IMPERIAL

# On the IR divergences on de Sitter space


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Base on work with A Achucarro, A Davis, S Melville, G Palma and D-G Wang

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Plus work in progress with G Kaplanek and T Colas

# Outline

- Introduction
- IR divergences
- Stochastic Inflation
- Wavefunction approach to stochastic inflation

# IR divergences on de Sitter

# Introduction

- Let's consider a massless scalar field $\phi$. The two point function is given by

$$\langle \phi_k^2 \rangle' = \frac{H^2}{2k^3}(1 + k^2/(a_0 H_0)^2) \to \frac{H^2}{2k^3}$$

- Two point function becomes constant on super horizon scales

# Introduction

- Let's consider a massless scalar field $\phi$. The two point function is given by

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- Let's consider interactions. This are computed using the In-In formalism

$$\langle \phi(\mathbf{k}_1, t_0) \ldots \phi(\mathbf{k}_n, t_0) \rangle =$$

$$\langle T \left( e^{+i \int_{-\infty(1+i\epsilon)}^{t_0} dt H_{\rm int}(t)} \right) \phi(\mathbf{k}_1, t_0) \ldots \phi(\mathbf{k}_n, t_0) T \left( e^{-i \int_{-\infty(1+i\epsilon)}^{t_0} dt H_{\rm int}(t)} \right) \rangle$$

# Introduction

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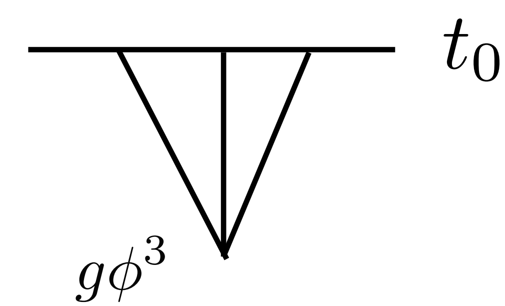
$$\langle \phi_k^2 \rangle' = \frac{H^2}{2k^3}(1 + k^2/(a_0 H_0)^2) \to \frac{H^2}{2k^3}$$

- Let's include a cubic interaction $V = \frac{g}{3!}\phi^3$




$$\langle \phi^3(t_0) \rangle = -i \int_{-\infty}^{t_0} dt \langle [\phi^3(t_0), \mathcal{H}_{\mathrm{int}(t)}] \rangle$$

- At late times this leads to

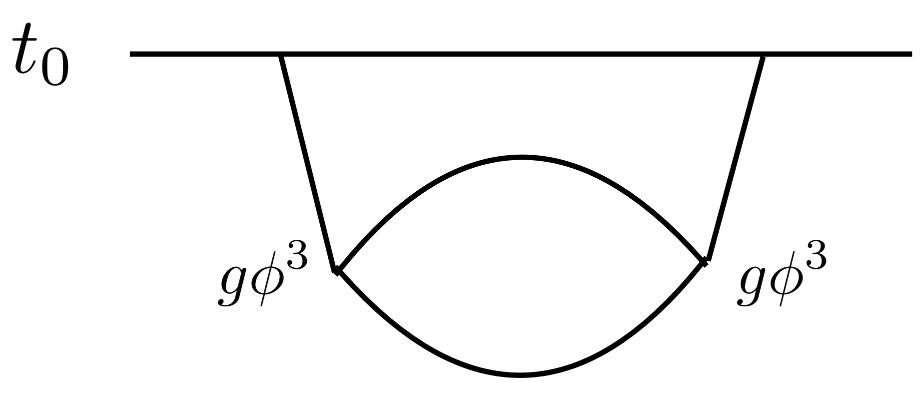
$$\langle \phi_{k_1}\phi_{k_2}\phi_{k_3} \rangle' \sim \frac{H^3}{12}\frac{(k_1^3 + k_2^3 + k_3^3)}{k_1^3 k_2^3 k_3^3}(-2/3 + \gamma_E + \log(k_1 + k_2 + k_3)/(a_0 H_0))$$

# Introduction

- Interaction does not decay after horizon exit

$$\langle \phi_{k_1}(t_0)\phi_{k_2}(t_0)\phi_{k_3}(t_0)\rangle' \\ = i\phi_{k_1}(t_0)\phi_{k_2}(t_0)\phi_{k_3}(t_0)\int dt a(t)^3 \phi^*_{k_1}(t)\phi^*_{k_2}(t)\phi^*_{k_3}(t) + \mathrm{cc}$$

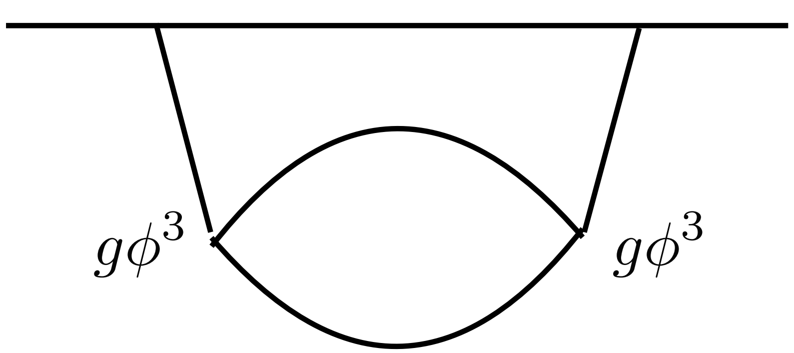
- This problem implies that now loops get a time dependence




$$\langle \phi_k^2 \rangle \sim \frac{H^2}{2k^3}\frac{g^2}{72\pi^2 H^2}\log(kL)\log(2k/(a_0H_0))^2$$

# IR divergences

- This arise due to interactions not decaying after horizon exit.
- For example, interactions with two derivatives are IR safe
- Different types of IR divergences
  - Time integrals
  - Momentum integrals




$$\sim \frac{H^2}{2k^3}\frac{g^2}{72\pi^2 H^2}\log(k/a_0H_0)^2\log(kL)$$

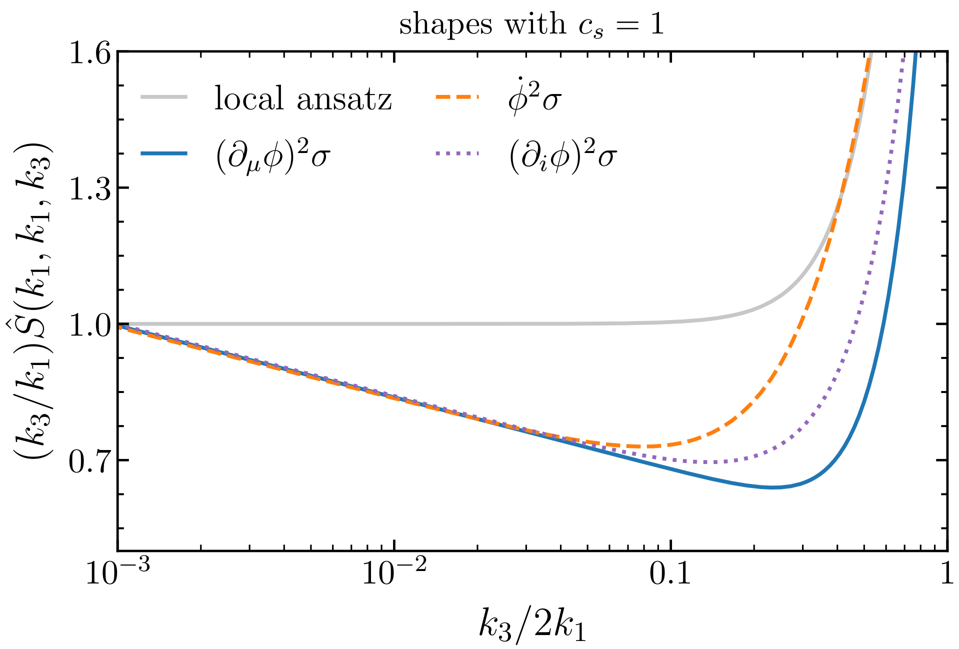
# IR divergences and inflation

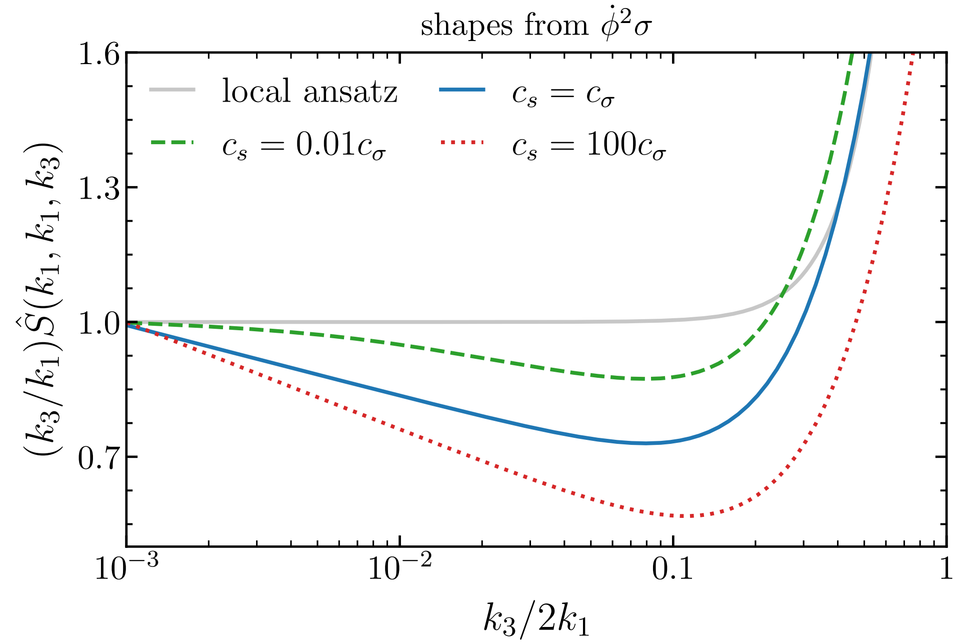
- The existence of IR divergences imply that perturbation theory breaks down at late times.
- IR divergences do not appear in single field inflation due to the non-linearly realised shift symmetry.
- Curvature field is constant on superhorizon scales
- Eg. In $\dot{\phi}^3$ $\langle \phi_{k_1} \phi_{k_2} \phi_{k_3} \rangle' = i \phi_{k_1}(t_0) \phi_{k_2}(t_0) \phi_{k_3}(t_0) \int dt a(t)^3 \dot{\phi}^*_{k_1}(t) \dot{\phi}^*_{k_2}(t) \dot{\phi}^*_{k_3}(t) + \mathrm{cc}$

$$\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \rangle \sim \frac{H^3}{k_1 k_3 k_3 k_T^3} \mathrm{Im}(2i + 2(k_T)/(a_0 H_0) - i(k_T)^2/(a_0 H_0)^2)$$

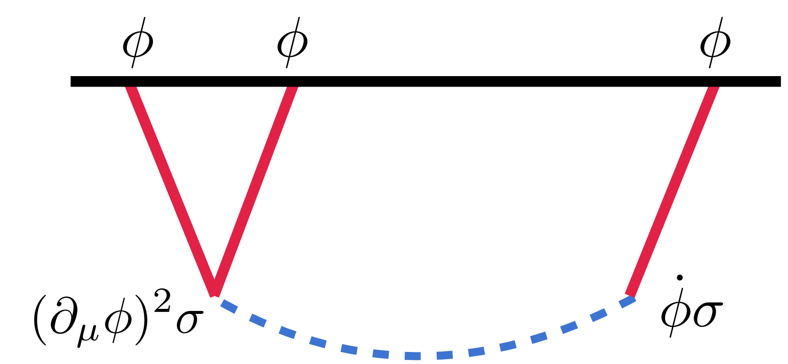
*Maldacena 2003*
*Urakawa Tanaka 2009,2013*
*Senatore Zaldarriaga 2012*
*Assassi, Baumann Green 2015*

# Light spectator fields




- IR divergences can be important for phenomenology
- Superhorizon growth of all correlation functions
- If the field has a mass the superhorizon growth will eventually stop




$$S(k_1,k_2,k_3) \propto \frac{1}{k_1^3k_2^3k_3^3}\Big[(\gamma_E - 3 - \log(-k_t\eta_0))(k_1^3+k_2^3+k_3^3) + k_te_2 - 4e_3 + (k_2^3+k_3^3)\log(-2k_1\eta_0) + (k_1^3+k_3^3)\log(-2k_2\eta_0) + (k_1^3+k_2^3)\log(-2k_3\eta_0)\Big]$$

*Chen and Wang '09*
*Arkani-Hamed and Maldacena '15*
*Wang, Pimentel and Achucarro '22*

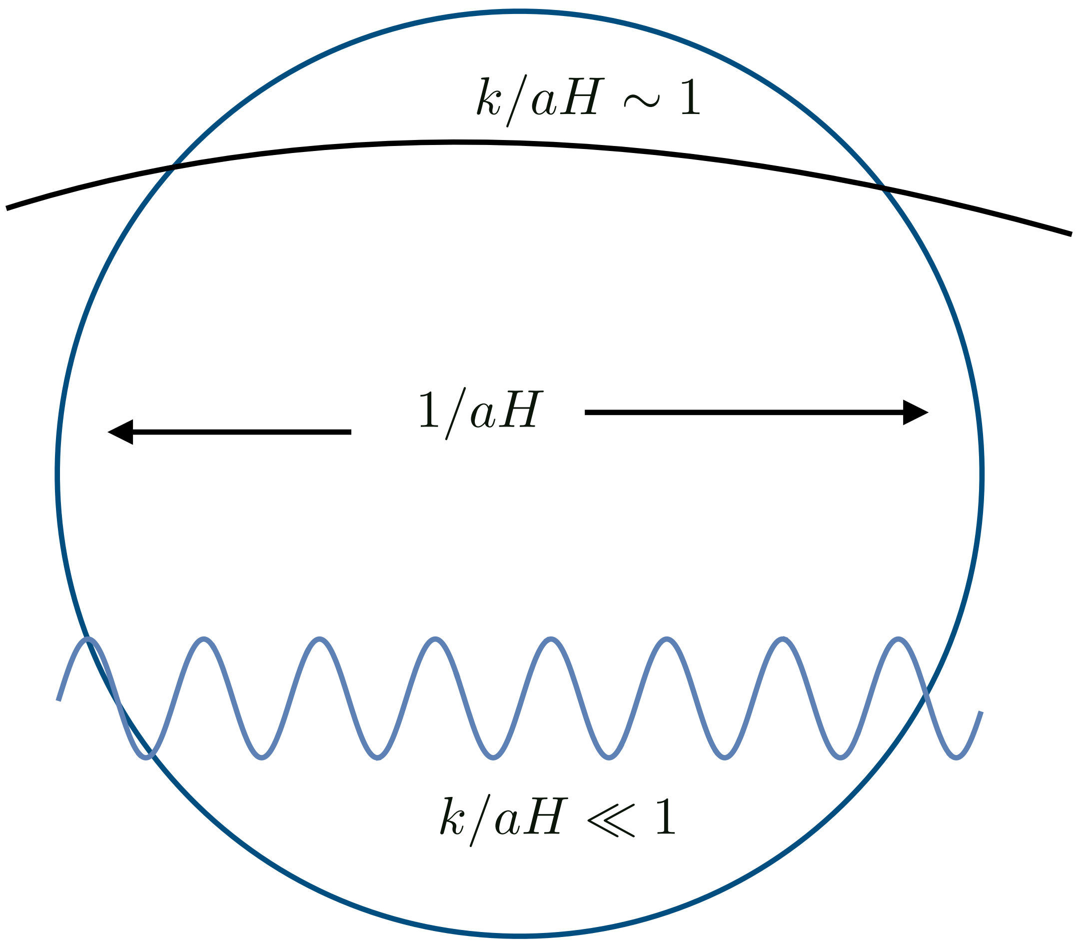
# IR divergences

- IR Divergences are ubiquitous in dS spaces, but do not appear on single-field inflation.
- They appear also with massive fields (Eg. Three point function with conformally coupled fields)
- The history is different for other fields coupled to inflation
- Two different origins. At tree-level only 'time' IR secular divergences $\log(-k\eta_0)$
- It is possible to regularise them using boundary counterterms (very similar to renormalised perturbation theory in AdS/CFT)

*SC, Davis and Melville 2020*

# Stochastic inflation

# Fluctuations during inflation

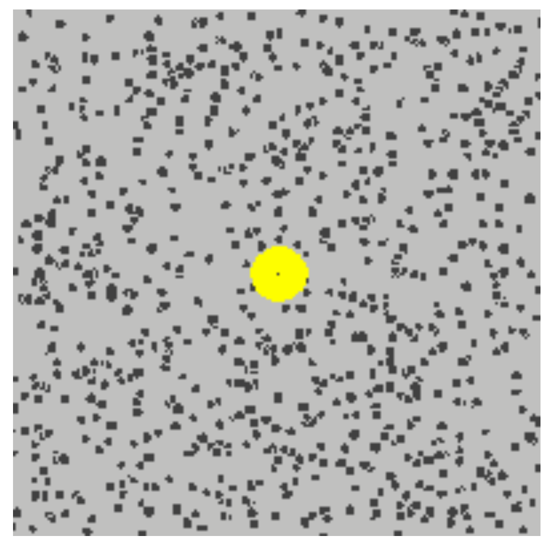



- Fluctuations grow until they become super horizon
- Once there they freeze
- New perturbations keep leaving the horizon

Superhorizons perturbations can be thought of as classical perturbations being seeded by quantum short wavelength perturbations

# Stochastic inflation

- Dynamics of long wavelength perturbations as brownian motion



- Short modes act as random Gaussian noise encoding short distance effects.

# Fokker-Planck equation

- Langevin equation can be written as a Fokker-Planck equation

$$\frac{d}{dt}P(\phi,t) = \frac{1}{3H}\frac{\partial}{\partial\phi}(V'(\phi)P(\phi,t)) + \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\phi^2}P(\phi,t)$$

- Solving the associated Fokker-Planck equation we find that,

$$\lim_{t\to\infty} P(\phi) \sim \exp\left(-\frac{8\pi^2 V(\phi)}{3H^4}\right)$$

Eg $\quad V(\phi) = \frac{\lambda}{4}\phi^4$

$$P(\phi) \sim \exp\left(-\frac{\pi^2}{6H^4}\lambda\phi^4\right)$$

- Probability for the field depends on the classical potential.

- Highly non linear and goes beyond perturbation theory

# Loops from FP equation

- We can solve the Fokker-Planck equation pertubatively $\langle \phi^n \rangle = \int d\phi \phi^n P(\phi)$

$$\langle \phi^2 \rangle = \frac{H^2}{4\pi^2} \log a - \frac{\lambda H^2}{144\pi^4} (\log a)^3 + \frac{\lambda^2 H^2}{2880\pi^6} (\log a)^5 + \mathcal{O}(\lambda^3 (\log a)^7$$

- First term is the variance

$$\Lambda(t) = \epsilon a(t) H$$

UV

$$\sigma^2 \equiv \langle \phi_l(\mathbf{x}=0)^2 \rangle = \int_{\mathbf{k}} \frac{H^2}{2k^3} = \frac{H^2}{4\pi^2} \log(\Lambda L_{\rm IR}) \ .$$

IR

$$L_{\rm IR}^{-1} = a(t_i) H$$

*Starobinsky '83,*
*Starobinsky and Yokoyama '92*

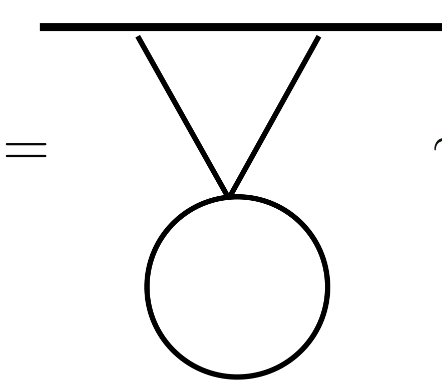
# Loops from FP equation

- We can solve the Fokker-Planck equation pertubatively

$$\langle \phi^2 \rangle = \frac{H^2}{4\pi^2} \log a - \frac{\lambda H^2}{144\pi^4} (\log a)^3 + \frac{\lambda^2 H^2}{2880\pi^6} (\log a)^5 + \mathcal{O}(\lambda^3 (\log a)^7$$

- Second correspond to loop correction to the two point function

$$\langle \phi^2 \rangle_{1-\mathrm{loop}} =$$



$$\sim \frac{\lambda}{4\pi^2} \frac{H^2}{k^3} \log(k/aH) \log\left( \frac{k}{aH} (aHL)^2 \right)$$

- Time dependence also comes from the loop IR divergence

*Tsamis and Woodard ‘05*
*Tokuda and Tanaka ’19*
*Baumgart and Sundrum ‘19*

# Perturbation theory vs stochastic inflation

- Perturbation theory contains IR divergences associated to the fact that perturbations keep entering the horizon

- Perturbation theory breaks down for $\lambda(\log a)^2 \gg 1$

- The common lore is that stochastic inflation sums all loops rendering finite results

- At late times there is an equilibrium solution

$$V(\phi) = \frac{\lambda}{4}\phi^4 \qquad \langle \phi^2 \rangle \sim \frac{H^2}{\lambda^{1/2}}$$

- Equilibrium solution is de Sitter invariant

*Starobinsky and Yokoyama 1994*
*Baumgart and Sundrum 2019*
*Gorbenko and Senatore 2019*
*Cohen and Green 2020*

- We would like to understand better how to go from perturbation theory to the stochastic theory

- This can be phrased as ''what does the stochastic theory computes"

- Might look like the semiclassical regime but we would like to understand this better

Fokker-Planck Equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \phi}\left(\frac{V_\phi}{3H}P\right) + \frac{H^3}{8\pi^2}\left(\frac{\partial^2 P}{\partial \phi^2}\right)$$

- stochastic effects
- non-perturbative
- equilibrium from re sum

Cosmological bootstrap

$$P = |\Psi|^2$$

$$\Psi[\phi] = \exp\left[\sum_{n=2}^{\infty}\frac{1}{n!}\int_{\mathbf{k}_1,\ldots,\mathbf{k}_n}\psi_n(\mathbf{k}_1,\ldots,\mathbf{k}_n)\;\phi_{\mathbf{k}_1}\cdots\phi_{\mathbf{k}_n}\right]$$

- related to correlators
- perturbative
- secular growth

*Burgess et al '09, '10, '14*
*Gorbenko and Senatore '19*
*SC, Davis and Wang '23*

# Wavefunction method

# Semiclassical approximation

- When dealing with a quantum system, when we expect the nature of the system to be well approximated by the classical trajectory

$$\Psi[\phi(\vec{x})] = \int_{\substack{\Phi(\eta_0)=\phi \\ \Phi(-\infty)=0}} \mathcal{D}\Phi \exp\left(-\frac{i}{\hbar}S[\Phi]\right) \simeq \exp\left(-\frac{i}{\hbar}S_0[\Phi_{\rm cl}]\right)$$

$$S[\Phi] = S_0[\Phi_{\rm cl}] + \hbar S_1[\Phi_{\rm cl}] + \dots$$

- The semiclassical action can be non linear and the approximation can go beyond perturbation theory

- Our goal: In which sense are is stochastic inflation related to the semiclassical approximation.

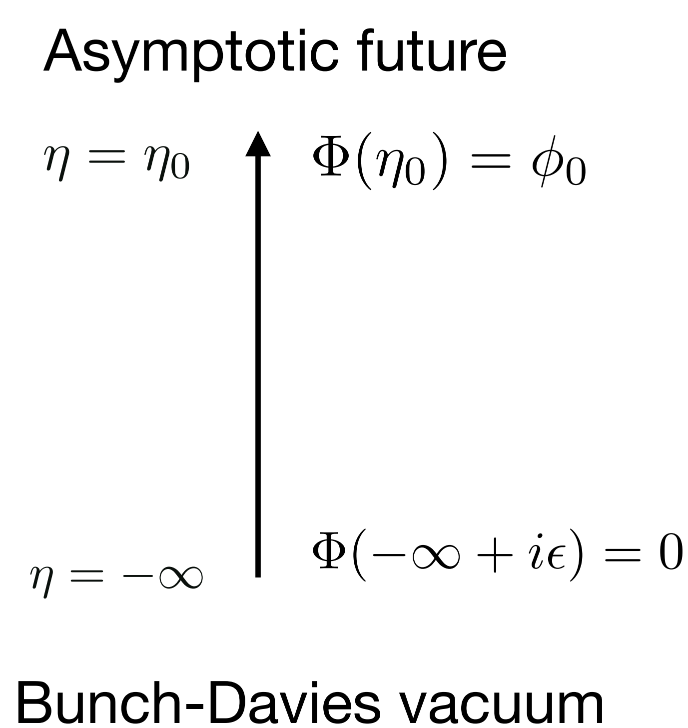
# Wavefunction method

Let’s consider scalar field on de Sitter evolving through the Schrödinger equation

$$i\hbar \frac{d}{dt}\Psi[\phi] = H\Psi[\phi]$$

$$\Psi[\phi(\vec{x})] = \int_{\substack{\Phi(\eta_0)=\phi \\ \Phi(-\infty)=0}} \mathcal{D}\Phi \exp\left(-\frac{i}{\hbar}S[\Phi]\right)$$

Boundary conditions




We expand the action $$S[\Phi] = S_0[\Phi_{\rm cl}] + \hbar S_1[\Phi_{\rm cl}] + \ldots$$

# Wavefunction method

Given a classical solution

$$\Psi[\phi(\vec{x})] = \int_{\substack{\Phi(\eta_0)=\phi \\ \Phi(-\infty)=0}} \mathcal{D}\Phi \exp\left(-\frac{i}{\hbar}S[\Phi]\right) \simeq \exp\left(-\frac{i}{\hbar}S[\Phi_{\rm cl}]\right)$$

bulk-to-boundary

$$(\Box - m^2)\Phi = -\frac{1}{\sqrt{-g}}\frac{\delta S_{\rm int}}{\delta \Phi} \rightarrow \Phi_{\rm cl}(\eta, \mathbf{k}) = \phi_{\mathbf{k}} K(k,\eta) + \frac{i}{\hbar}\int {\rm d}\eta' G(k;\eta,\eta') \frac{\delta S_{\rm int}}{\delta \Phi_{\mathbf{k}}(\eta')}\Bigg|_{\Phi=\Phi_{\rm cl}}$$

bulk-to-bulk

Path integral Boundary conditions

$$(\Box - m^2)K_\varphi(\mathbf{x},t) = 0, \quad \text{with} \lim_{t\to t_0} K(\mathbf{x},t) = 1, \quad \lim_{t\to-\infty(1+i\epsilon)} Ki(\mathbf{x},t) = 0$$

$$(\Box - m^2)G(x,x') = \frac{i}{\sqrt{-g}}\delta(t-t')\delta^{(3)}(\mathbf{x}-\mathbf{x}') \quad \text{with} \lim_{t,t'\to-\infty(1+i\epsilon)} G(x,x') = \lim_{t,t'\to t_0} G(x,x') = 0$$

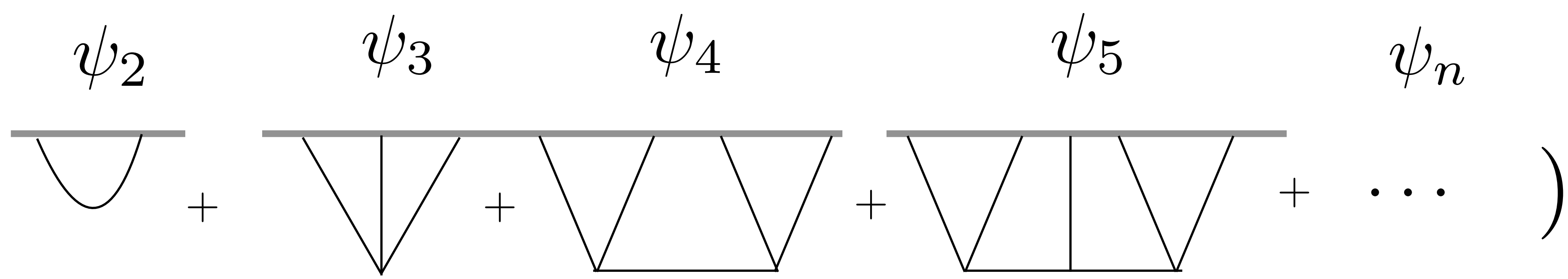
# Wavefunction method

- The semiclassical action generates **only tree-level** wavefunction coefficients

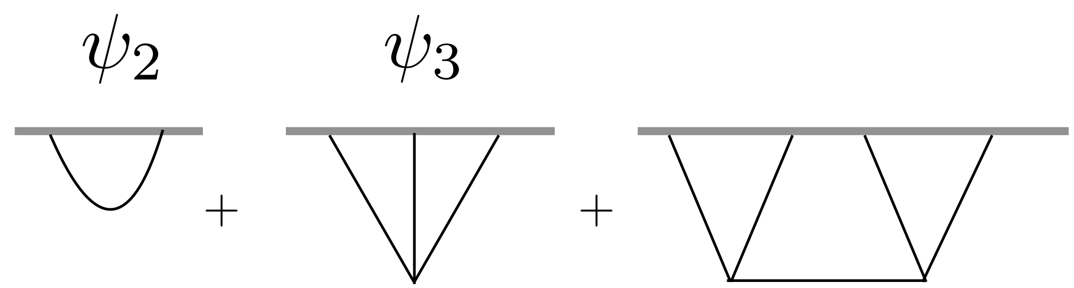
$$\Psi[\phi(\vec{x})] \simeq \exp\left(-\frac{i}{\hbar}(S_0[\Phi_{\rm cl}] + \hbar S_1[\Phi_{\rm cl} + \dots]\right)$$

$$S_0[\Phi_{\rm cl}] = \frac{1}{2!}\int_{\mathbf{k}} \psi_2 \phi_k^2 + \frac{1}{3!}\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \psi_3 \phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_4} + \frac{1}{4!}\int_{\mathbf{k}_1,\cdots,\mathbf{k}_4} \psi_4 \phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\phi_{\mathbf{k}_4} + \dots$$

$$\Psi[\phi_{\rm cl}] \sim \exp\Big($$



# Wavefunction coefficients

$\Psi[\phi_{\rm cl}] \sim \exp\Big($  $+ \cdots + \quad \Big)$



$$\psi_2' = \frac{ik^2}{H^2\eta_0} - \frac{k^3}{H^2} + \mathcal{O}(\eta_0)$$

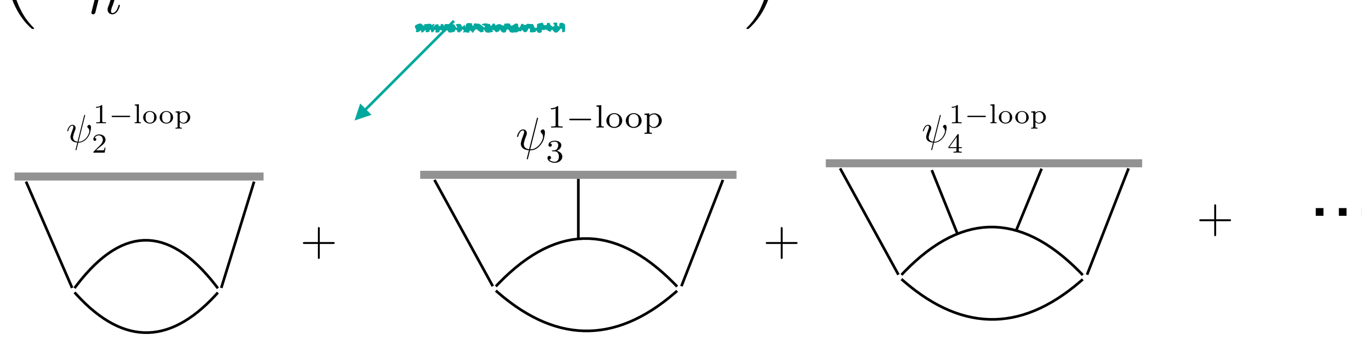
$$\psi_3' = -\frac{i}{3H^4\eta_0^3} - \frac{i(k_1^2+k_2^3+k_3^2)}{2H^4\eta_0} - \frac{1}{18H^4}(k_1^2(k_2+k_3)+k_1)(k_2^2-k_2k_3+k_3^2)$$
$$-\frac{1}{18H^4}(k_1^3+k_2^3+k_3^3)(8+6\gamma_E-3i\pi+\log((k_1+k_2+k_3)\eta_0)) + \mathcal{O}(\eta_0)$$

- Exchange diagrams are in general very cumbersome
- All IR divergent terms except the last one are phases

# 1-loop wavefunction

- Loop corrections are generated by quantum terms and are proportional to higher powers of $\hbar$

$$\Psi[\phi(\vec{x})] \simeq \exp\left(-\frac{i}{\hbar}(S_0[\Phi_{\rm cl}] + \hbar S_1[\Phi_{\rm cl}] + \dots\right)$$




- Quantum corrections are generated by the functional determinant as in the usual Wilsonian EFT

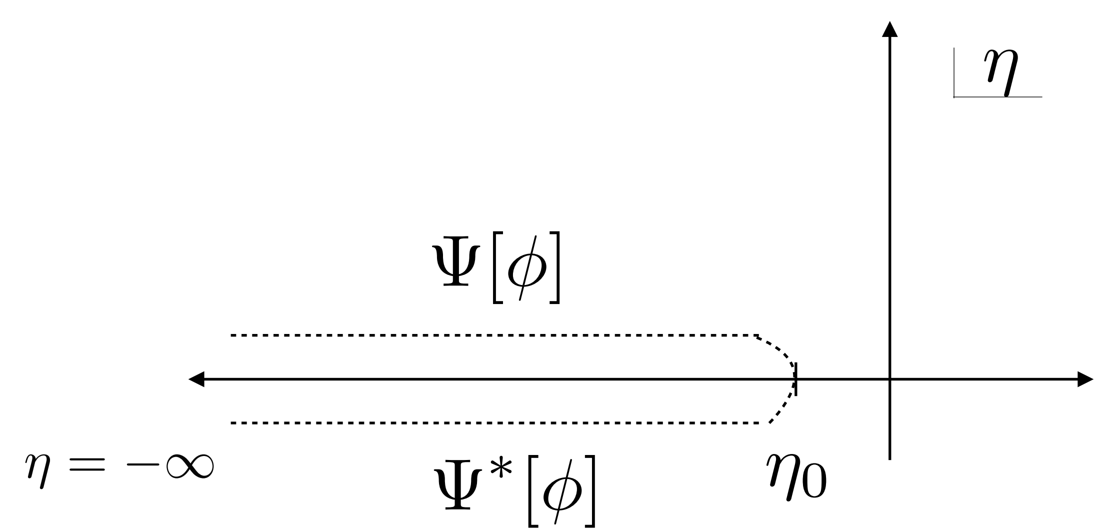
$$S_1[\Phi_{\rm cl}] = i{\rm Tr}\log\left(\frac{\delta^2 S}{\delta\Phi^2}\right)$$

$$S_1 \supset -ig^2\int d\eta a(\eta)^4 \int d\eta' a(\eta')^4 \int_{\mathbf{k},\mathbf{q}} G(k,\eta,\eta')G(|\mathbf{k}+\mathbf{q}|,\eta,\eta')\Phi(k,\eta)\Phi(k,\eta')$$

# Correlation functions

- The wave function is not an observable by itself but we can define $$P[\phi_k] \sim |\Psi[\phi]|^2$$
- The correct prescription is defined using the In-In formalism
- Correlation functions are computed using the Born rule

$$\langle \phi_{\vec{k}_1} \cdots \phi_{\vec{k}_n} \rangle = \frac{\int \mathcal{D}\phi \;\; \phi_{\vec{k}_1} \cdots \phi_{\vec{k}_n} \, |\Psi[\phi, \eta_0]|^2}{\int \mathcal{D}\phi \;\; |\Psi[\phi, \eta_0]|^2} .$$




- Eg. $$\langle \phi_k^2 \rangle = \frac{1}{2\mathrm{Re}(\psi_2)} = \frac{H^2}{2k^3}$$

# Correlation functions

- Then we have

$$\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' = -\frac{1}{2\,\mathrm{Re}\,\psi_2'(k)}\,,$$

$$\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \rangle' = -\frac{\mathrm{Re}\,\psi_3'(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)}{4\mathrm{Re}\,\psi_2'(k_1)\ \mathrm{Re}\,\psi_2'(k_2)\ \mathrm{Re}\,\psi_2'(k_3)}$$

$$\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \rangle' = \frac{\mathrm{Re}\,\psi_4'(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)}{8\prod_{a=1}^{4} \mathrm{Re}\,\psi_2'(k_a)} - \langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \rangle_{\mathrm{d}}'\,,$$

- Eg. 3-point correlation

$$\langle \phi_{k_1} \phi_{k_2} \phi_{k_3} \rangle' \sim \frac{H^3}{12} \frac{(k_1^3 + k_2^3 + k_3^3)}{k_1^3 k_2^3 k_3^3} (-2/3 + \gamma_E + \log(k_1 + k_2 + k_3)\eta_0)$$

- Phases do not contribute to the correlation function

- Correlation function has a secular divergence

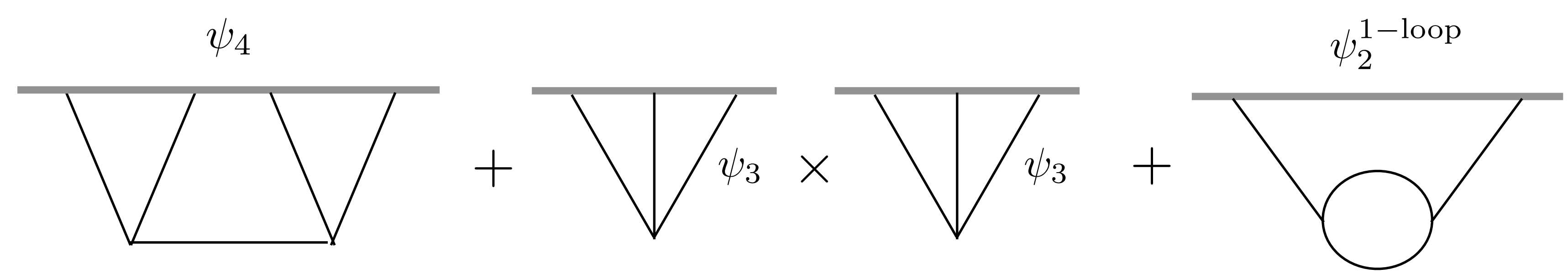
# Loops from wavefunctions

- Different wave functions coefficients gets mixed up when computing correlation functions

- Loop corrections contains term from the classical and the quantum effective action

- Eg. $V = g\Phi^3/3!$

$$\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle^{1-\mathrm{loop}} =$$

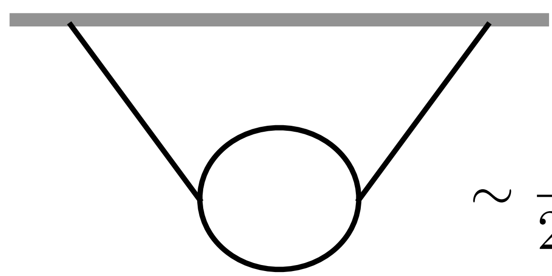



$$\sim \frac{g^2}{8k^3} \log(kL) \log(-2k\eta_0)^2 \quad \sim -\frac{5g^2}{18k^3} \log(kL) \log(-2k\eta_0)^2 \quad \sim \frac{g^2}{2k^3} \log(-2k\eta_0)^2$$

# Loops in correlators

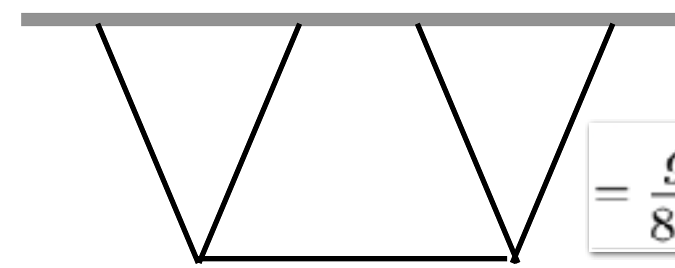
From Wavefunction to Correlators: One-Loop example for $g\Phi^3$

$\psi_2^{1-\text{loop}}$



$$\sim \frac{g^2}{2k^3}\log(-2k\eta_0)$$

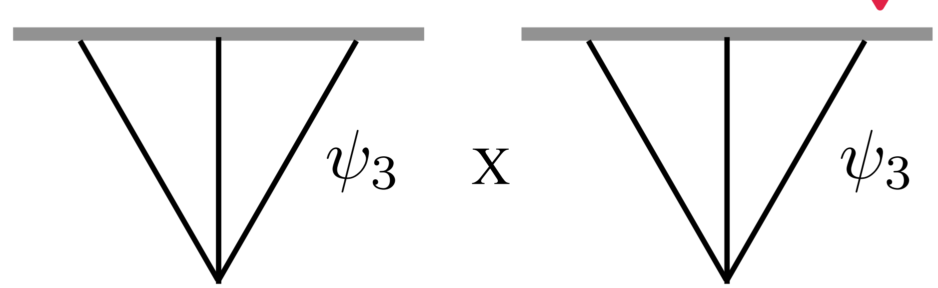
$\psi_4$



$$= \frac{g^2}{8k^3}\frac{-5}{18\pi^2}\log(kL)\log(-2k\eta_0)^2$$

$$\langle\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\rangle'_{1-\text{loop}} = \frac{\text{Re}\,\psi'^{1-\text{loop}}_{\mathbf{k}_1\mathbf{k}_2}}{2\text{Re}\,\psi_2'(k_1)\text{Re}\,\psi_2'(k_2)} - \frac{1}{8\text{Re}\,\psi_2'(k_1)\text{Re}\,\psi_2'(k_2)}\int_{\mathbf{p}}\frac{\text{Re}\,\psi_4'(\mathbf{k}_1,\mathbf{k}_2,\mathbf{p},-\mathbf{p})}{\text{Re}\,\psi_2'(p)}$$
$$+\frac{1}{8\text{Re}\,\psi_2'(k_1)\text{Re}\,\psi_2'(k_2)}\int_{\mathbf{p}}\left[\frac{\text{Re}\,\psi_3'(\mathbf{k}_1,\mathbf{p},-\mathbf{p}-\mathbf{k}_1)\text{Re}\,\psi_3'(\mathbf{k}_2,-\mathbf{p},\mathbf{p}+\mathbf{k}_1)}{\text{Re}\,\psi_2'(p)\text{Re}\,\psi_2'(|\mathbf{p}+\mathbf{k}_1|)}\right.$$
$$\left.+\frac{\text{Re}\,\psi_3'(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2)\text{Re}\,\psi_3'(\mathbf{k}_1+\mathbf{k}_2,\mathbf{p},-\mathbf{p})}{\text{Re}\,\psi_2'(p)\text{Re}\,\psi_2'(|\mathbf{k}_1+\mathbf{k}_2|)}\right]$$

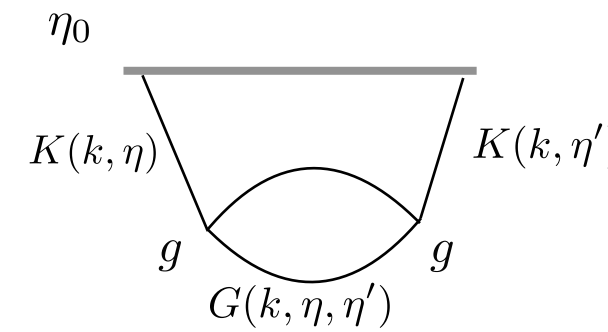
Classical loops




$$= \frac{g^2}{8k^3}\frac{1}{3\pi^2}\log(kL)\log(-2k\eta_0)^2$$

# Wavefunction is IR safe

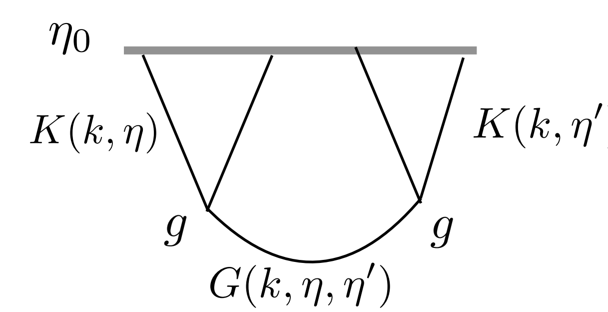
- Classical contributions have a IR loop divergence
- What we saw is a consequence of a more general result: there are no IR loop divergences in wave function coefficients




$$\psi_n^{L-\text{loop}} \sim \int d\eta_1 ... d\eta_m a(\eta_1)^4 ... a(\eta_m)^4 K(k_1,\eta_1) ... K(k_n,\eta_m)$$

$$\int_{\mathbf{p_1},...,\mathbf{p_L}} G(p_1,\eta_a,\eta_b) ... G(p_L,\eta_c,\eta_d) G(|\mathbf{p_x}+\mathbf{k_y}|,\eta_e,\eta_f) ...$$

$$\lim_{p\to 0} G(p,\eta,\eta') = -\frac{i}{6}H^2(\eta^3+\eta'^3) + \mathcal{O}(p)$$




- The highest IR divergence terms are in the semiclassical wavefunction

$$\Psi[\phi(\vec{x})] \simeq \exp\left(-\frac{i}{\hbar}(S_0[\Phi_{\rm cl}] + \hbar S_1[\Phi_{\rm cl}] + \dots\right)$$

*SC, Davis, Wang (23)*

*Gorbenko and Senatore 2019*

# Power counting argument

- To compare to the stochastic formalism let us take a look at the long wavelength correlation functions.

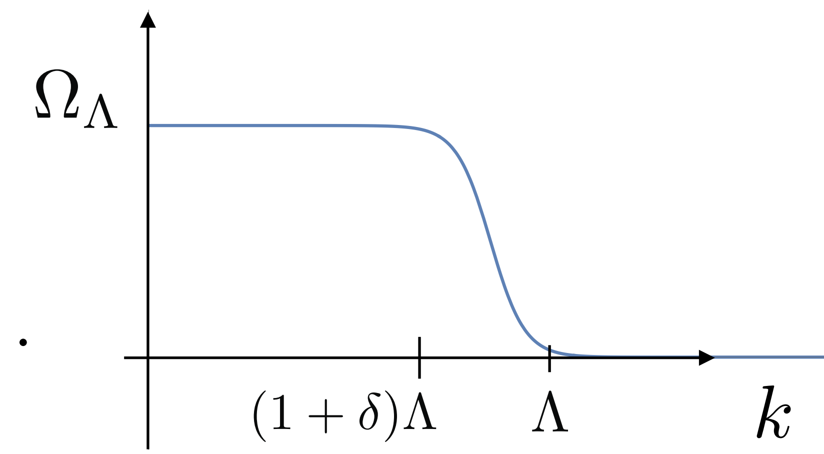
$$\Lambda(t) = \epsilon a(t) H$$

UV

$$\sigma^2 \equiv \langle \phi_l(\mathbf{x}=0)^2 \rangle = \int_{\mathbf{k}} \frac{\Omega_\Lambda(k)}{2\mathrm{Re}\psi_2(k)} = \frac{H^2}{4\pi^2} \log(\Lambda L_{\mathrm{IR}}) \ .$$

IR

$$L_{\mathrm{IR}}^{-1} = a(t_i) H$$




- Thus

$$\langle \phi_l^2 \rangle = \frac{H^2}{4\pi^2} \log a(t) \quad \Rightarrow \quad \langle \phi^2 \rangle = \frac{H^2}{4\pi^2} \log a - \frac{\lambda H^2}{144\pi^4} (\log a)^3 + \frac{\lambda H^2}{64\pi^6} (\log a)^2 + \mathcal{O}(\lambda^3 (\log a)^7)$$

# Semiclassical limit

- We have seen that the semiclassical wavefunction grows larger than all quantum corrections

- We can understand this as defining the coupling $\tilde{\lambda} = \lambda(\log a)^2$

$$\Psi \sim \exp\left(i\tilde{S} + \frac{i}{a}\tilde{S}_1 + \mathcal{O}(a^{-2})\right)$$

- In the limit $\lambda \to 0,\ \log a \to \infty,\ \tilde{\lambda} \to \text{const.}$ the semiclassical action dominates and we can neglect quantum corrections

$$\langle\phi^2\rangle = \frac{H^2}{4\pi^2}\log a\left(1 - \frac{\tilde{\lambda}}{36\pi^2} + \frac{\tilde{\lambda}}{16\pi^2}(\log a)^{-1} + \mathcal{O}(\tilde{\lambda}^3)\right)$$

# Beyond perturbation theory

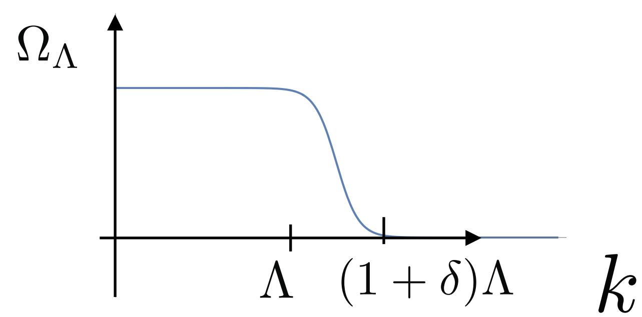
# Beyond perturbation theory

- We can remove the short wavelength modes directly from the partition function and consider a simpler object

$$P_\Lambda[\phi,t] = e^{W_0[\phi]+W_I[\phi]}$$

$$W_0 = \int_{\mathbf{k}} \mathrm{Re}\ \psi_2(k)\Omega_\Lambda^{-1}(k)\ \phi_{\mathbf{k}}\phi_{-\mathbf{k}}\ ,$$

$$W_I = \sum_{n=3}^{\infty}\frac{1}{n!}\int_{\mathbf{k}_1,\ldots,\mathbf{k}_n} 2\mathrm{Re}\ \psi_n^\Lambda(\mathbf{k}_1,\ldots,\mathbf{k}_n,t)\ \phi_{\mathbf{k}_1}\cdots\phi_{\mathbf{k}_n}$$




- This is the semiclassical limit of

$$P[\phi_l] = \int D\phi_l\ \delta\left(\phi_l - \int \frac{d^3k}{(2\pi)^3}\Omega_k\phi_k\right)|\Psi(\phi_k)|^2$$

# Beyond perturbation theory

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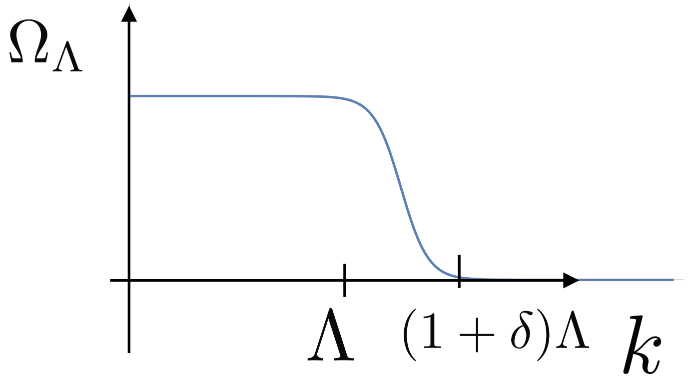



- The time derivative is $$\frac{d}{dt}P_\Lambda[\phi,t] = \frac{\partial}{\partial t}P_\Lambda[\phi,t] + \dot{\Lambda}\frac{\partial}{\partial \Lambda}P_\Lambda[\phi,t]$$

# Fokker-Planck from the wave function

- The partial derivative is given by the Hamiltonian unitary evolution

$$\frac{\partial P[\phi,\chi,t]}{\partial t} = [H,P] = -\int d^3x \frac{\delta}{\delta\phi(\vec{x})}\left(\Pi_\phi P[\phi,\chi,t]\right)$$

- Can be written as a field derivative over the long wavelength momentum

$$\frac{\partial P_\Lambda}{\partial t} = -\frac{1}{a^3}\int d^3x d^3x' \int_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\Omega_\Lambda(k)\frac{\delta}{\delta\phi_l(\mathbf{x})}\Pi_l(\mathbf{x},t)P_\Lambda = -\int d^3x \frac{\delta}{\delta\phi_l(\mathbf{x})}\left(\dot{\Phi}_l(\mathbf{x},t)P_\Lambda\right)$$

- In general this term is non linear $\dot{\Phi} \approx -\frac{V'(\Phi)}{3H}$

# Polchinski RG equation

- In QFT one can study the effect of removing high energy modes directly from the regulated generating functional

$$Z_\Lambda = \int D\varphi e^{-\frac{1}{2}\int_{\mathbf{k}} \frac{1}{\Omega_\Lambda}\varphi_{\mathbf{k}} G_k^{-1}\varphi_{-\mathbf{k}} - S_{\rm int}}$$

- Moving the cutt-off doesn't affect observable thus,

$$\Lambda \frac{d}{d\Lambda} Z_\Lambda = 0$$

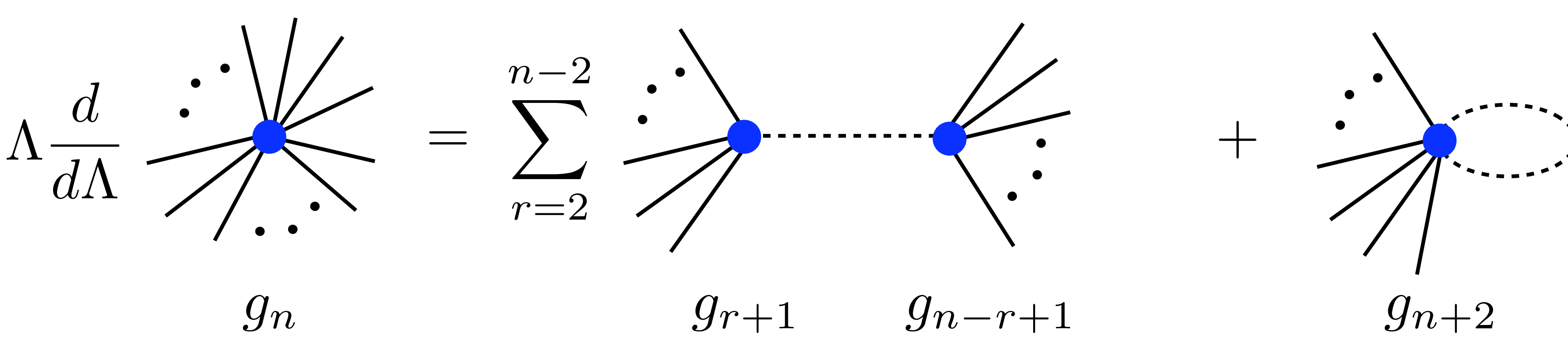
- This implies an equation for the interactions

$$\Lambda \frac{d}{d\Lambda} e^{-S_{\rm int}^{\Lambda}} = -\frac{1}{2}\int \frac{d^d k}{(2\pi)^d} \frac{d\Omega_\Lambda}{d\ln\Lambda} G_k \frac{\delta^2}{\delta\varphi_{\mathbf{k}}\delta\varphi_{-\mathbf{k}}} e^{-S_{\rm int}}$$

# Polchinski RG equation

- If we rewrite it as,

$$\Lambda \frac{d}{d\Lambda} S_{\text{int}}^{\Lambda} = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{d\Omega_\Lambda}{d \ln \Lambda} G_k \left( \left( \frac{\delta}{\delta \phi_k} S_{\text{int}} \right)^2 + \frac{\delta^2 S_{\text{int}}}{\delta \varphi_{\mathbf{k}} \delta \varphi_{-\mathbf{k}}} \right)$$




- Flow equation for integrating out high energy degrees of freedom

# Polchinsky equation for correlators

- In the case of cosmology we have to taken into account the different contour

- At tree level we can use that for $P_\Lambda[\phi, t] = e^{W_0[\phi]+W_I[\phi]}$

$$\frac{d}{d\ln\Lambda}\int D\phi P_\Lambda[\phi, t] = 0$$

- This implies a Polchinsky RG type equation.

$$e^{W_0}\frac{de^{W_I}}{d\log\Lambda} = \frac{1}{4}\int_{\mathbf{k}}\frac{d\Omega_\Lambda}{d\log\Lambda}\frac{1}{\mathrm{Re}\psi_2}\left[\left(\frac{\delta^2}{\delta\phi_{\mathbf{k}}\delta\phi_{-\mathbf{k}}}e^{W_I}\right)e^{W_0} - 2\frac{\delta}{\delta\phi_{\mathbf{k}}}\left(e^{W_0}\frac{\delta e^{W_I}}{\delta\phi_{-\mathbf{k}}}\right)\right]$$

- Extra boundary terms, allows matching with correlation functions

# Fokker-Planck equation

- The Polchinski equation can be rewritten as

$$\frac{d}{d\ln\Lambda}P_\Lambda = -\frac{1}{4}\int_{\mathbf{k}}\frac{1}{\mathrm{Re}\psi_2}\frac{d\Omega_\Lambda}{d\ln\Lambda}\frac{\delta^2}{\delta\phi_{\mathbf{k}}\delta\phi_{-\mathbf{k}}}P_\Lambda$$

- Joining both terms we recover the Fokker-Planck equation

$$\frac{d}{dt}P_\Lambda[\phi,t] = \frac{\partial}{\partial t}P_\Lambda[\phi,t] + \dot{\Lambda}\frac{\partial}{\partial\Lambda}P_\Lambda[\phi,t] \quad \left\{ \begin{array}{l} \dot{\Lambda}\dfrac{\partial}{\partial\Lambda}P_\Lambda[\phi,t] = \dfrac{H^3}{8\pi^2}\dfrac{\partial^2 P_\Lambda}{\partial\phi^2} \\ \dfrac{\partial}{\partial t}P_\Lambda[\phi,t] = -\dfrac{\partial}{\partial\phi}\left(\Pi_\Lambda P_\Lambda\right) = \dfrac{\partial}{\partial\phi}\left(\dfrac{V_\phi}{3H}P_\Lambda\right) \end{array} \right.$$

- Fokker-Planck equation is a summation of (semi classical) terms

# Fokker-Planck equation

- Joining both terms we recover the usual Fokker-Planck equation

$$\frac{d}{dt}P_\Lambda[\phi,t] = \frac{\partial}{\partial\phi}\left(\frac{V_\phi}{3H}P_\Lambda\right) + \frac{H^3}{8\pi^2}\frac{\partial^2 P_\Lambda}{\partial\phi^2}$$

*Starobinsky ’83,*
*Starobinsky and Yokoyama ‘92*

- We can check by solving in perturbation theory

$$\langle\phi^2\rangle = \frac{H^2}{4\pi^2}\log a - \frac{\lambda H^2}{144\pi^4}(\log a)^3 + \frac{\lambda^2 H^2}{2880\pi^6}(\log a)^5 + \mathcal{O}(\lambda^3(\log a)^7$$

*SC, Davis, Wang 2023*

Matches with classical loops

- Higher order quantum corrections are subleading and not considered in this equation. See however

*Gorbenko and Senatore ’19*
*Mirbabayi ’20*
*Green and Cohen ‘21*

# Quantum corrections

- In order to go beyond the semiclassical wavefunction we need to to compute the 2PI effective action

- The effect of the long modes acts as a mass for the short modes dressing the propagators

$$\left(-\Box + \frac{\lambda}{6}\langle\phi_r^2\rangle\right) G^K(x,y) = i\delta(x,y) + \mathcal{O}(\lambda^2)$$

- This changes the diffusion term

$$\text{Diff} = \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\phi^2}\left(\left(1 + \frac{2\lambda}{H^2}(\log\Lambda - \psi(3/2))\phi^2\right) P(\phi)\right) \; ,$$

*Gorbenko and Senatore '19*
*Mirbabayi '20*
*Green and Cohen '21*
*SC and Colas (In progress)*

# Conclussions

- We have showed how to recover the Fokker-Planck equation from the wavefunction of the universe approach
- The relation between the semiclassical action and the stochastic theory becomes manifest
- The resumption of IR divergent terms is achieved through a Fokker-Planck equation
- This approach is non perturbative and can help to compute the whole PDF for a given EFT
- Many directions to follow (compute corrections, phenomenological implications)

# In-In formalism

- This method is equivalent to the usual in-in formalism.
- Defining the partition function

$$Z[J_1, J_2] = \int D\phi_k \int_{\rm BD}^{\phi_k} \mathcal{D}\Phi_1 \int_{\rm BD}^{\phi_k} \mathcal{D}\Phi_2 e^{iS[\Phi_1]-iS[\Phi_2]} e^{i\int J_1\Phi_1 - i\int J_2\Phi_2}$$

- Solving the e.o.m with sources

$$Z[J_1, J_2] = \int \mathcal{D}\phi_k \exp\Big(-\int_{\mathbf{k}} \psi_\varphi^{(2)}\phi_k^2 + i\int_{\mathbf{k}}\int dt\,(J_1(t,\mathbf{k})K(t,\mathbf{k}) - J_2(t,\mathbf{k})K^*(t,\mathbf{k}))\,\phi_k$$
$$-\frac{1}{2}\int_{\mathbf{k}}\int dtdt' J_1(t,\mathbf{k})G(t,t',\mathbf{k})J_1(t',\mathbf{k}) - J_2(t,\mathbf{k})G^*(t,t',\mathbf{k})J_2(t',\mathbf{k})\Big)$$

- This leads to

$$Z[J_1, J_2] = \exp\Big(-\frac{1}{2}\int_{\mathbf{k}}\int dtdt'[J_1(t)\Delta_{++}(\mathbf{k},t,t')J_1(t') - J_2(t)\Delta_{--}(\mathbf{k},t,t')J_2(t') + J_1(t)\Delta_{+-}(\mathbf{k},t,t')J_2(t')]\Big)$$

# General loop results

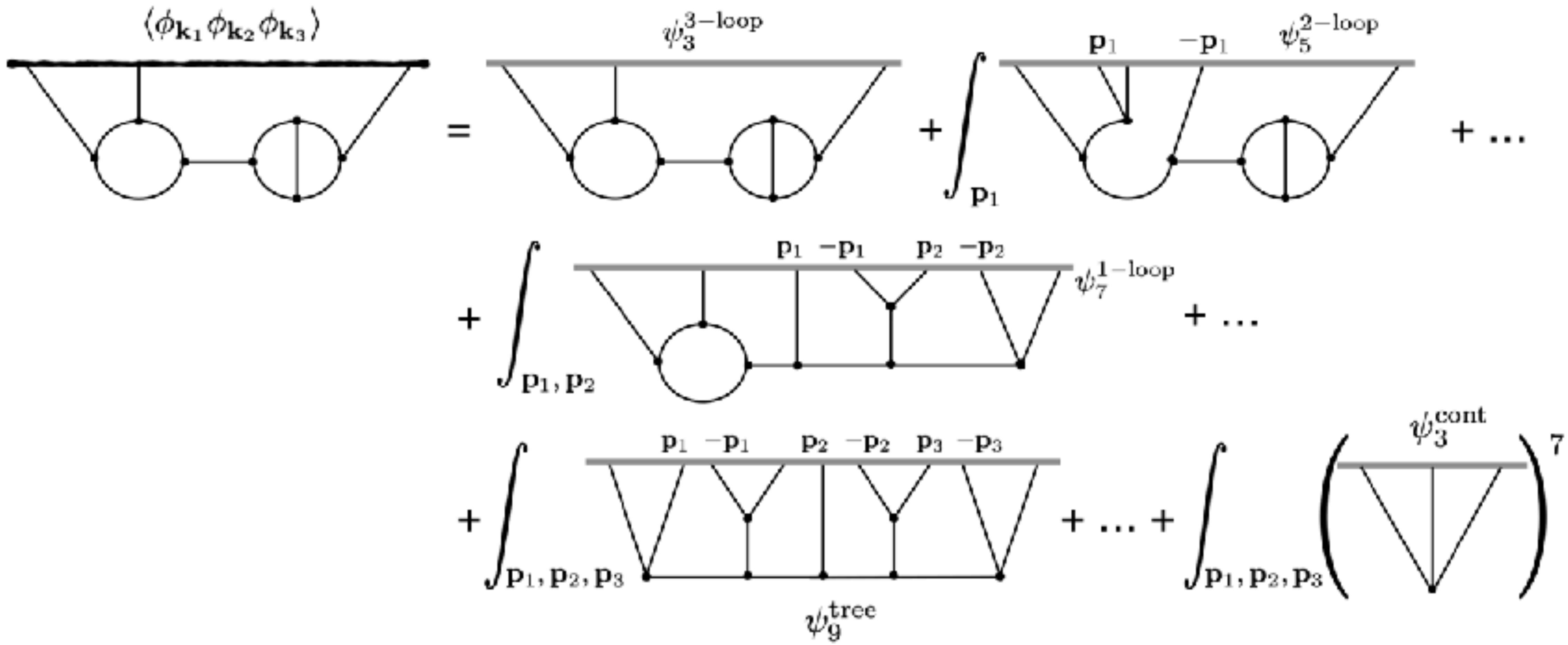
$$\langle\phi^n\rangle_{L-\text{loop}} \sim \frac{\text{Re}\,\psi_n^{L-\text{loop}}}{(\text{Re}\psi_2)^n} + \int_{\mathbf{p_1},\ldots,\mathbf{p_a}} \frac{(\text{lower-order loop }\psi)}{\text{Re }\psi_2 \ldots \text{Re }\psi_2} + \ldots$$
$$+ \int_{\mathbf{p_1},\ldots,\mathbf{p_b}} \frac{(\text{exchange }\psi)}{\text{Re }\psi_2\ldots\text{Re }\psi_2} + \ldots + \int_{\mathbf{p_1},\ldots,\mathbf{p_L}} \frac{(\text{contact Re }\psi_3)^V}{(\text{Re}\psi_2)^{(3V+n}/2}$$

$$\propto \lambda^V \log(kL_{\text{IR}})^L \log(-k\eta_0)^V$$

in agreement with Baumgart, Sundrum 2019

IR-divergent correlators are always dominated by Classical loops

IR divergence ⟷ Superhorizon physics ⟷ Semi-classical effect



$\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \rangle$
=
$\psi_3^{3-\text{loop}}$
+
$\int_{\mathbf{p}_1}$
$\mathbf{p}_1$
$-\mathbf{p}_1$
$\psi_5^{2-\text{loop}}$
+ …
+
$\int_{\mathbf{p}_1, \mathbf{p}_2}$
$\mathbf{p}_1$ $-\mathbf{p}_1$ $\mathbf{p}_2$ $-\mathbf{p}_2$
$\psi_7^{1-\text{loop}}$
+ …
+
$\int_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3}$
$\mathbf{p}_1$ $-\mathbf{p}_1$ $\mathbf{p}_2$ $-\mathbf{p}_2$ $\mathbf{p}_3$ $-\mathbf{p}_3$
$\psi_9^{\text{tree}}$
+ … +
$\int_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3}$
$\psi_3^{\text{cont}}$
7