha_s}\right] \int_{\Gamma} \prod_{e\in\mathcal{E}^{(L)}} \left[\frac{dy_e}{y_e}\, y_e^{\beta_e}\right] \mu_d(y)\, \frac{\mathfrak{n}_{\delta}(x+X,y)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} \left[q_{\mathfrak{g}}(x+X,y)\right]^{\tau_{\mathfrak{g}}}}$$

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The asymptotic structure of $\mathcal{I}_{\mathcal{G}}$ is captured by:

1. a nestohedron, which is determined by the underlying cosmological polytope $\mathcal{P}_{\mathcal{G}}$, and whose facets are fixed via subgraphs

$$\mathfrak{W}^{\left(j_1\ldots j_{n_s^{(\mathfrak{g})}+n_e^{(\mathfrak{g})}}\right)} = \begin{pmatrix} \lambda^{\left(j_1\ldots j_{n_s^{(\mathfrak{g})}+n_e^{(\mathfrak{g})}}\right)} \\ \mathfrak{e}_{\left(j_1\ldots j_{n_s^{(\mathfrak{g})}+n_e^{(\mathfrak{g})}}\right)} \end{pmatrix}, \qquad \lambda^{\left(j_1\ldots j_{n_s^{(\mathfrak{g})}+n_e^{(\mathfrak{g})}}\right)} = \sum_{s\in\mathcal{V}_{\mathfrak{g}}} \alpha_s + \sum_{e\in\mathcal{E}^{(L)}} \beta_e - \sum_{\mathfrak{g}'\in(\text{tubings})} \tau_{\mathfrak{g}'}$$

The integral diverges in the direction $\mathfrak{e}$ if the related λ is ≥ 0

2. the contour of the loop integration Γ, which selects the divergent directions among the $\mathfrak{W}$'s of the nestohedron

Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

[P.B., F. Vazão; 24]

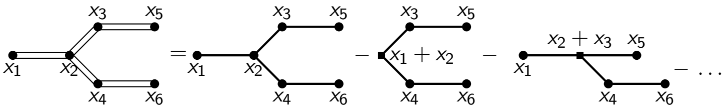
$$\mathcal{I}_{\mathcal{G}} = \int_0^{+\infty} \prod_{s\in\mathcal{V}} \left[\frac{dx_s}{x_s}\, x_s^{\alpha_s}\right] \int_{\Gamma} \prod_{e\in\mathcal{E}^{(L)}} \left[\frac{dy_e}{y_e}\, y_e^{\beta_e}\right] \mu_d(y)\, \frac{\mathfrak{n}_{\delta}(x+X, y)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} \left[q_{\mathfrak{g}}(x+X, y)\right]^{\tau_{\mathfrak{g}}}}$$

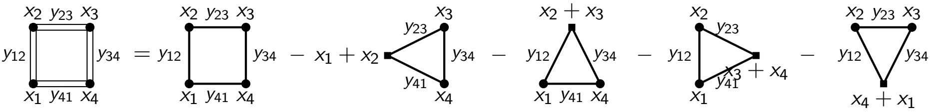
This combinatorial picture allows to:

1. straightforwardly determine both the directions along which $\mathcal{I}_{\mathcal{G}}$ can diverge and their degree of divergence;
2. straightforwardly compute leading and subleading divergences (both in the IR and in the UV) via sector decomposition;
3. the leading divergence in the IR are associated to the restriction of the underlying cosmological polytope onto special hyperplanes;
4. write a systematic substraction that produces IR-finte quantities.

Towards IR-finte computables

[P.B., F. Vazão; 24]

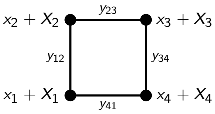




■ indicates that two sites are collapsed into one, and their site-weight integration measure is shifted

Towards IR-finte computables

[**P.B.**, Vazão, '24]



Towards IR-finte computables

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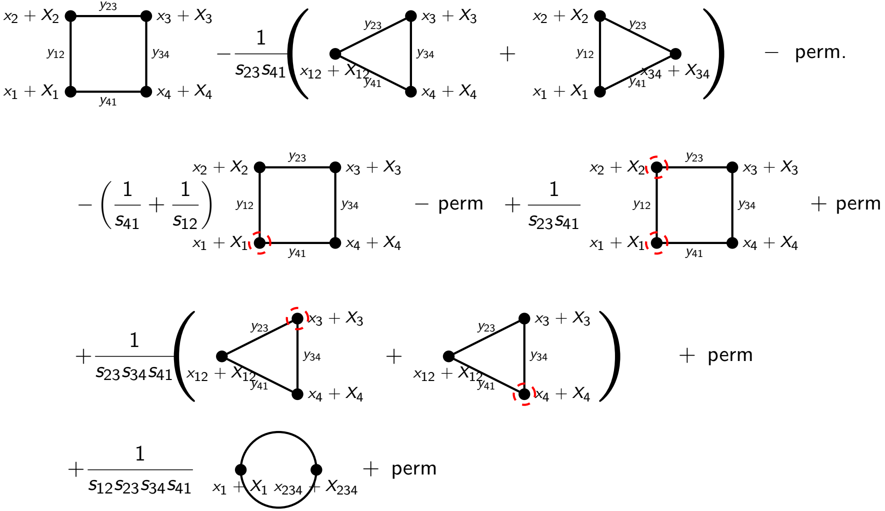

Towards IR-finte computables

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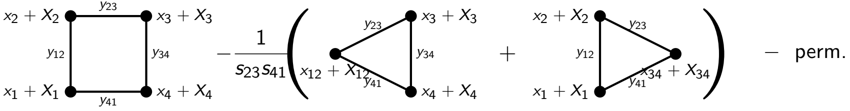
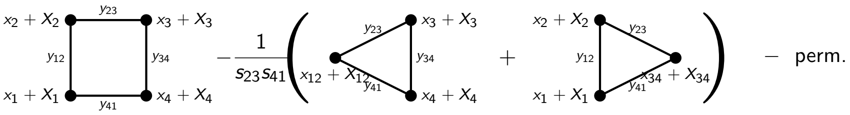


Flat-space limit: This is the known result in scattering amplitudes which can be obtained from a IR-finite observable: the Wilson loop with a Lagrangian insertion.

Towards IR-finte computables

[**P.B.**, Vazão, '24]



Flat-space limit: This is the known result in scattering amplitudes which can be obtained from a IR-finite observable: the Wilson loop with a Lagrangian insertion.

- Our systematic procedure automatically returns an IR finite flat-space limit despite the cosmological box integral might not a IR-divergent loop integral.

One loop corrections without integration

Understanding the space of functions

[P.B., G. Brunello, M. K. Mandal, P. Mastrolia, F. Vazão; 24]

$$\mathcal{I}_{\mathcal{G}} = \int_0^{+\infty} \prod_{s \in \mathcal{V}} \left[\frac{dx_s}{x_s} x_s^{\alpha_s} \right] \int_\Gamma \prod_{e \in \mathcal{E}^{(L)}} \left[\frac{dy_e}{y_e} y_e^{\beta_e} \right] \mu_d(y) \frac{\mathfrak{n}_\delta(x+X, y)}{\prod_{\mathfrak{g} \subseteq \mathcal{G}} \left[q_{\mathfrak{g}}(x+X, y) \right]^{\tau_{\mathfrak{g}}}}$$

Understanding the space of functions

[P.B., G. Brunello, M. K. Mandal, P. Mastrolia, F. Vazão; 24]

$$\mathcal{I}_{\mathcal{G}} = \int_0^{+\infty} \prod_{s\in\mathcal{V}} \left[\frac{dx_s}{x_s} x_s^{\alpha_s}\right] \int_\Gamma \prod_{e\in\mathcal{E}^{(L)}} \left[\frac{dy_e}{y_e} y_e^{\beta_e}\right] \mu_d(y) \frac{\mathfrak{n}_\delta(x+X,y)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} [q_{\mathfrak{g}}(x+X,y)]^{\tau_{\mathfrak{g}}}}$$

can be expressed in terms of

(twisted period integrals)

$$\mathcal{I}^{(j)}_{\{\tau_{\mathfrak{g}}\}} := \int_\Gamma \mu_d\, \varphi \quad , \qquad \varphi := \frac{\prod_{e\in\mathcal{E}^{(L)}} dy_e}{\prod_{\mathfrak{g}\in\mathfrak{G}^{(j)}\cup\{e\}} [q_{\mathfrak{g}}(y)]^{\tau_{\mathfrak{g}}}},$$

Each of these integrals can be expressed as a *finite* linear combination of master integrals

$$\mathcal{I}^{(j)}_{\{\tau_{\mathfrak{g}}\}} := \sum_{j=1}^{\nu} c_j \mathcal{J}_j, \qquad d\mathcal{J} = d\mathbb{A}\,\mathcal{J}$$

Canonical form: $d\mathcal{J} = \varepsilon d\mathbb{A}\,\mathcal{J} \Rightarrow \mathcal{J} = \mathbb{P}\mathrm{exp}\left\{\varepsilon \int_\Gamma d\mathbb{A}\right\} \mathcal{J}_\circ$

Understanding the space of functions

[P.B., G. Brunello, M. K. Mandal, P. Mastrolia, F. Vazão; 24]

1. The system of differential equations has a block-triangular form, and for each block can be rewritten in terms of a higher order differential equation for a single $\mathcal{J}_j$;

Understanding the space of functions

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2. The factorisation property of such higher order operator determines the type; of solutions

Understanding the space of functions

[P.B., G. Brunello, M. K. Mandal, P. Mastrolia, F. Vazão; 24]

1. The system of differential equations has a block-triangular form, and for each block can be rewritten in terms of a higher order differential equation for a single $\mathcal{J}_j$;
2. The factorisation property of such higher order operator determines the type; of solutions
3. Expressing the integrand in terms of triangulations of the cosmological polytope or its restrictions, allows to simplify the problem;

Understanding the space of functions

[P.B., G. Brunello, M. K. Mandal, P. Mastrolia, F. Vazão; 24]

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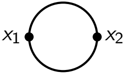

Loop integration $\quad \log,\ \mathrm{Li}_2 \quad \Longrightarrow \quad$ Site-weight integration $\quad {}_2F_1,\ {}_3F_2 \quad$ General power-law FRW cosmologies

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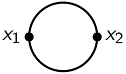

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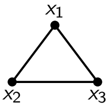

Loop integration — Polylogs, Elliptics

Conclusion

First clues on constraints on cosmological processes:
perturbative unitarity, flat-space limit,
factorisations, higher-codimensions singularities

General framework to have a direct formulation
with IR safe observables

We scratched the surface of the one-loop structure:
first glimpses of its analytic structure and its space of functions.